Approximation of Fourier Series of a Function Belonging to the Class $Lip(\alpha)$ by $(\bar{N}, p_n, q_n)(E, s)$ - Summability Method

Laxmidhar Jena[1]*, Susanta Kumar Paikray[2] and Minakshi Dash[3]

[1]*Research Scholar, Department of Mathematics, Ravenshaw University Cuttack - 753003, India*

[2]*Department of Mathematics, VSS University of Technology, Burla - 768018, India*

[3]*Department of Mathematics, Ravenshaw University Cuttack - 753003, India*

Corresponding author: Susanta Kumar Paikray

Abstract: **- Approximation of periodic functions by different linear summation methods have been studied by many researchers. Further, for sharpening the estimate of errors out of the approximations several product summability methods were introduced by different researchers. In this paper a new theorem has been established on $(\bar{N}, p_n, q_n)(E, s)$-summability of Fourier series of a function belonging to $Lip(\alpha)$-class that generalizes several known results.**

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I. INTRODUCTION

Approximation of periodic functions of a given period arose in connection with the convergence of Fourier series. The object of such study is to estimate the minimum error obtained out of the approximations of a function in a given interval. The most important trigonometric polynomials used in approximation theory are obtained by different linear summation methods of Fourier series of 2π periodic functions in the real line R (i.e, Cesàro mean, Nörlund mean, matrix mean etc.). Much advances in the theory of trigonometric approximation has been studied by different researchers for periodic function of Lipschitz class. The degree of approximation of functions belonging to Lip(α), Lip(α, r), Lip($\xi(t)$, r), ($r \geq 1$)-classes through trigonometric Fourier series expansion by uging different summability methods has been studied by various investigators like Mishra [3], Paikray [5], Lal [6] and Mishra *et. al* [8] and many others. Recently, Nigam [2] has established degree of approximation of a function belonging to weighted W(L_r, $\xi(t)$) class by $(C, 1)(E, q)$ mean. In an attempt to sharpening the estimate of errors and to have an advance study in this direction, here have established a new theorem on $(\bar{N}, p_n, q_n)(E, s)$-summability mean of Fourier series of a function $f \in Lip(\alpha)$.

II. DEFINITIONS AND NOTATIONS

Let $\sum u_n$ be a given infinite series with the sequence of partial sum $\{s_n(f)\}$. Let $\{p_n\}$ and $\{q_n\}$ be sequences of positive real numbers such that,

$$P_n = \sum_{k=0}^{n} p_k \text{ and } Q_n = \sum_{k=0}^{n} q_k$$

and let

$$r_n = p_0q_n+p_1q_{n-1}+\ldots..+p_nq_0 \neq 0 \ (p_{-1}=q_{-1}=r_{-1}=0)$$

The sequence to sequence transformation (see [4]),

$$t_n^N = \frac{1}{r_n}\sum_{k=0}^{n} p_{n-k}q_k s_k$$

defines the (N, p_n, q_n) mean of the sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ and $\{q_n\}$, and it is denoted by t_n^N.

Similarly, we define the extended Riesz mean as the sequence to sequence transformation,

$$t_n^{\bar{N}} = \frac{1}{r_n}\sum_{k=0}^{n} p_k q_k s_k \qquad (2.1)$$

where $r_n = q_0p_0+q_1 p_1+\ldots..+q_np_n \neq 0$; $p_{-1}=q_{-1}=r_{-1}=0$ defines the $(\bar{N}, p_n, q_n)$ mean of sequence $\{s_n\}$ generated by the sequence of coefficients $\{p_n\}$ and $\{q_n\}$, and let it be denoted by $t_n^{\bar{N}}$.

If $\lim_{n\to\infty} t_n^{\bar{N}} \to s$ then the series $\sum u_n$ is $(\bar{N}, p_n, q_n)$ summable to s.

Analogous to regularity conditions of Riesz summability [1], we have the necessary and sufficient conditions for regularity of $(\bar{N}, p_n, q_n)$ summability are

1. $\frac{p_kq_k}{r_n} \to 0$, f or each integer $k \geq 0$ as $n \to \infty$ and
2. $|\sum_{k=0}^{n} p_k q_k| < C|r_n|$ where C is any positive integer independent of n.

The sequence to sequence transformation,

$$E_n^q = \frac{1}{(1+q)^n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}s_v \quad (2.2)$$

defines the sequence $\{E_n^q\}$ of (E, q) mean of the sequence $\{s_n\}$.

If $E_n^q \to s$ *as* $n \to \infty$ then $\sum u_n$ summable to s with respect to (E, q) summability and (E, q) method is regular (see [1]).

Now we define, a new composite transformation $(\bar{N}, p_n, q_n)$ over (E, s) of $\{s_n\}$ as,

$$T_n^{\bar{N}E} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k \ (E_k^s) = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+s)^k}\sum_{v=0}^{k}\binom{k}{v}s^{k-v}s_v\right\} \quad (2.3)$$

If $T_n^{\bar{N}E} \to s$ as $n \to \infty$, then $\sum u_n$ summable to s by $(\bar{N}, p_n, q_n)$ (E, s) summability method.

