An Integral Method to Solve Phase-Change Problems With/Without Mushy Zones

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Abstract: **An integral method is developed to solve phase-change problems with/without mushy zones. The method can be viewed as a modified version to the classical approach of Goodman's method in which, *three non-linear ordinary differential equations* were obtained at the end of each time, solved iteratively using complicated numerical schemes. Solving these equations lead to the unknowns in the overall problem. A major modification to Goodman's method is carried out by dealing with finite domain and taking the mushy zone into consideration. In the present paper, the proposed method starts by assuming temperature profile for liquid and solid phase in such a way that some –*not all*– of the boundary conditions will be satisfied. Some mathematical manipulation will lead to *two non-linear algebraic equations* at each time step, their solution numerically by a proposed scheme will lead to the unknowns that appear throughout the whole process.**

Keywords: Phase change problems, Goodman's method, Vaporization problems, Mushy zone models.

Nomenclature

U_i	***Initial temperature***
U_m	***Melting temperature***
α	***Thermal diffusivity***
K	***Conductivity***
C	***Heat capacity***
ρ	***Density***
L	***Latent heat***
$R_1(t)$	***Right boundary of the liquid phase and the left boundary for the mushy zone***
$R_2(t)$	***Left boundary of the liquid phase***
$R_3(t)$	***Right boundary of the mushy zone and left boundary of the solid***
ε & γ	***Mushy zone parameters***

I. INTRODUCTION

Moving boundary problems have wide range of engineering and industrial applications such as metal casting; ground freezing problems and thermal storage **[1-2]**. When dealing with different types of Stefan problems, which originally the basis of moving boundary problems, it should be taken into consideration the criteria of the type underhand. In case of existence of mushy zone between solid and liquid, the situation is then changed completely as in [**3**]. Since the early stages of modeling and simulation of Stefan problems, different restrictions had been taken into consideration such as the geometry. The difficulty in solving moving boundary problems refers mainly to the non-linearity caused by the presence of nonlinear boundary condition at the moving boundary [**4-5**]. Approximate methods such as heat balance integral method [**6**] and the moment integral method **[7]** had been applied for solving such problems but with sharp interface. In recent years, efforts concentrate on the use of numerical techniques due to rapid development in computer technology and its high performance. Boundary element method becomes one of the most popular numerical methods, which applied to a wide range of engineering and industrial applications [**8-9**]. An integral method is developed to solve phase-change problems with/without mushy zones. The method can be viewed as a modified version to the classical approach of Goodman's method in which, ***three non-linear ordinary differential equations*** were obtained at the end of each time, solved iteratively using complicated numerical schemes. Solving these equations lead to the unknowns in the overall problem. A major modification to Goodman's method is carried out by dealing with finite domain and taking the mushy zone into consideration. In the present paper, the proposed method starts by assuming temperature profile for liquid and solid phase in such a way that some –*not all*– of the boundary conditions will be satisfied. Some mathematical manipulation will lead to ***two non-linear algebraic equations*** at each time step, their solution iteratively by a proposed scheme will lead to the unknowns that appear throughout the whole process.

II. MATHEMATICAL FORMULATION

A solid medium of length, ℓ initially at a uniform temperature, U_i the boundary $x=0$ subjected to a high input heat flux therefore three phases occurs. In the present paper, we mainly consider the third stage in which the vapor will be removed upon formation, therefore, the problem still remains two phase with mushy zone separating liquid and solid, respectively **[10]**.

