Hopf Bifurcation and Chaotic Response in Nonlinear Dynamics of Firing-Rate Recurrent Networks of Neurons

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***Abstract*— In-depth analysis of the nonlinear dynamics and chaotic behaviour of interconnection of neurons has been made in order to investigate the learning capabilities of this interconnection. Firing-rate recurrent neural networks are used to study the neuronal behaviour in a population of neurons. Dynamical behaviour of these network models is investigated in order to seek their capability to represent the presence of chaos in nervous system. Study of chaos and other phenomena of nonlinear dynamics in these network models can provide a significant help in investigating the learning mechanism. It is found that the response of the network highly depends on its parameters. Such type of model exhibits all types of dynamics namely converging, oscillatory, and chaotic with the variation in the synaptic weights.**

***Keywords*— Recurrent Neural Networks, Firing Rate, Nonlinear Dynamics, Hopf Bifurcation, Chaos.**

I. Introduction

The brain is one of the most complex objects in the universe. Although many attempts have been made to investigate and model the functionalities of the brain, the exact working of it is still unknown. The research in the field of computational neuroscience is aimed to know about the brain with more intricacy and to develop more realistic models of its constituents. These models are important tools for characterizing what nervous systems do, determining how they function, and understanding why they operate in particular ways. As most of these models are dynamical in nature, theory of dynamical systems is useful in gaining new insights into the operation of nervous system. The primary step for understanding the brain dynamics is to understand the dynamical behaviour of mathematical models of individual neurons. The most important part of this study is the bifurcation analysis of the neurons and their networks. Certain bifurcations in the membrane potential result in neural excitability, spiking, and bursting. Revealing these bifurcations in neuron models helps in knowing various functions of the brain. Such types of studies include the analysis of chaotic behaviour of neural systems. These neural systems can be individual neurons or their interconnections. The ongoing research in this regard is to examine the role of chaos in learning. Exploring dynamics of biological neuron models is helpful not only in neuroscience studies but also in neural network applications. Capabilities of existing artificial neural networks are extremely less as compared to that of a human brain. Artificial neural networks mimic only a negligibly small part of the actual activities in brain. It is logical to seek the possibilities of improvements in artificial neural networks by incorporating more of biological facts.

In literature, different dynamical models are proposed to represent biophysical activities of neurons. Commonly used models for the study of spiking and bursting behaviours of neurons include integrate-and-fire model and its variants [5], [25], FitzHugh-Nagumo model [6], Hindmarsh-Rose model [14], [10], Hodgkin-Huxley model [10], [11], and Morris-Lecar model [20]. A short review of these models is provided by Rinzel in [21] - [23]. An excellent comparison of more than twenty neurocomputational properties of the most popular spiking and bursting models have been made in [14]. Bifurcation phenomena in individual neuron models including the Hodgkin-Huxley, Morris-Lecar and FitzHugh-Nagumo models have been investigated in the literature [14], [22], [4]. Rinzel and Ermentrout [22] studied bifurcations in the Morris-Lecar model by treating the externally applied direct current as a bifurcation parameter. Effect of noise on the dynamics of biological neuron models has been investigated in [19].

After studying the dynamics of individual neurons, the next step to study brain dynamics is to analyse the neuronal behaviour in a network of neurons. A neuron communicates with other neurons via electrical impulses called spikes. From various experiments, it has been well established that neuronal activities show many characteristics of chaotic behaviour. Some researchers believe that this sort of behaviour is necessary for the brain to engage in continual learning – categorizing a novel input into a novel category rather than trying to fit it into an existing category [24], [8], [1]. Freeman developed a mathematical model for EEG signals generated by the olfactory system in rabbits [8]. He suggested that the learning and recognition of novel odours, as well as the recall of familiar odours can be explained through chaotic dynamics of the olfactory cortex. Chaotic response in the models of single neurons has been observed in [17] and a similar analysis on coupled neurons has been performed in [18]. Attempts have been made to represent the neuronal dynamics of biological neural networks in terms of artificial neural network type of structures with some extent to their intricacies. Chaotic dynamics based neural networks have also been proposed to capture some of the characteristics of learning in the brain [2].

In this paper work, nonlinear dynamical analysis of a firing-rate recurrent neural network of three neurons has been carried out and it is observed that its dynamical behaviour exhibits Hopf bifurcation and becomes chaotic at some set of parametric values. This study supports the role of chaos in continual learning– categorizing a novel input into a novel category rather than trying to fit it into an existent category.

