# Some Fully Symmetric Quadrature Rules for Numerical Integration of Complex Cauchy Principal Value Integral

R. N. Das[1], G. Pradhan[2], Swagatika Das[3*]

*[1]Retd. Head of the Department of Mathematics and Computer Science, G.M (Autonomous) college, Sambalpur, India*
*[2]Head of the Department of Mathematics & Humanities, College of Engineering & Technology, Bhubaneswar-751003, India*
*[3]Department of Mathematics, R.C.M. Science College, Khallikote-751003, India*

***Abstract:*** **Some fully symmetric quadrature rules of Newton-cotes type have been constructed for approximate evaluation of Cauchy principal value of both real and complex integrals having the point of singularity at the mid-point of the range of integration. The algebraic degrees of precision of the rules are eight, twelve and sixteen. Asymptotic error estimate of each rule has been derived. Some standard test integrals have been numerically integrated and it is seen that the accuracy is attained up to fifteen decimal places.**

***Key Word:*** **Complex Cauchy principal value integral, Quadrature rule, degree of precision, asymptotic error estimate.**

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## I. INTRODUCTION

In this paper, we consider the approximation of Cauchy principal value integral of the type

$$\text{PV} \int_{z_0-h}^{z_0+h} \frac{f(z)}{z-z_0}\,\mathrm{d}z \qquad (1)$$

where, it is assumed that the function $f(z)$ is analytic in a simply connected domain:

$$\Omega = \{z\epsilon C : |z - z_0| \leq \rho = r|h|\,;\, r > 1\}$$

containing the line segment : z = $z_0 + ht;\ \ -1 \leq t \leq 1$.

Cauchy principal value of an integral of an analytic function $f(z)$ is defined in Saff and Snider [14] as

$$\text{PV} \int_{L} \frac{f(z)}{z-s}\,ds = \lim_{\varepsilon\to 0}\left\{\int_{L_1} \frac{f(z)}{z-s}\,\mathrm{d}z + \int_{L_2} \frac{f(z)}{z-s}\,\mathrm{d}z\right\} \qquad (2)$$

(i) where s is a point on the path of integration i.e. the line segment L having the end points $z_0 - h$ and $z_0 + h$ .
(ii) $L_1$ and $L_2$ are the line segments having the end points $z_0 - h,\ s - \varepsilon$ and $s + \varepsilon, z_0 + h$ respectively.

If the limit (2) exists then, this limiting value is defined as Cauchy principal value of the integral on the left side of the equation (2).

The numerical evaluation of Cauchy principal value integral is now an important branch of numerical integration, as efficient evaluation of such integrals are often encountered in research of applied mathematics viz: theory of aerodynamics, scattering theory, crack problem in plane elasticity, the singular eigen function method in neutron transport and many other field of physical sciences.

Recently, a one parameter quadrature rule has been constructed by Das and Hota [9] to evaluate approximately the Complex Cauchy principal value integral given in equation (1) and the algebraic degree of precision of this rule is at least eight for any arbitrary value of the parameter $\alpha$ in the interval (0,1); but the degree of precision of the said rule increases from eight to ten if the parameter $\alpha = \frac{1}{\sqrt{3}}$. It may be noted that, the evaluation of the derivative of the function $f(z)$ at $z = z_0$ is not required at the time of approximation of the integral (1) unlike the rule given by Milovanovic *et al* [13].

Later on, a set of three quadrature rules has been proposed for approximation of the integral (1) by Das and Hota [10], which are obtained from the one parameter rule (Ref:[9]) for values of $\alpha = \sqrt[4]{\frac{9}{35}}, \sqrt{\frac{5}{21}}$ and $\sqrt[4]{\frac{3}{35}}$ . Each of these rules is a six point rule and the precision of each is eight. Das *et al* [11] further continued to formulate quadrature rules of precision ten from the said rules of precision eight by employing the method of extrapolation, which are linear combination of two rules of lower but equal precision and are called as ***composite rules.***

In this paper, we intend to construct few more rules of Newton-Cotes type of algebraic precision ***eight***, ***twelve*** and ***sixteen*** for approximate evaluation of the Complex Cauchy principal value integral of the type given in equation (1).

In the coming section, we have given a complete description of construction of the quadrature rule, which is of precision eight and to avoid mere repetition we have only stated the rules of algebraic degree of precision twelve and sixteen.