$$\alpha_\ell \frac{\partial^2 u_\ell}{\partial x^2} = \frac{\partial u_\ell}{\partial t},\ R_2(t) < x < R_1(t) \tag{1}$$

$$\alpha_S \frac{\partial^2 u_S}{\partial x^2} = \frac{\partial u_S}{\partial t},\ R_3(t) < x < \ell \tag{2}$$

$$u_S(\ell, t) = U_i \tag{3}$$

$$u_\ell(R_2(t), t) = U_V \tag{4}$$

$$\frac{\partial u_\ell}{\partial x} = \frac{\gamma}{[R_3(t) - R_1(t)]} \tag{5}$$

$$\left(K_\ell \frac{\partial u_\ell}{\partial x} - K_S \frac{\partial u_S}{\partial x}\right) = \rho L \left[\begin{matrix} \varepsilon \dfrac{dR_1(t)}{dt} \\ + (1-\varepsilon)\dfrac{dR_3(t)}{dt} \end{matrix}\right] \tag{6}$$

In this formulation, we have five unknowns, the temperature distribution in the liquid and solid phases and three moving boundaries, $R_2(t), R_1(t)$ and $R_3(t)$.

III. THE PROPOSED METHOD

Step: 1 Assume temperature profiles for the phases that appear throughout the whole process:

$$u_\ell(x,t) = \left[\left(\frac{U_m}{R_1 - R_2}\right) + B_2 (x - R_1)^2\right](x - R_2) + \left[\left(\frac{U_v}{R_2 - R_1}\right) + B_2 (x - R_2)^2\right](x - R_1) \tag{7}$$

$$u_S(x,t) = \left[\frac{U_m}{R_3 - \ell} + B_3 (x - R_3)^2\right](x - \ell) + \left[\frac{U_i}{R_3 - \ell} + B_3 (x - \ell)^2\right](R_3 - x) \tag{8}$$

Step: 2 Integrate equation (7) with respect to the space variable x from $x = R_2$ to $x = R_1$, and equation (8) with respect to the space variable x from $x = R_3$ to $x = \ell$ yields:

$$\alpha_\ell \left\{\left(\frac{\partial u_\ell}{\partial x}\right)_{x=R_{11}} - \left(\frac{\partial u_\ell}{\partial x}\right)_{x=R_2}\right\} = \frac{d}{dt}\left\{\int_{x=R_2}^{x=R_1} u_\ell(x,t)dx\right\} + u_\ell(R_1, t)\frac{dR_1}{dt} - u_\ell(R_2, t)\frac{dR_2}{dt} \tag{9}$$

$$\alpha_S \left\{\left(\frac{\partial u_S}{\partial x}\right)_{x=\ell} - \left(\frac{\partial u_S}{\partial x}\right)_{x=R_3}\right\} = \frac{d}{dt}\left\{\int_{x=R_3}^{x=\ell} u_S(x,t)dx\right\} - u_S(R_3, t)\frac{dR_3}{dt} \tag{10}$$

Step: 3 Derivation a relation between moving boundary velocities and potential derivative. It is known that the total derivative of temperature at the moving boundaries is zero;

$$\frac{D}{Dt} u_\ell(R_2, t) = 0 \tag{11}$$

Equation (11) can be written in an expanded form as follow:

$$\frac{\partial u_\ell(R_2, t)}{\partial x}\frac{dR_2}{dt} + \frac{\partial u_\ell(R_2, t)}{\partial t} = 0 \tag{12}$$

Therefore; the velocities at R_3 and R_1 can be written as follow:

$$\left(\frac{dR_2}{dt}\right)_{x=R_2} = -\alpha_\ell \left(\frac{\partial^2 u_\ell / \partial x^2}{\partial u_\ell / \partial x}\right)_{x=R_2} \tag{13}$$

$$\left(\frac{dR_1}{dt}\right)_{x=R_1} = -\alpha_\ell \left(\frac{\partial^2 u_\ell / \partial x^2}{\partial u_\ell / \partial x}\right)_{x=R_1} \tag{14}$$

Similarly for solid phase;

$$\left(\frac{dR_3}{dt}\right)_{x=R_3} = -\alpha_S \left(\frac{\partial^2 u_S / \partial x^2}{\partial u_S / \partial x}\right)_{x=R_3} \tag{15}$$