II. INTRODUCTION TO NONLINEAR DYNAMICS AND CHAOS

A dynamical system consists of a rule which specifies how a system evolves and an initial condition at which the system starts. The most common form of rules is a set of differential equations [21]. A dynamical system is said to be nonlinear if it is described by nonlinear differential equations. A nonlinear system exhibits various types of dynamics including converging, oscillatory and chaotic. For dynamical analysis of nonlinear systems, eigenvalue analysis, Lyapunov exponents, and bifurcation diagrams are three major tools. These tools are used to detect the qualitative change in the dynamical behaviour of the system when a parameter is changed. This phenomenon is called bifurcation.

A. Eigenvalue Analysis

Consider the following nonlinear dynamical system

$$dx(t)/dt = F(x(t); \mu) \tag{1}$$

where $x(t) = x_1(t), x_2(t), \ldots, x_n(t)$ is the state vector and μ is a parameter. Equilibrium points are obtained from the condition $dx/dt = 0$. Therefore, for any μ, the equilibrium point $x_e(\mu)$ satisfy the following algebraic equation

$$F(x_e(\mu); \mu) = 0 \tag{2}$$

Jacobian matrix evaluated at the equivalent point is $J(\mu)$. Eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the characteristic equation

$$det(\lambda I - J(\mu)) = 0 \tag{3}$$

where I is the $n \times n$ identity matrix. Suppose in a neighbourhood of a particular value μ_0 of the parameter μ there is a pair of eigenvalues of $J(\mu)$ of the form $\lambda_{real}(\mu) \pm i\lambda_{imag}(\mu)$ such that $\lambda_{real}(\mu_0) = 0$, $\lambda_{imag}(\mu_0) = 0$, no other eigenvalue of $J(\mu_0)$ has a pair of pure imaginary eigenvalues. Suppose further that the rate of change of the real part of eigenvalues is nonzero at μ_0, i.e.,

$$d\lambda_{real}/d\mu \ (at\ \mu = \mu_0) = 0 \tag{4}$$

then a limit cycle bifurcates from the origin with an amplitude that grows proportional to $|\mu|^{½}$ while its period tends to $2\pi/\lambda_{imag}$ as $\mu \rightarrow \mu_0$. This bifurcation is called Hopf bifurcation.

B. Lyapunov Exponents

Trajectories of chaotic systems are very sensitive to the initial conditions. Starting from slightly different initial conditions the trajectories diverge exponentially. Let d_0 denote the distance between the two initial states of a continuous dynamical system. Then, for a chaotic motion, at a later time,

$$d(t) = e^{\mu t} d_0 \tag{5}$$

The measure of the divergence of trajectories is obtained by averaging the exponential growth at many points along a trajectory [27]. To define this, first a reference trajectory is obtained. Then, a point on an adjacent trajectory is selected and $d(t)/d_0$ is measured. After this, a new $d_0(t)$ is selected on new adjacent trajectory. Lyapunov exponent is computed as

$$\mu(x(0)) = \lim_{N\rightarrow\infty} \frac{1}{t_N - t_0} \sum_{k=1}^{N} ln \frac{d(t_k)}{d_0(t_k - 1)} \tag{6}$$

There are n Lyapunov exponents for an n-dimensional nonlinear dynamical system. To define these Lyapunov exponents, an n dimensional sphere cantered at a point on a reference trajectory is chosen. Let the radius of this trajectory be δ_0. At a later time, an n-dimensional ellipsoid is constructed with the property that all the trajectories emanating from the previously chosen sphere pass through this ellipsoid. Let the n semiaxes of the ellipsoid be denoted by $\delta_i(t)$, Lyapunov exponent is computed as

$$\mu(x(0)) = \lim_{t\rightarrow\infty} \left(\lim_{\delta_i(0)\rightarrow 0} \left(\frac{1}{t} ln \frac{\delta_i(t)}{\delta_i(0)} \right) \right); i = 1,2, \ldots \ldots, n \tag{7}$$

A system is chaotic if there exists at least one positive Lyapunov exponent. Plot of the largest Lyapunov exponent with respect to the bifurcation parameter gives the range of the parameter for which there exists at least one positive Lyapunov exponent. Thus, system exhibits chaotic behaviour for this range of the parameter.

