Some Results Conncered to the I-Function of Fractional Calculus

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Abstract:- **The main objects of this paper is to derive the results for the I- function involving the Riemann-Liouville, the Weyl and such other fractional calculus operators as those based on the Cauchy- Goursat integral formula. The results derived in this paper are basic in nature and may include a number of known and new results as special cases.**

Keyword: **- Riemann-Liouville, the Weyl Oprators, H-function, G-function, and I function**

I. INTRODUCTION AND PRELIMINARIES

In view of the generality of the I-function, on specializing the various parameters, we can obtain from our results, several results involving a remarkably wide variety of useful functions, which are expressible in terms of H-function, G-function, Fox's Wright function, generalized mittag-Leffler functions and their various special cases. Thus, the results presented in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problems of science, engineering, mathematical physics etc.

In 1961, Charles Fox [3] introduced a function which is more general in than the Meijer's G-function and this function is well known in the literature of special functions as Fox's H-function.

The functions are defined and presented by means of the following Mellin-Barnes type contour integral:

$$H(z) = H^{m,n}_{p,q}[z] = H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{1}$$

Where $$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}. \tag{2}$$

An account of the convergence conditions for this integral can be found in the paper by Fox [3].

Remark: If we set $\alpha_i = \beta_j = 1, i = 1,2,..., q; j = 1,2,..., p$ in (1), which reduces to the well-known G-function Fox (1961) [31].

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,1)_{1,p}\\(b_j,1)_{1,q}\end{matrix}\right.\right] = H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,1),...,(a_p,1)\\(b_1,1),...,(b_q,1)\end{matrix}\right.\right] = G^{m,n}_{p,q}\left[z\left|\begin{matrix}a_1,...,a_p\\b_1,...,b_q\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s ds \tag{3}$$

If we set $n = 0, m = q$ in above equation, reduces to the MacRobert's E-function Fox (1961) [3].

$$G^{q,0}_{p,q}\left[z\left|\begin{matrix}a_1,...,a_p\\b_1,...,b_q\end{matrix}\right.\right] = E[q;b_j : p;a_j : z] \tag{4}$$

The I-function which is more general than the Fox's H-function, defined by Saxena (2008) [8], by means of the following Mellin-Barnes type contour integral:

$$I[z] = I^{m,n}_{p_i,q_i:r}[z] = I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{5}$$

Where $$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}. \tag{6}$$

For details regarding existence conditions and variour parameter restrictions of I-function we may refer Saxena (2008) [8]

Remark: If we take $r = 1, p_1 = p, q_1 = q$ in (5), which reduces to the well-known Fox's H-function Fox (1961) [3].

$$I^{m,n}_{p_1,q_1:1}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_1}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_1}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_j)_{m+1,q}\end{matrix}\right.\right]$$

$$=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\alpha_1),...,(a_p,\alpha_p)\\(b_1,\beta_1),...,(b_q,\beta_q)\end{matrix}\right.\right] \quad (7)$$

Two of the most commonly encountered tools in the theory and applications of fractional calculus are provided by the Riemann-Liouville and the Weyl operators which are respectively defined by Erdelyi (1954) [2], Podlubny (1999) [6], Samko et al. (1993) [7], H.M.Shrivastava et al. (1992) [10], Chaurasia et al. (2010) [1]

$$R^{\upsilon}_{x}(f(x))=\frac{1}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1} f(t)dt, \mathrm{Re}(\upsilon)>0$$

$$=\frac{d^n}{dx^n}R^{\upsilon+n}_{x}(f(x)),-n<\mathrm{Re}(\upsilon)\geq 0; n\in N \quad (8)$$

and

$$W^{\upsilon}_{x}(f(x))=\frac{1}{\Gamma(\upsilon)}\int_x^\infty (t-x)^{\upsilon-1} f(t)dt, \mathrm{Re}(\upsilon)>0$$

$$=\frac{d^n}{dx^n}W^{\upsilon+n}_{x}(f(x)),-n<\mathrm{Re}(\upsilon)\geq 0; n\in N \quad (9)$$

Provided that the defining integrals exist.

The following definition Nishimoto (1996) [5], Nishimoto (1991) [4], Shrivastava et al. (1985) [11] of a fractional differintegral is based on the familiar Cauchy-Goursat integral formulae:

If the function $f(z)$ is analytic inside and on C,

Where $C=\{C^-,C^+\}$, C^- is a contour along the cut joining the points z and $-\infty+i\Im(z)$, which starts from the point at $-\infty$, encircles the point z once contour-clockwise, and returns to the point at $-\infty$, C^+ is a contour along the cut joining the points z and $\infty+i\Im(z)$, which starts from the point at ∞, encircles the point z once contour-clockwise, and returns to the point at ∞,

$$f_{\upsilon}(z)=\frac{\Gamma(\upsilon+1)}{2\pi i}\int_C \frac{f(t)}{(t-z)^{\upsilon+1}}dt, \quad (10)$$

where

$$\upsilon\in C\setminus\overline{z};\overline{z}=\{-1,-2,...,...\}$$
(11)

$$f_{-n}(z)=\lim_{\upsilon\to -n}\{f_{\upsilon}(z)\}, n\in N \quad (12)$$

$-\pi\leq\arg(t-z)\leq\pi$ for C^-,

and

$0\leq\arg(t-z)\leq 2\pi$ for C^+,

then $f_{\upsilon}(z), \mathrm{Re}(\upsilon)>0$ is said to be the fractional derivative of $f(z)$ of order υ and $f_{\upsilon}(z), \mathrm{Re}(\upsilon)<0$ is said to be the fractional integral of $f(z)$ of order $-\upsilon$, provided $\left|f_{\upsilon}(z)\right|<\infty, \upsilon\in R.$

II. MAIN RESULTS

In this section we will evaluate the certain results for the I-function involving the Riemann-Liouville, the Weyl and such other fractional calculus operators as those based on the Cauchy- Goursat integral formula. The results are presented in the form of theorems stated below:

Theorem 2.1

$$R^{\upsilon}_{x}\left\{z^{\lambda-1}I^{m,n+1}_{p_i,q_i:r}\left[\omega z^k\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}$$

$$=z^{\upsilon+\lambda-1}I^{m,n+1}_{p_i+1,q_i+1:r}\left[\omega z^k\left|\begin{matrix}(1-\lambda,k),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})]_{+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(1-\lambda-\upsilon,k)\end{matrix}\right.\right]$$

(13)

where

$k>0, \mathrm{Re}(\upsilon)>0, \mathrm{Re}(\lambda)+k\min_{1\leq j\leq m}[\mathrm{Re}(b_j/\beta_j]>0, i=1,2,..., r$

provided that each member of the above equation exists.

Proof: Taking LHS

$$R^{\upsilon}_{x}\left\{z^{\lambda-1}I^{m,n+1}_{p_i,q_i:r}\left[\omega z^k\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\} \quad (14)$$

By using the above definitions interchanging the order of summations and integrations and setting and setting t = zξ

Finally, using the definition (3), we get the desired result.

Theorem2.2

$$W_x^{\upsilon}\left\{z^{-\lambda} I_{p_i,q_i:r}^{m,n+1}\left[\omega z^{-k}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right\}$$

$$= z^{\upsilon-\lambda} I_{p_i+1,q_i+1:r}^{m,n+1}\left[\omega z^{-k}\left|\begin{matrix}(\upsilon-\lambda+1,k),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(1-\lambda,k)\end{matrix}\right.\right], \quad (15)$$

where

$k>0, \mathrm{Re}(\lambda)+k\min_{1\le j\le n}[\mathrm{Re}(\alpha_j-1)/\beta_j]>\mathrm{Re}(\upsilon)>0, i=1,2,...,r$
provided that each member of the equation exists.

Proof: Proof of this theorem follows similarly.

III. SPECIAL CASES

As I- function is the most generalized special function, numerous special cases with potentially useful transcendental functions such as Mittag-Leffler function, Bessel functions, Whittaker functions, hypergeometric functions, generalized hypergeometric function, Meijer's G-function, Fox-Wright function, Fox's H-function and their special cases can be deduced by assigning suitable values to the parameters. Some interesting special cases of the main results are given below:

(i) If we put $r=1, p_1=p, q_1=q$ in theorems (2.1) and (2.2), which lead to the well-known results for H-function given by Chaurasia, et al. (2010) [1]:

Corollary 3.1

$$R_x^{\upsilon}\left\{z^{\lambda-1} H_{p,q}^{m,n+1}\left[\omega z^{k}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]\right\}$$

$$= z^{\upsilon+\lambda-1} H_{p+1,q+1}^{m,n+1}\left[\omega z^{k}\left|\begin{matrix}(1-\lambda,k),(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)]_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q},(1-\lambda-\upsilon,k)\end{matrix}\right.\right] \quad (17)$$

Where
$k>0, \mathrm{Re}(\upsilon)>0, \mathrm{Re}(\lambda)+k\min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j]>0,$
provided that each member of the above equation exists.

Corollary 3.2

$$W_x^{\upsilon}\left\{z^{-\lambda} H_{p,q}^{m,n+1}\left[\omega z^{-k}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q}\end{matrix}\right.\right]\right\}$$

$$= z^{\upsilon-\lambda} H_{p+1,q+1}^{m,n+1}\left[\omega z^{-k}\left|\begin{matrix}(\upsilon-\lambda+1,k),(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m};(b_j,\beta_j)_{m+1,q},(1-\lambda,k)\end{matrix}\right.\right], \quad (18)$$

where
$k>0, \mathrm{Re}(\lambda)+k\min_{1\le j\le n}[\mathrm{Re}(\alpha_j-1)/\beta_j]>\mathrm{Re}(\upsilon)>0,$
provided that each member of the equation exists.

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