Let $s_n \to s$ implies $E_n^s(s_n) \to s$ as $n \to \infty$. Hence (E, s) method is regular. Now we may write, $T_n^{\bar{N}E} = t_n^{\bar{N}}\left(E_n^s(s_n)\right) \to s$ as $n \to \infty$. Therefore, $(\bar{N}, p_n, q_n)$ (E, s) method is also regular.

Remark 1. If we put $q_n = 1$, in equation (2.1) then $(\bar{N}, p_n, q_n)$-summability method reduces to $(\bar{N}, p_n)$-summability and for $p_n = 1$, it reduces to $(\bar{N}, q_n)$-summability. If we put $p_n = 1$ then $(\bar{N}, p_n)$-summability reduces to $(C, 1)$-summability.

Let f is 2π periodic and belonging to L^r $[0, 2\pi]$, $r \geq 1$, be a signal function with the partial sum $s_n(f)$, defined by

$$s_n(f) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \quad (2.4)$$

A signal (function) $f \in Lip(\boldsymbol{\alpha})$, if

$$|f(x+t) - f(x)| = O(|t|^{\alpha}) \ for \ 0<\alpha\leq 1, \ t>0.$$

The L_∞-norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

and L_r-norm of a function $f : R \to R$ is defined by

$$\|f\|_r = \left(\int_{[0,2\pi]}|f(x)|^r dx\right)^{\frac{1}{r}}, \ r \geq 1.$$

The degree of approximation of a function $f{:}R \to R$ by a trigonometric polynomial (t_n) of order n under $\|.\|_\infty$ is defined by

$$\|t_n - f(x)\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$$

and the error $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_m \|t_m - f\|_r$$

We use the following notations throughout the paper:

$$\varphi(t) = f(x + t) + f(x - t) - 2f(x)$$

$$K_n(t) = \frac{1}{2\pi r_n}\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^n}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}s_v\right\}$$

III. KNOWN THEOREM

Theorem-1. [2] If a function f is 2π periodic belonging to the class $Lip(\alpha)$, then its degree of approximation by $(C, 1)(E, q)$ summability mean of Fourier series is given by

$$\left\|C^1 E_n^q - f\right\|_\infty = O\{(n+1)^{-\alpha}\}, \ (0<\alpha<1), \quad (3.1)$$

Where $C^1 E_n^q$ represents the $(C, 1)$ transform of (E, q) transform of $s_n(f)$.

IV. MAIN THEOREM

Theorem-2 If a function f is 2π periodic belonging to the class $Lip(\alpha)$, then its degree of approximation by $(\bar{N}, p_n, q_n)(E, q)$-summability mean of Fourier series $s_n(f)$ is given by

$$\left\|T_n^{\bar{N}E} - f\right\|_\infty = O\{(n+1)^{-\alpha}\}, \ (0<\alpha<1), \quad (4.1)$$

Where $T_n^{\bar{N}E}$ represents the $(\bar{N}, p_n, q_n)$ transform of (E, s) transform of $s_n(f)$.

To prove the above theorem, first we need to prove the following lemmas.

Lemma-1. $|K_n(t)| = O(n), \ for \ 0 \leq t \leq \frac{1}{n+1}$.

Proof. For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin nt \leq n \sin t$, thus

$$|K_n(t)| = \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_k q_k \frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}(2v+1)\frac{\sin\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_k q_k \frac{1}{(1+q)^k}(2k+1)\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right|$$

$$\leq \frac{(2n+1)}{2\pi r_n}\sum_{k=0}^{n} p_k q_k \frac{1}{(1+q)^k}\left|\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\right|$$

$$= \frac{(2n+1)}{2\pi r_n}\sum_{k=0}^{n} p_k q_k \frac{1}{(1+q)^k}(1+q)^k$$

$$= O(n).$$

Lemma-2. $|K_n(t)| = O\left(\frac{1}{t}\right), \ for \ \frac{1}{n+1} < t \leq \pi$.

Proof: For $\frac{1}{n+1} < t \leq \pi$ using Jordans lemma, $\sin\frac{t}{2} \geq \frac{t}{\pi}$ and $|\sin nt| \leq 1$ we have

$$|K_n(t)| = \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n}\left\{\frac{1}{(1+q)^n}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{\sin\left(v+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\frac{1}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{2\pi r_n}\left|\sum_{k=0}^{n} p_k q_k \left\{\frac{1}{(1+q)^k}\sum_{v=0}^{k}\binom{k}{v}q^{k-v}\left(\frac{\pi}{t}\right)\right\}\right|$$

$\leq \frac{1}{2\pi r_n}(\frac{\pi}{t}) \sum_{k=0}^{n} p_k q_k \ \frac{1}{(1+q)^k} \left|\sum_{v=0}^{k}\binom{k}{v} q^{k-v}\right|$

$= \frac{1}{2t r_n}\sum_{k=0}^{n} p_k q_k$

$= O\left(\frac{1}{t}\right).$

Proof of Theorem-3. By equation (2.3) we have,

$\left|T_n^{\bar{N}E} - f\right| =$
$\frac{1}{2\pi r_n}\sum_{k=0}^{n} p_k q_k \ \int_0^{\pi} \frac{\varphi(t)}{\sin\frac{t}{2}} \frac{1}{(1+q)^k}\left\{\sum_{v=0}^{k}\binom{k}{v} q^{k-v} \sin\left(v + \frac{1}{2}\right) t dt\right\}$

$= \int_0^{\pi} \varphi(t) K_n(t) \mathrm{dt}$

$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right] \varphi(t) K_n(t) \mathrm{dt}$

$= I_1 + I_2$.