C. Bifurcation Diagram

Bifurcation diagrams are pictorial representation of qualitative change in the dynamical behaviour of a system when a parameter is varied. This parameter is called bifurcation parameter and its values at which bifurcation takes place are called bifurcation points. The horizontal axis of a bifurcation diagram has the parameter and the vertical axis has some aspect of the solution, such as, the norm of the solution, the maximum and/or minimum values of one of the state variables, the frequency of a solution, or the average of one of the state variables.

III. DYNAMICAL ANALYSIS OF FIRING-RATE RECURRENT NEURAL NETWORK

Firing-rate recurrent neural networks are used to study the neuronal behaviour in a population of neurons. Dynamical behaviour of these network models is investigated in order to seek their capability to represent the presence of chaos in nervous system. Study of chaos and other phenomena of nonlinear dynamics in these network models can provide a significant help in investigating the learning mechanism. It is found that the response of the network highly depends on its parameters. Such type of model exhibits all types of dynamics namely converging, oscillatory and chaotic with the variation in the synaptic weights.

A. Models of Biological Neural Networks

Widespread synaptic connectivity is a characteristic of neural circuitry. Network models permit us to discover the computational potential of such connectivity, using both analysis and simulations. These networks have been considered to investigate the various tasks performed by them. These tasks include coordinate transformations needed in visually guided reaching, discriminatory amplification leading to models of simple and complex cells in primary visual cortex, amalgamation as a model of short-term memory, noise reduction, input selection, gain modulation, and associative memory [3]. There are two ways to simulate neural networks: one is based on the action potential and another one is based upon the firing rate. The first one presents noteworthy computational and interpretational challenges. Firing-rate models avoid the short time scale dynamics required to simulate action potentials and thus are much easier to simulate on computers [2].

B. Firing Rate Models

The construction of a firing-rate model proceeds in two steps. First, it is determine how the total synaptic input to a neuron depends on the firing rates of its presynaptic afferents. This is where the firing rates are used to approximate neural network functions. Second, the dependency of firing rate of the postsynaptic neuron on its total synaptic input is formed. Firing rate response curves are usually measured by injecting current (I_s) into the soma of a neuron. Letter u is used to symbolize a presynaptic firing rate and v to symbolize a postsynaptic rate [3].

C. Feedforward and Recurrent Networks

Examples of network models with feedforward and recurrent connectivity are shown in Figure 1. The feedforward network of Figure 1(a) has N_V output units with rates v_i $(i = 1, 2, 3, \ldots, N_V)$, denoted jointly by v determined by N_U input units with rates u_i $(i = 1, 2, 3, \ldots, N_U)$, denoted jointly by $\boldsymbol{u}$. The output firing rates are then determined by Equations 8 and 9.

$$\tau_r \frac{\partial v_i}{\partial t} = -v_i + F \sum_{j=1}^{N_u} W_{ij} u_j \qquad (8)$$

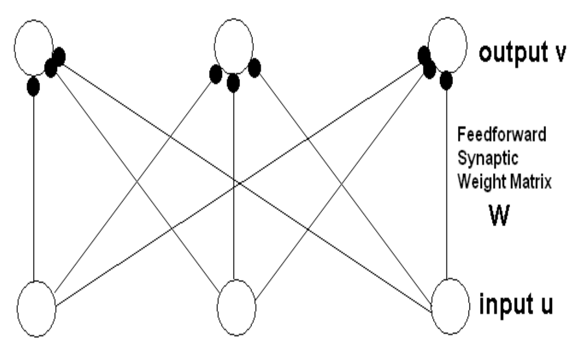
$$\tau_r \frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{v} + F(\boldsymbol{W}.\boldsymbol{u}) \qquad (9)$$

where F is any activation function. It is commonly taken to be a saturating function such as a sigmoid function. The recurrent network of Figure 1(b) also has two layers of neurons with rates $\boldsymbol{u}$ and v, but in this case the neurons of the output layer are interconnected with synaptic weights described by a matrix w. Matrix element w_{ij^0} describes the strength of the synapse from the output unit j^0 to output unit i. The output rates in this case are determined by Equations 10 and 11.

