Numerical Solution for Advection-Diffusion Equation Using Collocation Based New Radial Basis Function

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Abstract: **In the present paper, a composite radial basis function is developed by a comination between thin plate and logarithmic radial basis functions. This radial basis function is developed to overcome the singularity appeared when source and field points are the same. In this fuction the thin plate is used when evaluating diagonal elements of the matrix coefficients and the logarithmic function is used for evaluating the other elements. One- and two-dimensional test examples were solved and the present results were compared with the analytical results and gave a good agreement.**

Keywords: Collocation techniques, thin plate radial basis function, logarithmic radial basis function, advection-diffusion equation.

Nomenclature

$u(\mathrm{x},t)$:	**Temperature at position x and time t**
x:	**Vector position, $x = (x_1, x_2, x_3, \ldots, x_d)$**
d:	**Dimension of the problem**
∇:	**Gradient differential operator**
Ω:	**Problem domain**
$\partial\Omega$:	**Boundary of the domain**
k:	**Diffusion coefficient**
v:	**Vector contains advection coefficients**
$\varphi_1(r_{ij})$:	**Thin plate radial basis function**
$\varphi_2(r_{ij})$:	**Logarithmic basis function**
ε:	**Control parameter**
r_{ij}:	**Distance between points i and j**
$\mathrm{P}_e = (v/k)$:	**Peclet number**

I. INTRODUCTION

A significant progress of the numerical methods has been achieved in the recent years. For long time the researchers recognised problems when using a mesh-based method such as finite element method [1-2]. One way for their efforts to overcome these problems was the automatic mesh generation. Automatic mesh generation is difficult to be fully generated in a wide range of engineering applications. The second way to overcome such problems was developing the meshless methods [3]. The initial idea of meshless methods dates back to the smooth particle hydrodynamics method for modeling astro-physical phenomena [4]. Several domain type meshfree methods such as element free Galerkin method [5], reproducing kernel particle method [6], the point interpolation method [7] and the meshless Petrov-Galerkin method [8] have been proposed and achieved remarkable progress in solving a wide range of static and dynamic problems for solid and structures. Advection-diffusion equation is one of the most important partial differential equations and observed in a wide range of engineering and industrial applications [9]. It has been used to decsribe heat transfer in a draining film [10], water transfer in soil [11], dispersion of tracers in porous media [12], contaminant dispersion in shallow lakes [13], the spread of solute in a liquid flowing through a tube, long-range transport of pollutants in the atmosphere [14] and dispersion of dissolved salts in groundwater [15]. Accurate numerical solution of the advection-diffusion equation is usually characterized by a dimensionless parameter, called Peclect number. These results become increasingly difficult as the Peclect number increases due to onset of spurious oscillations or excessive numerical damping if finite difference [16] or finite element formulations are used [17]. In the present paper, a composite CRBF is developed by a comination between thin plate and logarithmic radial basis functions. In this fuction the thin plate is used when evaluating diagonal elements of the matrix coefficients and the logarithmic function is used for evaluating the other elements. One- and two-dimensional test examples were solved and the present results were compared with the analytical results and gave a good agreement.

II. COLLOCATION BASED ON THE NEW CRBF

2.1. The New Radial Basis Function

In the collocation techniques, the approximate solution is applied at the scattared nodes over the domain of interest. After collocations, two different $(N \times N)$ matrices are obtained. When using logarithmic radial basis function, singularity appears in the diagonal elements of the matrices. In [18] a new radial basis functions were developed and based on a combination between multiquadric and logarithmic radial basis functions, while in [19] a new radial basis function with collocation and cartesian grid method was developed to solve partial differential equations. The results were very good compared with other methods. A problem had been

encountered with that function, was the choice of the shape parameter in the multiquadirc radial basis function. To overcome such difficulty, herein in, the mutliquadric is replaced by the thin plate spline radial basis function and the new function is applied to the advection-diffusion problem in one-, and two-dimensions. The new radial basis function herein, named as composite radial basis function (**CRBF**). The new **CRBF** is assumed as follows:

$$\Phi\left(r_{ij}\right) = (1-\varepsilon)\varphi_1\left(r_{ij}\right) + \varepsilon\varphi_2\left(r_{ij}\right) \tag{1}$$

$$\varphi_1\left(r_{ij}\right) = \text{TPS} = r_{ij}^{2m} \tag{2}$$

$$\varphi_2\left(r_{ij}\right) = \text{LBF} = r_{ij}^{2m}\log\left(r_{ij}\right) \tag{3}$$