Step: 4 Analytical treatment of equation (9), this step started by differentiating assumed potential profile for liquid phase, then, substituting into the right hand side, one can ensure that this side will equal to zero. The first term in the right hand side is integrated first based on the assumed profile, then differentiating the result w. r. t. time, meanwhile, make use of equations (13) and (14), then after long mathematical manipulation, one can obtain the following equation:

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$$\left(\frac{U_m+U_v}{2}\right)\left[\frac{-6\alpha_\ell B_2\left(R_1-R_2\right)}{\left(\frac{U_m-U_v}{R_1-R_2}\right)+B_2\left(R_1-R_2\right)^2}+\frac{6\alpha_\ell B_2\left(R_2-R_1\right)}{\left(\frac{U_m-U_v}{R_1-R_2}\right)+B_2\left(R_2-R_1\right)^2}\right]=$$

$$\frac{\eta_1(t)B_2}{\eta_2(t)+\eta_3(t)B_2}+\frac{\varsigma_1(t)B_2}{\eta_2(t)+\varsigma_2(t)B_2} \qquad (16)$$

Where

$$\eta_1(t)=-6\alpha_\ell U_v\left(R_2-R_1\right)$$

$$\eta_2(t)=\left(\frac{U_v-U_m}{R_2-R_1}\right)$$

$$\eta_3(t)=\left(R_2-R_1\right)^2$$

$$\varsigma_1(t)=-6\alpha_\ell U_m\left(R_2-R_1\right)$$

$$\varsigma_2(t)=\left(R_1-R_2\right)^2$$

Step: 5 Similar procedure are carried out again but on equation (10), the result for this step is as follow:

$$\left[\left(\ell-R_3\right)^2+\left(R_3-\ell\right)^2\right]\left[\left(\frac{U_m-U_i}{R_3-\ell}\right)-B_3\left(R_3-\ell\right)\right]= \qquad (17)$$

$$2\left(\ell-R_3\right)\left[\frac{5}{3}B_3R_3^3-5B_3\ell R_3^2+5B_3\ell^2R_3+\frac{1}{3}B_3\ell^2-\frac{1}{2}U_m+U_i\right]$$

It is clear from equation (16), that it contains three unknowns one of them is the unknown function in the assumed profile, given by equation (7). Meanwhile, equation (17) contains two of them is the unknown function in the assumed profile given by equation (8), therefore the procedure for solution will be illustrated in the next section.

IV. SOLUTION PROCEDURE

(1) Specify initial input data for both mushy zone parameters, ε and γ

(2) Assume initial position for both R_1 and R_3

(3) From equation (16) evaluate B_2

(4) Evaluate the error in equation (5) making use of equation (7)

(5) Repeat steps (1) to (4) till satisfying step (4) with prescribed tolerance

(6) From equation (17) evaluate B_3 and then the error in equation (6) but taking into consideration the last and accurate value for the first term in the left-hand side. If the error is not within a prescribed error, update the moving boundaries location, if it is satisfied, then go to the next time step.

V. NUMERICAL RESULTS

To test the proposed method and check its applicability, three test problems are solved herein. Two types of problems are solved, the first two problems with sharp interface, i.e.; without mushy zone and the last one take the existence of the mushy zones into consideration. Also in the third problem, two major parameters that control the mushy zone are studied well from their effect on the formation of the mushy zone and the movement of the overall moving fronts that appear throughout the whole process.

5.1 Problems Without Mushy Zones

A sub-cool medium exposed to a large heat flux until vaporization, therefore, three phases appear. The medium is exposed to two different cases of heat flux, constant and linear. Summary of the two problems are shown in table (1) **[10]**. In these problems, it is expected appearing two moving interfaces, the first one separating liquid and solid phases while, the second separates gas and liquid respectively. The location of these moving interfaces is shown in figures (1) and (2) respectively. The location of the moving boundaries separating solid-liquid and liquid-gas for case problem (1) are shown in figure (1), while the same results for case problem (2) are shown in figure (2). The results in both figures are compared with numerical results from the source and sink method. A good agreement between the two methods is obtained. It is also clear from these figures that, the curvature upward for liquid-solid interfaces increases for linear heat flux and decreases for constant case. The downward curvature increases for liquid-gas interface in case of linear heat flux and decreases but still clear in case of constant flux. In the next subsection existence of the mushy zone is taken into consideration and the results from the present method only due to the lake of available results for such a problem.