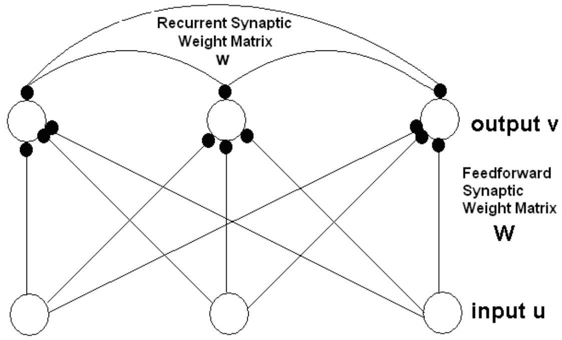
$$\tau_r \frac{\partial v_i}{\partial t} = -v_i + F\left(\sum_{j=1}^{N_u} W_{ij} u_j + \sum_{j'=1}^{N_v} w_{ij'} v_{j'}\right) \qquad (10)$$

$$\tau_r \frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{v} + F(\boldsymbol{W}.\boldsymbol{u} + \boldsymbol{w}.\boldsymbol{v}) \qquad (11)$$

A firing-rate recurrent neural network with $N_V = 3$ and $W = 0$ is considered for this study. Dynamics of this model is studied at different values of synaptic strength w. Condition for Hopf bifurcation is determined with the help of eigenvalue analysis of the linearized system around its equilibrium points. The presence of chaos is investigated by calculating the largest Lyapunov exponent and plotting bifurcation diagram, time responses and phase portraits at some relevant values of parameters. The nonlinear differential equations for the above model are given in Equations 12-14.



(a) Feedforward network



(b) Recurrent network

Fig. 1. Feedforward and Recurrent networks. In the case of feedforward networks, the neurons of the output layer are not interconnected while they are interconnected with synaptic weights in case of the recurrent network.

$$\frac{\partial x(t)}{\partial t} = f_1(w_{12}y(t) + w_{13}z(t)) - \alpha_1 x(t) \quad (12)$$
$$\frac{\partial x(t)}{\partial t} = f_2(w_{21}x(t) + w_{23}z(t)) - \alpha_2 y(t) \quad (13)$$
$$\frac{\partial z(t)}{\partial t} = f_3(w_{31}x(t) + w_{32}y(t)) - \alpha_3 z(t) \quad (14)$$

The response function f_i is given in Equation 15.

$$f_i(s) = \frac{1}{1+e^{-\beta_i(s-\theta_i)}} \quad (15)$$

$x(t)$, $y(t)$, and $z(t)$ are the output spike-rates (i.e., elements of v) of neurons represented by subscripts 1, 2, and 3 respectively. These quantities are interpretable as short-term average of firing-rates of respective neurons. θ_i is the threshold and β_i is the slope of transfer function of neuron i. α_i is the decay rate of the neuron i. w_{ij} is the synaptic strength of the connection from neuron j to neuron i.

D. Eigenvalue Analysis of the Model

The equilibrium state (x_e, y_e, z_e) of the system is given by the solution of the following set of equations

$$f_1(w_{12}y_e(t) + w_{13}z_e(t)) - \alpha_1 x_e(t) = 0 \quad (16)$$
$$f_2(w_{21}x_e(t) + w_{23}z_e(t)) - \alpha_2 y_e(t) = 0 \quad (17)$$
$$f_3(w_{31}x_e(t) + w_{32}y_e(t)) - \alpha_3 z_e(t) = 0 \quad (18)$$

Linearizing the system around the equilibrium points (x_e, y_e, z_e), we get the following system matrix:

$$J = \begin{bmatrix} -\alpha_1 - u & \beta_1 w_{12}\, a & \beta_1 w_{13}\, a \\ \beta_2 w_{21}\, b & -\alpha_2 - u & \beta_2 w_{23}\, b \\ \beta_3 w_{31}\, c & \beta_3 w_{32}\, c & -\alpha_3 - u \end{bmatrix}$$

where

$$a = F(\beta_1(w_{12}\, y_e + w_{13}\, z_e) - \theta_1)$$
$$b = F(\beta_1 (w_{21}\, x_e + w_{23}\, z_e) - \theta_2)$$
$$c = F(\beta_1(w_{31}\, x_e + w_{32}\, y_e) - \theta_3)$$

Here, $F(a)$ is given by

$$F(a) = \frac{e^{-a}}{(1+e^{-a})^2} \quad (19)$$

We can form the characteristic equation by substituting the above matrix J in $|\lambda I - J| = 0$. Thus, we get the following characteristic equation