## II. FORMULATION OF THE RULE

For the construction of the rule of precision eight, the following set of nine nodes are chosen:

$$z_0, z_0 \pm h, z_0 \pm \frac{h}{2}, z_0 \pm ih, z_0 \pm i\frac{h}{2}; i = \sqrt{-1}.$$

Let the rule based on these points be denoted by

$$R_1(f) = w_{10}f(z_0) + w_{11}\{f(z_0+h) - f(z_0-h)\} + w_{12}\left\{f\left(z_0+\frac{h}{2}\right) - f\left(z_0-\frac{h}{2}\right)\right\} + w_{13}\{f(z_0+ih) - f(z_0-ih)\} + w_{14}\left\{f\left(z_0+i\frac{h}{2}\right) - f\left(z_0-i\frac{h}{2}\right)\right\}. \quad (3)$$

It may be noted here that,

$$I((z-z_0)^{2k}) = R_1((z-z_0)^{2k}); \quad \text{for k= 1,2,3,4…}$$

i.e. for all the even powers of $(z-z_0)$, as nodes are symmetrically situated with respect to line segment L .

Further in order to determine the weights in the rule $R_1$, we impose the conditions:

$$I((z-z_0)^k) = R_1((z-z_0)^k); \quad \text{for k= 0,1,3,5,7.} \quad (4)$$

From the conditions stated in equation (4), the following set of five linear equations in the unknowns $w_{10}$, $w_{11}$, $w_{12}$, $w_{13}$ and $w_{14}$ are obtained

$$\left.\begin{aligned} w_{10} &= 0, \\ 2w_{11} + w_{12} + 2iw_{13} + iw_{14} &= 2, \\ 2w_{11} + \frac{1}{4}w_{12} - iw_{13} - \frac{1}{4}iw_{14} &= \frac{2}{3}, \\ 2w_{11} + \frac{1}{16}w_{12} + iw_{13} + \frac{1}{16}iw_{14} &= \frac{2}{5}, \\ 2w_{11} + \frac{1}{64}w_{12} - iw_{13} - \frac{1}{64}iw_{14} &= \frac{2}{7}. \end{aligned}\right\} \quad (5)$$

On solving the set of equations in (5) we have

$$\left.\begin{aligned} w_{10} &= 0, \\ w_{11} &= \frac{218}{1575}, \\ w_{12} &= \frac{2624}{1575}, \\ w_{13} &= -\frac{13i}{1575}, \\ w_{14} &= -\frac{64i}{1575}. \end{aligned}\right\} \quad (6)$$

Thus, the quadrature rule proposed in the equation (3) is now given by

$$R_1(f) = \frac{218}{1575}\{f(z_0+h) - f(z_0-h)\} + \frac{2624}{1575}\left\{f\left(z_0+\frac{h}{2}\right) - f\left(z_0-\frac{h}{2}\right)\right\} - \frac{13}{1575} \times i\{f(z_0+ih) - f(z_0-ih)\} - \frac{64}{1575} \times i\left\{f\left(z_0+i\frac{h}{2}\right) - f\left(z_0-i\frac{h}{2}\right)\right\} \quad (7)$$

which is practically a 8-point rule integrating all polynomials of degree **at most 8**.

*Degree of Precision of the rule* $R_1(f)$ *:*

If $E_1(f)$ denotes the truncation error in approximating the integral $I(f)$ given in equation (1) by the rule $R_1(f)$ as stated in equation (7), then

$$I(f) = R_1(f) + E_1(f). \quad (8)$$

It is found that

$$E_1((z-z_0)^k) = 0 \qquad \text{for k= 0(1)8}$$

and

$$E_1((z-z_0)^9) = \frac{-7}{90} \times h^9 \neq 0.$$

Hence, the rule $R_1(f)$ exactly integrates all polynomials of degree eight or less. Thus the degree of precision of the rule $R_1(f)$ is eight.