5.2. *Problems with Mushy Zone*S

The numerical data input of this problem is taken from S. G. Ahmed **[4]**. In this problem, aluminum occupies mould of length, ℓ initially at uniform temperature U_i . An input heat flux is applied at $x=0$; therefore, three phases are appearing. The vapor is removed upon formation and so the problem still two phase with mushy zone separating liquid and solid. The thermo-physical is shown in table (2). The results of this problem are shown in figures (3-7). Let us start discussion by figure (3) which shows the variation of solid-liquid and ablated surface against time. One major observation from the computation is the effect of the mushy zone parameters $\varepsilon \,\&\, \psi$ on the movement of these moving interfaces. As it is clear that the mushy zone parameter ε has no direct effect on the movement of $R_1(t)$, while its effect appears on the movement of $R_3(t)$. Therefore, three different cases of ε are used in the computations and plotted as seen in figure (3). Also, one can conclude that by increasing ε and at the same time step the moving boundary location increases. Mushy zone thickness against time and for three different values of mushy zone parameter ε is plotted against time as can be seen in figure (4). As seen from this figure, the thickness increases by increasing the mushy zone parameter ε at the same time step, and this prove the same results from figure (3). Following up the results of test problem (3), figure (5) shows the variation of the left hand side of liquid phase, $R_2(t)$ against time. It is important herein to mention that mushy zone parameters have no effect on the movement of this moving interface. To complete the results of this problem, temperature variation in both solid and liquid phases are plotted in figures (6) and (7), respectively. One important notation should be taken into consideration is that the last point in the liquid temperature variation is completely different from the starting point in the solid phase temperature variation, this occurs due to the width of the mushy zone.

CONCLUSION

The suggested method based mainly on the first basic principle of the classical heat balance integral method. The mathematical manipulation and solution procedure are completely different from the classical approach. The suggested method has advantages over the classical approach as follow:

(7) In classical approach ***three non-linear ordinary differential equations*** are obtained, while in the present method only ***two non-linear algebraic equations*** are obtained. Therefore less time consumed at each time step to find the solution.

(8) The classical approach did not take mushy zone into consideration while the present method takes it.

(9) The present method has the flexibility to solve phase change problem with and without mushy zone.

(10) The present method has the advantage to analyze the direct and indirect effects of the mushy zone parameters.

Finally, we can conclude that the present method gives the promise to solve higher dimensional problems and ensure that its results are very close to the real physical behavior of the problem underhand and its wide range of applications.