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \text{……..(20)}$$

where

$$A = \alpha_1 + \alpha_2 + \alpha_3$$

$$B = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 - (\beta_1 \beta_2 w_{12} w_{21}\, ab + \beta_2\beta_3 w_{23} w_{32}\, bc + \beta_1\beta_3 w_{31} w_{13}\, ca)$$

$$C = \alpha_1 \alpha_2 \alpha_3 - (\alpha_3\beta_1\beta_2 w_{12} w_{21}\, ab + \alpha_2\beta_3 \beta_1 w_{31} w_{13}\, bc + \alpha_1\beta_3\beta_2 w_{23} w_{32}\, ac) - \beta_1\beta_2\beta_3\, abc\, (w_{23} w_{31} w_{13} + w_{21} w_{32} w_{13})$$

By applying Routh-Hurwitz criterion to investigate the values of A, B and C, for Hopf bifurcation, it is found that the system exhibits Hopf bifurcation if $A = 0$, $AB - C = 0$ or $C = 0$. w_{13} is considered as bifurcation parameter. Following values of other parameters are used: $\beta_1 = 7, \beta_2 = 7, \beta_3 = 15, \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7, \alpha_1 = 0.65, \alpha_2 = 0.42, \alpha_3 = 0.1, w_{12} = 1, w_{21} = 1, w_{23} = 0.1, w_{31} = 1$ and $w_{32} = 0.02$. Therefore, at these values of parameters, A, $AB - C$ and C are plotted against the bifurcation parameter w_{13} as shown in Figure 2. It is observed that at $w_{13} = -5.2$, $AB - C$ becomes zero. This indicates the possibility of Hopf bifurcation which is confirmed from time responses and phase portraits.

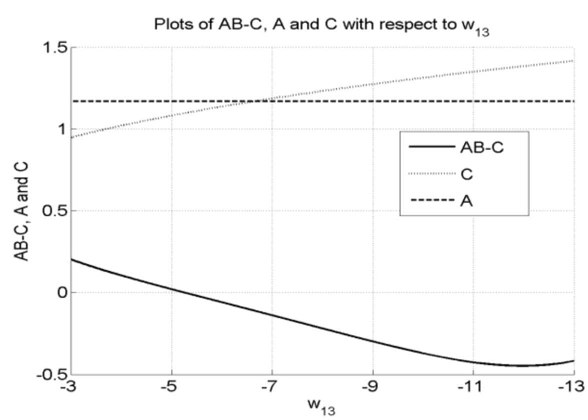
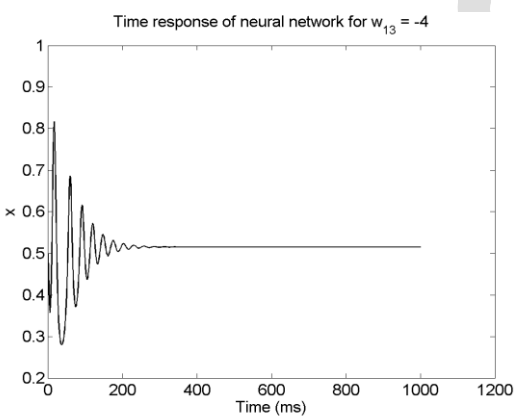


Fig. 2. Plots of A, $AB - C$, and C of the Routh array with respect to w_{13} . A, B, and C are the coefficients of the characteristic equation of the firing-rate recurrent neural network. It is observed that at

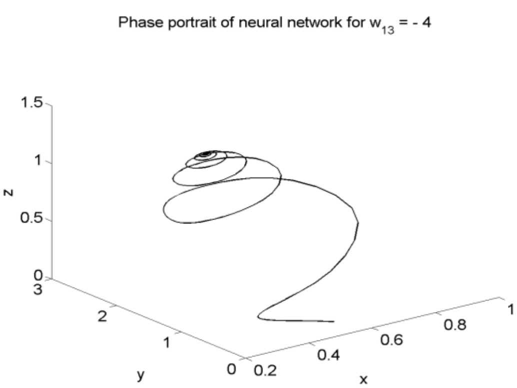
$w_{13} = -5.2$, $AB - C$ becomes zero and therefore Hopf bifurcation takes place at this value of w_{13}.