2.2 Mathematical Formulation

Consider the advection-diffusion equation:

$$\frac{\partial u\left(\mathrm{x},t\right)}{\partial \mathrm{t}} = k\nabla^2\left(x,t\right) + v.\nabla\left(x,t\right)$$

(4)

with the following associated boundary and initial conditions:

$$c_1\left(x,t\right) + c_2\left(x,t\right) = f\left(x,t\right), \quad x \in \partial\Omega, \ t > 0 \tag{5}$$

$$u\left(x,t\right) = u_0\left(t\right) \tag{6}$$

The physical parameter which describes the relation between the diffusion coefficient k and the advection coefficient v is the Peclet number defined as $P_e = \left(v/k\right)$. According to the values of P_e, equation (4) behaves as parabolic differential equation for small values of P_e and for large values of P_e it behaves as hyperbolic differential equation [20]. The discretization of equations (4-6) starts with the approximation of time derivatives using forward difference and for the spatial derivatives, the Θ-weighted Crank-Nicholson scheme is used [21]. Therefore, equations (4) and (5) will take the following form:

$$\mathrm{u}\left(\mathrm{x},t+\Delta t\right) - k\Theta\Delta t\nabla^2 u\left(\mathrm{x},t+\Delta t\right) - \Delta t\Theta v\nabla u\left(\mathrm{x},t+\Delta t\right)$$
$$= u\left(\mathrm{x},t\right) + \Delta t\left(1-\Theta\right)k\nabla^2 u\left(\mathrm{x},t\right) + \Delta t\left(1-\Theta\right)v.\nabla u\left(\mathrm{x},t\right) \tag{7}$$

Re-arrange equation (7) in terms of time notation, leads to:

$$\left(u\left(\mathrm{x},t\right) - k\Theta\Delta t\nabla^2 u\left(\mathrm{x},t\right) - \Delta t\Theta v\nabla u\left(\mathrm{x},t\right)\right)^{n+1}$$
$$= \left(u\left(\mathrm{x},t\right) + \Delta t\left(1-\Theta\right)k\nabla^2 u\left(\mathrm{x},t\right) + \Delta t\left(1-\Theta\right)v.\nabla u\left(\mathrm{x},t\right)\right)^{n} \tag{8}$$

The notation u^{n+1} stands for the potential at time $\left(t^n + \Delta t\right)$, Δt is the time step size. To simplify dealing with equation (8), let us introduce the following simplifications into consideration:

$$\alpha = -k\Theta\Delta t$$
$$\beta = -\Delta t\Theta v$$
$$\varsigma = \Delta t\left(1-\Theta\right)k \tag{9}$$
$$\xi = \Delta t\left(1-\Theta\right)v$$

By making use of equation (9) into equation (8), the later takes the following simplified form:

$$\left(1 + \alpha\nabla^2 + \beta\nabla\right)^{n+1} u\left(\mathrm{x},t\right) = \left(1 + \varsigma\nabla^2 + \xi\nabla\right)^{n} u\left(\mathrm{x},t\right) \tag{10}$$

Equation (10) can be re-written as follows:

$$H_L\left\{u\left(\mathrm{x},t\right)^{n+1}\right\} = H_R\left\{u\left(\mathrm{x},t\right)^{n}\right\} \tag{11}$$

where

$$H_L = \left(1 + \alpha\nabla^2 + \beta\nabla\right)^{n+1}$$
$$H_R = \left(1 + \varsigma\nabla^2 + \xi\nabla\right)^{n} \tag{12}$$

The next step, is to approximate the solution by making use of equation (1) into equation (11), yeilds:

$$\sum_{j=1}^{N} \lambda_j^{n+1} H_L \Theta\left(r_{ij}\right) = \sum_{j=1}^{N} \lambda_j^{n} H_R \Theta\left(r_{ij}\right) \tag{13}$$

To simplify computation procedure, let us introduce the following:

$$H_L \Theta\left(r_{ij}\right) = \Psi_1\left(r_{ij}\right)$$
$$H_R \Theta\left(r_{ij}\right) = \Psi_2\left(r_{ij}\right) \tag{14}$$