In the same vain (as the procedure is a mere repetition), we have constructed the rules $R_2(f)$ and $R_3(f)$ which are given by

$$R_2(f) = \frac{126281}{1501500}\{f(z_0+h) - f(z_0-h)\} + \frac{1444392}{1501500}\left\{f\left(z_0+\frac{2h}{3}\right) - f\left(z_0-\frac{2h}{3}\right)\right\} - \frac{736047}{1501500}\left\{f\left(z_0+\frac{h}{3}\right) - f\left(z_0-\frac{h}{3}\right)\right\} - \frac{9316i}{1501500}\{f(z_0+ih) - f(z_0-ih)\} + \frac{246888i}{1501500}\left\{f\left(z_0+i\frac{2h}{3}\right) - f\left(z_0-i\frac{2h}{3}\right)\right\} - \frac{2438748i}{1501500}\left\{f\left(z_0+i\frac{h}{3}\right) - f\left(z_0-i\frac{h}{3}\right)\right\} \quad (9)$$

and

$$R_3(f) = \frac{247766270}{4307428125}\{f(z_0+h) - f(z_0-h)\} + \frac{14126417920}{861485625}\left\{f\left(z_0+\frac{h}{4}\right) - f\left(z_0-\frac{h}{4}\right)\right\} - \frac{3595149952}{1317566250}\left\{f\left(z_0+\frac{h}{2}\right) - f\left(z_0-\frac{h}{2}\right)\right\} + \frac{405899864064}{503969090625}\left\{f\left(z_0+\frac{3h}{4}\right) - f\left(z_0-\frac{3h}{4}\right)\right\} - \frac{12435551}{4307428125} \times i\{f(z_0+ih) - f(z_0-ih)\} + \frac{1104156876}{861485625} \times i\left\{f\left(z_0+i\frac{h}{4}\right) - f\left(z_0-i\frac{h}{4}\right)\right\} - \frac{2328437120}{1317566250} \times i\left\{f\left(z_0+i\frac{h}{2}\right) - f\left(z_0-i\frac{h}{2}\right)\right\} + \frac{53066373120}{503969090625} \times i\left\{f\left(z_0+i\frac{3h}{4}\right) - f\left(z_0-i\frac{3h}{4}\right)\right\} \quad (10)$$

Let $E_2(f)$ and $E_3(f)$ respectively denote the truncation errors associated with the rules $R_2(f)$ and $R_3(f)$.

Then

$$E_2(f) = I(f) - R_2(f) \quad (11)$$

and

$$E_3(f) = I(f) - R_3(f). \quad (12)$$

Now, we find that

$$E_2((z-z_0)^k) = \begin{cases} 0 & for\ k = 0(1)12 \\ -\frac{14912}{426465}h^{13} \neq 0 & for\ k = 13 \end{cases}$$

and

$$E_3((z-z_0)^k) = \begin{cases} 0 & for\ k = 0(1)16 \\ \frac{4609618412849}{161309491200000} h^{17} \neq 0 & for\ k = 17 \end{cases}$$

and from this we conclude that the algebraic degree of precision of the rules $R_2(f)$ and $R_3(f)$ are respectively twelve and sixteen.

It is noted here that, the degree of precision of such type of rules based on n nodes is n-1 in general.

## III. ERROR ANALYSIS

In this section, the asymptotic error estimate of the rule $R_1(f)$ given in (7) has been obtained and is given in Theorem-3.1.

Let $E_1(f)$ denote the truncation error associated with the rule $R_1(f)$ i.e.

$$E_1(f) = I(f) - R_1(f). \qquad (13)$$

We assume here that the function $f(z)$ is sufficiently differentiable in the open disc:

$$\Omega = \{z\epsilon C: |z - z_0| < \rho = r|h|\ ; r > 1\}\ . \qquad (14)$$

Under this assumption, $f(z)$ can be expanded in terms of the Taylor's series about the point $z = z_0$ in the disc $\Omega$ as

$$f(x) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad (15)$$

where $a_n = f^{(n)}(z_0)/(n)!$

are the Taylor's coefficients.

As the series given in (15) is uniformly convergent in $\Omega$, integrating the series (15) term by term and then equating it to the integral of $f$ we obtain:

$$I(f) = 2hf'(z_0) + \frac{2h^3}{3(3!)} f^{(3)}(z_0) + \frac{2h^5}{5(5!)} f^{(5)}(z_0) + \frac{2h^7}{7(7!)} f^{(7)}(z_0) + \frac{2h^9}{9(9!)} f^{(9)}(z_0) + \frac{2h^{11}}{11(11!)} f^{(11)}(z_0) + \cdots \quad . \qquad (16)$$