Since $f \in Lip(\alpha) \Rightarrow \varphi(t) \in Lip(\alpha)$, so by using Hölder inequality, we have

$I_1 = \int_0^{\frac{1}{n+1}} \varphi(t) K_n(t) \mathrm{dt}$

Now, using Lemma 1, we have

$|I_1| = \int_0^{\frac{1}{n+1}} |\varphi(t)||K_n(t)| \mathrm{dt}$

$\leq O(n) \int_0^{\frac{1}{n+1}} |\varphi(t)| \mathrm{dt}$

$= O(n) \int_0^{\frac{1}{n+1}} |t|^{\alpha} \mathrm{dt}$

$= O(n) \left[\frac{t^{\alpha+1}}{\alpha+1}\right]_0^{\frac{1}{n+1}}$

$= \frac{O(n)}{\alpha+1} \left[\frac{1}{(n+1)^{\alpha+1}}\right]$

$= O\left[\frac{1}{(n+1)^{\alpha+1}}\right]$ (4.2)

Next, using Lemma 2, we have

$I_2 = \int_{\frac{1}{n+1}}^{\pi} \varphi(t) K_n(t) \mathrm{dt}$

$|I_2| \leq \int_{\frac{1}{n+1}}^{\pi} |\varphi(t)||K_n(t)| \mathrm{dt}$

$\leq \int_{\frac{1}{n+1}}^{\pi} |t^{\alpha}| O\left(\frac{1}{t}\right) \mathrm{dt}$

$= \int_{\frac{1}{n+1}}^{\pi} t^{\alpha-1} \mathrm{dt}$

$= \left[\frac{t^{\alpha}}{\alpha}\right]_{\frac{1}{n+1}}^{\pi}$

$= O\left[\frac{1}{(n+1)^{\alpha}}\right]$ (4.3)

By using (4.2) and (4.3), we have

$\left|T_n^{\bar{N}E} - f\right| = O\left[\frac{1}{(n+1)^{\alpha}}\right]$

Thus, if $f \in Lip(\alpha)$ and $0 < \alpha < 1$, then

$\left\|T_n^{\bar{N}E} - f\right\|_{\infty} = \sup_{-\pi \leq x \leq \pi}\left|T_n^{\bar{N}E} - f\right| = O\left[\frac{1}{(n+1)^{\alpha}}\right]$

Which completes the proof of the theorem

REFERENCES

[1]. G. H. Hardy, Divergent Series, first ed., Oxford University Press, 1949.

[2]. H. K. Nigam, Degree of approximation of a function belonging to weighted $W(L_r, \xi(t))$ class by $(C, 1)(E, q)$ mean, Tamkang Journal of Mathematics, 42 (2011), 31-37.

[3]. M. Mishra, P. Palo, B. P. Padhy, P. Samanta, U. K. Misra, Approximation of Fourier series of a function of Lipschitz class by product means, Journal of Advances in Mathematics, 9(4) (2014), 2475-2484.

[4]. V. N. Mishra, K. Khatri, L. N. Mishra, Approximation of functions belonging to $Lip(\xi(t), r)$ class by $(N, p_n)(E, q)$ Summability of conjugate series of Fourier series, Journal of Inequalities and Applications, Article ID 296 (2012), doi: 10.1186/1029-242X-2012-296.

[5]. S. K. Paikray, R. K. Jati, U. K. Misra, N. C. Sahoo, On degree of approximation of Fourier series by product means, General Math. Notes., 13 (2012), 22-32.

[6]. S. Lal, On degree of approximation of conjugate of a function belonging to weighted $W(L_p, \xi(t))$ class by matrix summability mean of conjugate fourier series, Tamkang Journal of Mathematics, 31 (2000), 279-288.

[7]. V. N. Mishra and V. Sonovane, Approximation of functions of Lipschitz class by $(N, p^n)(E, 1)$ summability mean conjugate series of Fourier series, Journal of Classical Analysis, 6 (2) (2015), 137-151.

[8]. V. N. Mishra, V. Sonavane, L. N. Mishra, L_r-Approximation of signals (functions) belonging to weighted $W(L_r, \xi(t))$-class by $(C^1 . N_p)$ -summability method of conjugate series of its Fourier series. Journal of Inequality and Applications, Article ID 440, (2013) 1-15. doi: 10.1186/1029-242X-2013-440.