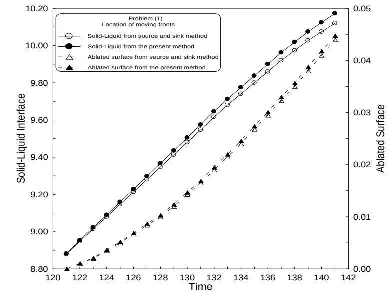

Figure 1: Problem (1) Solid-Liquid and Ablated interfaces

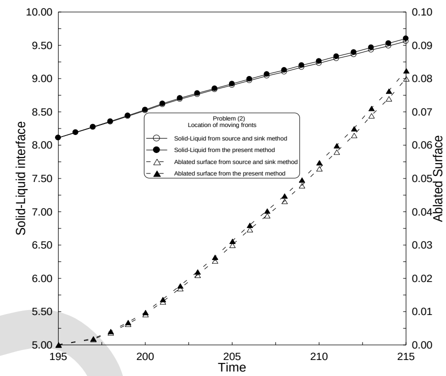

Figure 2: Problem (2) Solid-Liquid and Ablated interfaces

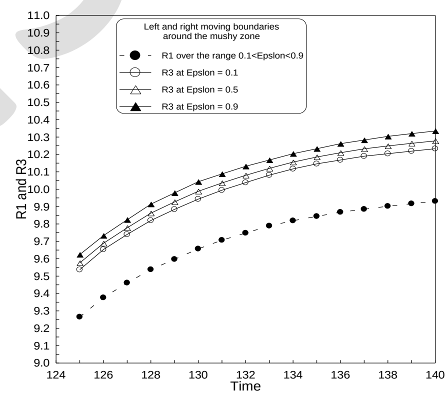

Figure 3: Problem (3) $R_1(t)$ and $R_3(t)$ location verses time over

$$0.1 < \varepsilon < 0.9$$

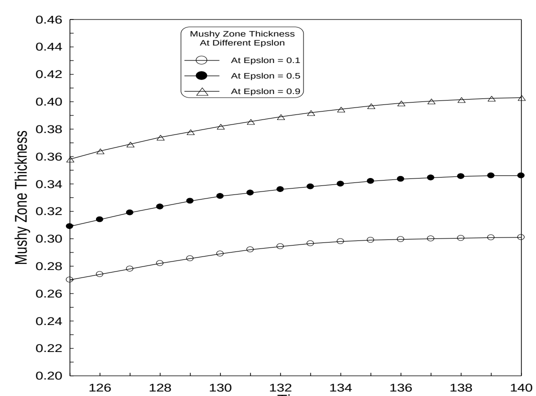

Figure 4: Problem (3) Mushy zone thickness verses time at two different ε

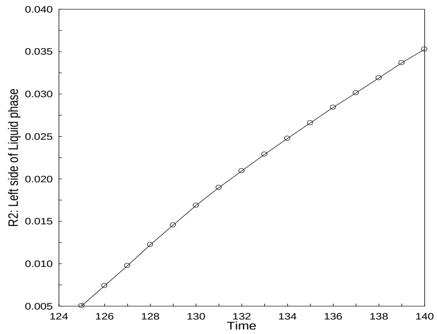

Figure 5: Location of $R_2(t)$, the left side of the liquid phase

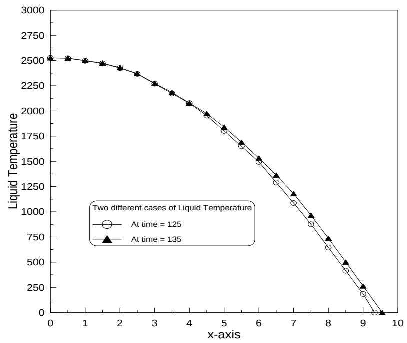

Figure 6: Problem (3) Liquid temperature

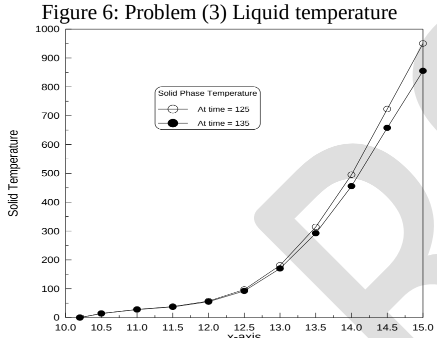

Figure 7: Problem (3) Solid temperature at different times

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Problem description	Input heat flux $G(t), W/m^2, t(\sec)$	Type of the problem	Exact solution
Three phase $U_i = 300K$ $U_i < U_m$	Constant $G(t) = 5\times10^6$ Linear $G(t) = 3\times10^4 t$	Combined ablation and Stefan	Not available

Table (1)

u_i	932 K
u_V	2543 K
L	376560 J/Kg
K	200 W/mK
ρ	2710 Kg/m^3
C	1200 $J/Kg\,K$

Table (2)

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