By making use of equation (14) into equation (13), the later takes the following simplified form:

$$\sum_{j=1}^{N} \lambda_j^{n+1} \Psi_1\left(r_{ij}\right) = \sum_{j=1}^{N} \lambda_j^{n} \Psi_2\left(r_{ij}\right) \tag{15}$$

III. SOLUTION PROCEDURE

Let us start the solution procedure by manuplating the right hand side of equation (15) as follows:

$$\left[u\left(x_j\right)\right]^T = \sum_{j=1}^{N} \lambda_j^{n} \Psi_2\left(r_{ij}\right) \tag{16}$$

To solve the system given by equation (16), let us re-write it as follows:

$$\lambda^n \Psi_2 = U\left(\mathrm{x}\right) \tag{17}$$

The solution of equation (17) takes the following form:

$$\lambda^n = \Psi_2^{-1} U\left(\mathrm{x}\right) \tag{18}$$

By making use of the solution of equation (18) into equation (15), then the solution at the n+1 time step will be:

$$\lambda^{n+1} = \Psi_1^{-1}\left(r_{ij}\right)\Psi_2\left(r_{ij}\right)\lambda^n \tag{19}$$

IV. RESULTS AND DISCUSSION

To test the validaty of the CRBF function, on the solution of certain class of partial differential equations, two advection-diffusion problems are solved in the next two subsections.

4.1-D advection-diffusion

The test problem consists of the governing equation wuth the associated boundary and initial conditions as follows:

$$\frac{\partial u(x,t)}{\partial t} = k\frac{\partial^2 u(x,t)}{\partial x^2} - v\frac{\partial u(x,t)}{\partial x},\ 0\leq x \leq 1,\ t > 0$$

With

$$u\left(x=0,t\right)=30$$
$$u\left(x=1,t\right)=30$$
$$u\left(x,t=0\right)=0$$

The analytical solution is given by [22]:

$$u^{ex}(x,t) = a\exp\left(bt+cx\right) \quad \& \quad c = \frac{v \pm \sqrt{v^2 + 4kb}}{2k}$$

The test problem is solved at three different cases of convection coefficients and fixed value of diffusion coefficient. The numerical data corresponding to the three different cases are shown in table (1).

Table 1: Numerical data for coefficients

	Case (1)	Case (2)	Case (3)
v	1	6	12
k	1	1	1

The results corresponding to the three cases are compared with the analytical results at time, $t = 0.5\,\text{sec}$ as shown in figure (1). From the figure, one can conclude that at small value of the convection coefficient, the temperature profile behave linearly and by growing up its value, the profile behaves in an exponentially decaying and this agrees well with the expected physical behavior. Also, it is clear that, there is a good agreement with the analytical results.
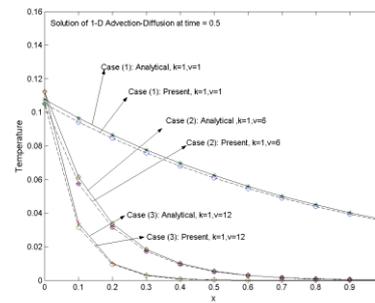

Figure 1: Results of test problem 1 at three different cases of k and v

4.2-D Advection-diffusion equation

To illustrate the application of the collocation method based on the proposed new CRBF for higher dimension, consider the following 2D advection-diffusion problem

$$\frac{\partial u(x,y,t)}{\partial t} = k_x \frac{\partial^2 u(x,y,t)}{\partial x^2} + k_y \frac{\partial^2 u(x,y,t)}{\partial y^2}$$
$$+ v_x \frac{\partial u(x,y,t)}{\partial x} + v_y \frac{\partial u(x,y,t)}{\partial y},\ 0 \leq x,y \leq 1$$

with the initial condition

$$u(x,y,0) = a\left(\exp(c_x x) + \exp(c_y y)\right)$$

and the boundary conditions

$$u(0,y,t) = a\left(\exp(bt) + \exp(bt + c_y y)\right),$$
$$u(1,y,t) = a\left(\exp(bt + c_x) + \exp(bt + c_y y)\right),$$
$$u(x,0,t) = a\left(\exp(bt + c_x x) + \exp(bt)\right),$$
$$u(x,1,t) = a\left(\exp(bt + c_x x) + \exp(bt + c_y)\right)$$