E. Time Responses and Phase Portraits

The system of three nonlinear differential Equations 12 – 14 are solved numerically. The time response and phase portraits of the system are plotted for different values of w_{13}. Its responses at different w_{13} show a qualitative change in the dynamics with the change of this parameter. It exhibits asymptotically stable response for $w_{13} = -4$, oscillatory response for $w_{13} = -5$ and chaotic response for $w_{13} = -6$. We have used following values of other parameters: $\beta_1 = 7, \beta_2 = 7, \beta_3 = 15, \theta_1 = 0.5, \theta_2 = 0.3, \theta_3 = 0.7, \alpha_1 = 0.65, \alpha_2 = 0.42, \alpha_3 = 0.1, w_{12} = 1, w_{21} = 1, w_{23} = 0.1, w_{31} = 1$ and $w_{32} = 0.02$. It is apparent from Figure 3 that at $w_{13} = -4$, the response converges to a fixed point attractor and therefore the response is stable. It is apparent from Figure 4 that at $w_{13} = -5$, the response is periodic and it does not converge to a fixed point attractor. In this case, attractor is a limit cycle. It shows an oscillatory response. Figure 5 shows that the response is chaotic at $w_{13} = -6$. For this parametric value, the response neither converges to a fixed point nor does it follow a limit cycle. A strange attractor is obtained in this case.



(a) Time response

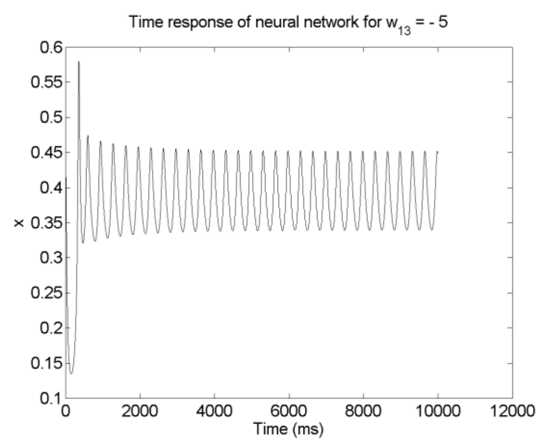


(b) Phase portrait

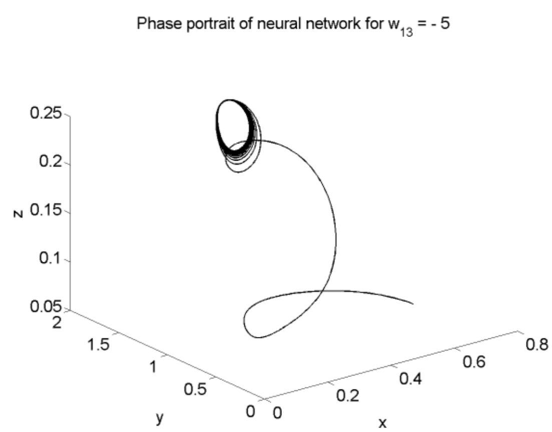
Fig. 3. Time response and phase portrait of the system for $w_{13} = -4$. The model exhibits a fixed-point attractor for this value of w_{13}.

F. Largest Lyapunov Exponents and Bifurcation Diagram

A standard way to determine the chaotic response is to find the presence of at least one positive Lyapunov exponent. The largest Lyapunov exponent is plotted against the bifurcation parameter and is shown in Figure 6(a). We found that the largest Lyapunov exponent becomes positive at $w_{13} = -6$. Dynamics of the system becomes chaotic at this value of w_{13}. In Figure 6(b), we show the bifurcation diagram for the model where $x(t)$ is plotted against w_{13}. It is observed that chaotic bifurcation takes place at $w_{13} = -6$. This is in agreement with the plot of largest Lyapunov exponent shown in Figure 6(a).