Again by expressing each term of the rule $R_1(f)$ given in (7) by Taylor's series expansion about the point $z = z_0$ in the disc $\Omega$ and then after simplification we obtain

$$R_1(f) = 2hf'(z_0) + \frac{2h^3}{3(3!)} f^{(3)}(z_0) + \frac{2h^5}{5(5!)} f^{(5)}(z_0) + \frac{2h^7}{7(7!)} f^{(7)}(z_0) + \frac{3h^9}{10(9!)} f^{(9)}(z_0) + \frac{11h^{11}}{42(11!)} f^{(11)}(z_0) + \cdots \quad . \qquad (17)$$

So, from the equations (13), (16) and (17) we have

$$E_1(f) = -\frac{7h^9}{90(9!)} f^{(9)}(z_0) - \frac{37h^{11}}{462(11!)} f^{(11)}(z_0) + \cdots \qquad (18)$$

from which, we have the following

*Theorem 3.1:* If $f(z)$ is analytic in a simply connected domain,

$\Omega = \{z\epsilon C: |z - z_0| \leq \rho = r|h|\ ; r > 1\}$, then

$$|\boldsymbol{E_1(f)}| \sim \boldsymbol{O}(|\boldsymbol{h}|^{\boldsymbol{9}}).$$

In the same vein (as in the case of the rule $R_1(f)$), the truncation errors associated with the rules $R_2(f)$ and $R_3(f)$ have been obtained and are given by

$$E_2(f) = -\frac{14912}{426465(13!)} h^{13} f^{(13)}(z_0) - \frac{29888}{1082565(15!)} h^{15} f^{(15)}(z_0) - \cdots$$

and $$E_3(f) = \frac{4609618412849}{161309491200000(17!)} h^{17} f^{(17)}(z_0) + \cdots \quad .$$

Hence we have

*Theorem 3.2:* If $f(z)$ is analytic in a simply connected domain

$\Omega = \{z\epsilon C: |z - z_0| \leq \rho = r|h|\ ; r > 1\}$, then

$$|\boldsymbol{E_2(f)}| \sim \boldsymbol{O}(|\boldsymbol{h}|^{\boldsymbol{13}})$$

and

$$|\boldsymbol{E_3(f)}| \sim \boldsymbol{O}(|\boldsymbol{h}|^{\boldsymbol{17}})\ .$$

## IV. NUMERICAL EXPERIMENT AND CONCLUSION

For numerical investigation, we have considered here two types of Cauchy principal value integrals depending on the position of singular point on the path of integration, which are:

(i) the point of singularity is at the mid-point of the range of integration,

(ii) the singularity at a point other than the mid-point of the interval of integration.

In case of the singular integral of the type (i), all the three quadrature rules: $R_1$, $R_2$ and $R_3$ can be applied directly to the Cauchy principal value integrals for their approximations, whereas, it is not so in case of second type of integrals.

First, we have chosen the following three integrals of type (i), out of which the first integral $I_1$ is a complex CPV integral and the other two i.e. $I_2$ and $I_3$ are real CPV integrals, which are frequently cited by the researchers for numerical experiment by the quadrature rules developed by them. The approximate values resulting from the numerical integration by the rules: $R_1(f), R_2(f)$ and $R_3(f)$ are depicted in Table-1 and Table-2 along with their exact values.

$$\boldsymbol{I_1 = PV \int_{-i}^{i} \frac{e^z}{z} dz}\ ,$$

$$\boldsymbol{I_2 = PV \int_{-1}^{1} \frac{e^x}{x} dx}\ ,$$

$$\boldsymbol{I_3 = PV \int_{-1}^{1} \frac{sinx}{x} dx}\ .$$

**Table-1: Numerical values of Complex CPV integral:**

$$\boldsymbol{I_1 = PV \int_{-i}^{i} \frac{e^z}{z} dz}$$

| | |
|---|---|
| $R_1(f)$ | **1.892 166** 353 085 104 $i$ |
| $R_2(f)$ | **1.892 166 140 73**9 960 $i$ |
| $R_3(f)$ | **1.892 166 140 734 366** $i$ |
| **Exact value** | **1.892 166 140 734 366** $i$ |