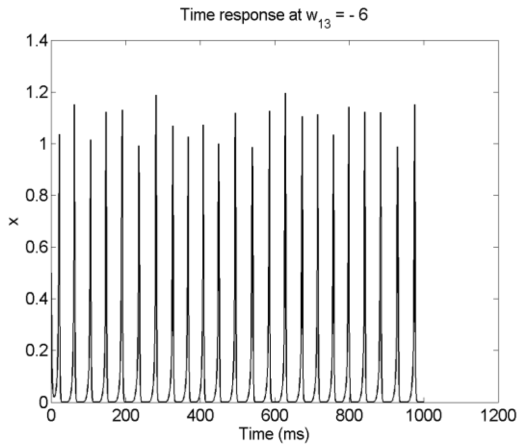


(a) Time response

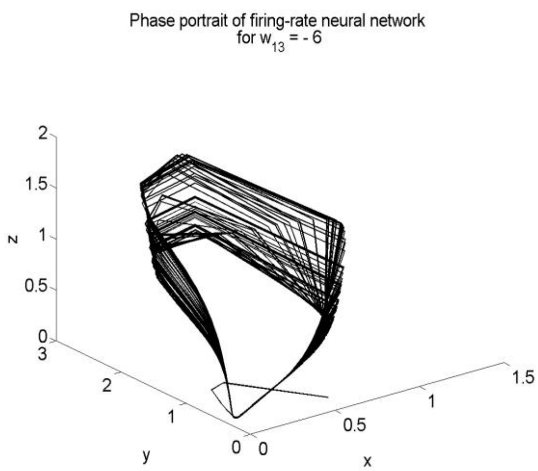


(b) Phase portrait

Fig. 4. Time response and phase portrait of the system for $w_{13} = -5$. The model exhibits a limit-cycle attractor for this value of w_{13}.

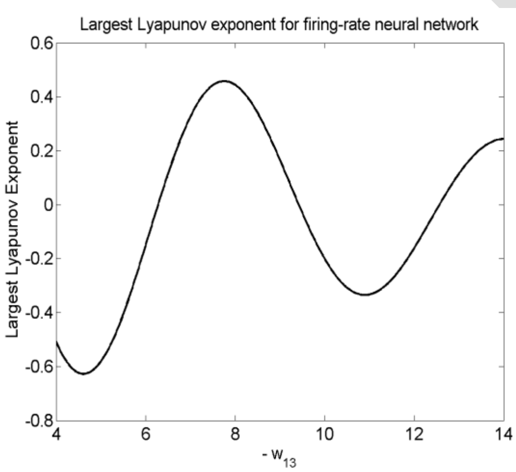


(a) Time response

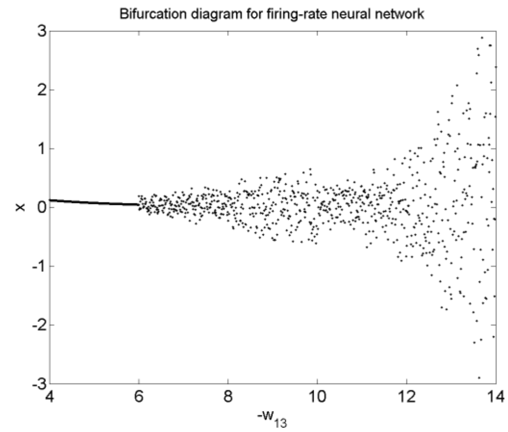


(b) Phase portrait

Fig. 5. Time response and phase portrait of the system for $w_{13} = -6$. The model exhibits chaotic behaviour for this value of w_{13}.



(a) Largest Lyapunov exponent



(b) Bifurcation diagram

Fig. 6. Plots of the largest Lyapunov exponent and bifurcation diagram for three-dimensional neural network model. (a)Largest Lyapunov exponent becomes positive at $w_{13} = -6$. Dynamics of the system becomes chaotic at this value of w_{13}. (b) Bifurcation diagram shows that the system dynamics changes from converging to periodic or chaotic as the bifurcation parameter is increased.

IV. CONCLUSIONS

One of the most effective approaches for the study of the nervous system is to look at its constituents as nonlinear dynamical systems. Dynamical analysis has been carried out on firing-rate recurrent neural networks. Eigenvalue analysis is performed for the detection of Hopf bifurcation and Lyapunov exponents are plotted for the study of chaos. It is found that a firing-rate recurrent neural network with three neurons exhibits chaotic attractor in addition to limit cycles and fixed-point attractors. It is observed that variation in synaptic strength causes qualitative change in its dynamical behaviour. Lyapunov exponent and eigenvalue analysis show that firing-rate recurrent neural network exhibit fixed points, limit cycles as well as chaotic (strange) attractors at various values of parameters. This research work can be extended in many directions. Dynamical analysis can be carried out in other neuron models (including stochastic neuron models) and their interconnections in order to study various characteristics of brain signals (e.g., bursting, chaos, stochastic resonance and threshold variability). Study of dynamics and learning capabilities of other biophysical neuron models and networks of synaptically coupled neurons can be performed for investigating biological significance of such type of learning.

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