**Table-2: Numerical values of Real CPV integrals:**

$$I_2 = PV\int_{-1}^{1}\frac{e^x}{x}\,dx \quad : \quad I_3 = PV\int_{-1}^{1}\frac{\sin x}{x}\,dx$$

| | | |
|---|---|---|
| $R_1(f)$ | **2. 114 501** 967 115 065 | **1.892 166** 353 085 104 |
| $R_2(f)$ | **2. 114 501 750 75**7 094 | **1.892 166 140 73**9 960 |
| $R_3(f)$ | **2. 114 501 750 751 45**5 | **1.892 166 140 734 366** |
| **Exact value** | **2. 114 501 750 751 457** | **1.892 166 140 734 366** |

Next, we consider the integral:

$$J_1 = PV\int_{-i}^{i}\frac{e^z}{z-\frac{i}{4}}\,dz \qquad (19)$$

which is of type (ii).

The integral of this type cannot be evaluated by using any one of the formulas $R_1(f)$, $R_2(f)$ and $R_3(f)$ directly, since the singularity is not at the mid- point of the range of integration. In such cases, we adopt the following method which may be called ***interval division method.***

*Interval division method:*

In this method the whole interval of integration is divided into two sub intervals in such a way that, the point of singularity (s) is at the mid-point of one of the two sub-intervals. For instance, if $s = {}^{i}/_{4}$ then we divide the line segment: $[-i, i] = [-i, {}^{-i}/_{2}] + [{}^{-i}/_{2}, i]$ so that, s is the mid-point of the sub-interval: $[{}^{-i}/_{2}, i]$.Then we express the integral under reference as:

$$J_1 = \int_{-i}^{-i/2}\frac{e^z}{z-\frac{i}{4}}\,dz + PV\int_{-i/2}^{i}\frac{e^z}{z-\frac{i}{4}}\,dz$$

$$= J_{11} + J_{12}\ ;\ \text{say.}$$

The singular point (${}^{i}/_{4}$) of the integrand in the integral $J_{11}$, if not in its range of integration, is close to one of its end points: namely the right of the end point i.e. ${}^{-i}/_{2}$ .In such a situation the singular point adversely affects the numerical integration, in general. In order to counter the adverse effect of the singular point on the numerical evaluation of the integral $J_{11}$, we have integrated this integrand by a high precision rule ( R ) of degree of precision 17 formulated by Das *et al* [8], for approximation of integrals of analytic functions along a directed line segment L, which is given here by

$$R = h\Big[\frac{-47548744}{268515}f(z_0) + \frac{3819749452}{73226278125}\{f(z_0+h) + f(z_0-h)\} + \frac{762997907456}{14645255625}\left\{f\left(z_0+\frac{h}{4}\right) + f\left(z_0-\frac{h}{4}\right)\right\} - \frac{4554955904}{861485625}\left\{f\left(z_0+\frac{h}{2}\right) + f\left(z_0-\frac{h}{2}\right)\right\} + \frac{10380713984}{12922284375}\left\{f\left(z_0+\frac{3h}{4}\right) + f\left(z_0-\frac{3h}{4}\right)\right\} - \frac{180872771}{73226278125}\{f(z_0+ih) + f(z_0-ih)\} + \frac{656033964032}{14645255625}\left\{f\left(z_0+i\frac{h}{4}\right) + f\left(z_0-i\frac{h}{4}\right)\right\} - \frac{2618398592}{861485625}\left\{f\left(z_0+i\frac{h}{2}\right) + f\left(z_0-i\frac{h}{2}\right)\right\}\ \frac{1554440192}{12922284375}\left\{f\left(z_0+i\frac{3h}{4}\right) + f\left(z_0-i\frac{3h}{4}\right)\right\}\Big] \quad . \qquad (20)$$

On the other hand, the second integral ($J_{12}$) is numerically integrated by the rule $R_3(f)$ of precision sixteen given in equation (10) of this paper.

Approximation of the integrals $J_{11}$ *and* $J_{12}$ and their sum which ultimately gives the approximation of the integral $J_1$ are presented in the Table-3 below.

**Table-3: Approximation of the integral $J_1$**

| **Integral** | **Approximate values** |
|---|---|
| $J_{11}$ | **-0.37715018652091+ $i$ 0.33664933159635** |
| $J_{12}$ | **-0.35970271788364+ $i$ 1.40871000015806** |
| $J_1 \approx J_{11} + J_{12}$ | **-0.7368529044045**5**+ $i$ 1.7453593317544**1 |
| **Exact value of $J_1$** | **-0.73685290440451+ $i$ 1.74535933175442** |

It appears that the method i.e. the ***interval division method*** is quite simple, straightforward and computationally effective in comparison to ***finite interval method*** [15].

This CPV integral ($J_1$) is also cited by Milovanovic *et al* [13],which is numerically evaluated by them by a quadrature rule obtained by integrating a suitable Hermite interpolating polynomial. They have obtained the approximate value of this integral correct up to seven decimal places and also, the quadrature rule constructed by them requires evaluation of the derivative of $f(z)$ at the point of singularity. So the quadrature rules $R_1, R_2$ and $R_3$ presented in this paper are preferred to the rule formulated by Milovanovic *et al* [13], as evaluation of derivative at the singular point is not required.

We have also numerically integrated few more integrals to confirm through numerical experiment, the computational efficiency and the accuracy obtained by *interval division method*, a short description of which is given in the immediately preceding paragraph. The integrals considered here are

$$J_2 = PV\int_{-1}^{1}\frac{e^x}{x-\frac{1}{4}}\,dx,$$

$$J_3 = PV\int_{-1}^{1} \frac{\cos x}{x-\frac{1}{4}}\, dx,$$

and $$J_4 = PV\int_{-1}^{1} \frac{\sin x}{x-\frac{1}{4}}\, dx .$$

Here we use the notation $J_k = J_{k1} + J_{k2}$ ; for k = 2,3,4.

The approximate values of these integrals along with their exact values are depicted in

Table-4.

**Table-4: The approximate values of integrals $J_2, J_3$ and $J_4$**

| $J_{21}$ | $J_{22}$ | $J_2 \approx J_{21} + J_{22}$ | Exact Value |
|---|---|---|---|
| − 0.249007762 44800 | 1.98725227626 094 | 1.73824451381 294 | 1.73824451381 299 |
| $J_{31}$ - 0.37715018652 091 | $J_{32}$ - 0.35970271788 365 | $J_3 \approx J_{31} + J_{32}$ - 0.73685290440 455 | - 0.73685290440 451 |
| $J_{41}$ 0.33664933159 635 | $J_{42}$ 1.40871000015 806 | $J_4 \approx J_{41} + J_{42}$ 1.74535933175 441 | 1.74535933175 442 |

It is pertinent to note here that, if the rule (20) is applied in compound form, then the accuracy of the approximate value of each of the integrals: $J_1, J_2, J_3$ and $J_4$ is likely to improve. We have tried with n=2 and 4 sub-divisions and observed that, the accuracy increases only to one more decimal place; this may be due to the round off error, otherwise we would have got the approximate value with higher decimal accuracy.

*Conclusion*

1. One positive advantage of such a procedure to integrate numerically the Cauchy principal value integral of the type (ii) is that, the rule (20) and (10) can be applied readily and directly as well by supplying the affix of the mid-point $(z_0)$ and h of the respective integrals to the computer program which is written once for all.
2. The other point of equal importance is that, the point of singularity is made the mid-point of the range of integration of the singular integral in ***interval division method***, so that the singular point does not coincide with any of the nodes of the quadrature rule applied for the approximation of this integral.
3. From the numerical experiments it is evident that, the singular point on the path of integration does not produce so much of adverse effect on the numerical results, since in each case $(J_1, J_2, J_3$ and $J_4)$, we get the results up to 13-decimal accuracy, otherwise the accuracy is up to 15-decimal places as it is seen from the numerical evaluation of the standard test integrals: $I_1$, $I_2$ and $I_3$ .
4. Also in this method, evaluation of the derivative of the function at the singular point is not required unlike the formulas obtained by other researchers in the past.
5. Finally the authors opine that, the quadrature rules formulated and the numerical technique proposed in this paper shall immensely help the researchers in the field of applied sciences, where evaluation of Cauchy principal value integrals is required and that cannot be evaluated analytically or too difficult to evaluate.

In later paper, some quadrature rules shall be formulated for evaluation of Hadamard finite part of complex hyper singular integral of the type:

$$HFP\int_{z_0-ih}^{z_0+ih} \frac{f(z)}{(z-z_0)^2}\, dz$$

All the numerical work have been performed by using the program written by authors in $C^{++}$ on Intel Core i3 in double precision.

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CONFLICT OF INTERESTS

Authors have declared that no competing interests exist.

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