The analytic solution is given [22]

$$u(x,y,t) = a\left(\exp(bt + c_x x) + \exp(bt + c_y y)\right)$$

$$c_x = \frac{v_x \pm \sqrt{v_x^2 + 4k_x b}}{2k_x}$$

and

$$c_y = \frac{v_y \pm \sqrt{v_y^2 + 4k_y b}}{2k_y}$$

The problem is solved using the collocation method based on the CBRF, and a uniform distribution of nodes over a unit square. The computational domain is discretized with different arrangement of collocation points with a time step $\delta t = 0.05\,\text{sec}$, the diffusion coefficient $k_x = 1.4, k_y = 1.7$, and the advection

coefficients are $v_x = 0.1$, $v_y = 0.1$. The results due to the present method are compared with the available analytical results. Let us start analyzing the results by plotting the variation of temperature against space variables at two different times $t = 0.1 \& 0.5\,\text{sec}$ are shown in figures (2) and (3), respectively.
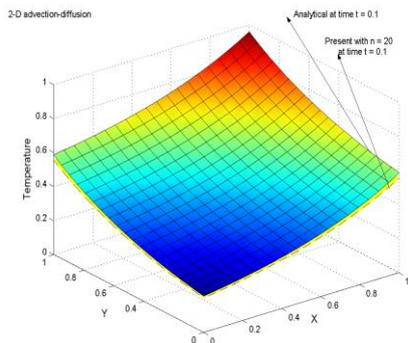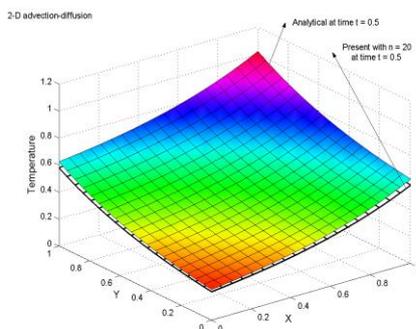

Figure 2: Temperature distribution at time t = 0.1

Figure 3: : Temperature distribution at time t = 0.5

From both figures (2) and (3), one observed a good agreement between the results due to the collocation based on the proposed CRBF and the analytical results. One can also observe that by growing up the time the error between the two results slightly increases but still in the allowed tollerence and can be accepted. This error can be dreased by increaing the number of scattered node throughout the domain. Finally, one can say that the collocation based on the present CBRF promises well in this field of research. To test the results of the collocation based on the CBRF, the L_2- relative error norms indicator is used for comparison and it is defined as:

$$L_2 - \text{REN} = \sqrt{\frac{\sum (u_{num} - u_{exact})^2}{\sum (u_{exact})^2}}$$

In the present computation, we check the comparison based on the innrease of the scattered nodes, and table (2) shows the effect of the inreasing the number of the scattered nodes and the corresponding relative error norm.

Table 2: Relative errors norm

Number of nodes	L_2-REN
81	368×10^{-5}
169	351×10^{-5}
256	319×10^{-5}
400	64×10^{-5}

From table (2), it is clear that by inreasing the number of scattered nodes, the the relative error norm decreases and this indicates that the error between the obtained results and the analytical one is acceptable. Also, one can conclude that the collocation based on the CBRF gives acceptable results and so can solve problems with no solutions availble to compare.

CONCLUSION

The present paper concerns mainly with collocation based on a new composite radial basis function. In the present paper, a composite CRBF is developed by a comination between thin plate and logarithmic radial basis functions. In this fuction the thin plate is used when evaluating diagonal elements of the matrix coefficients and the logarithmic function is used for evaluating the other elements. Finally, we can say that the collocation method is a very simple meshless method, also the results due to the the present CBRF is modified compared with the previous versions derived at 2006 and 2008.

REFERENCES

[1]   Chongjiang DU. An element-free Galerkin method for simulation of stationary two-dimensional shallow flows in rivers. Comput Meth Appl Mech Eng 2000;182:89–107.

[2]   Lee JK, Froechlich DC. Review of literature on the finite-element solution of the equation of two-dimensional surface water flows in the horizontal plane. US Geological Survey Circular 1009, 1986.

[3]   Zhu T, Zhang J, Atluri SN. A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and nonlinear boundary value problems. Eng Anal Boundary Elements, 1999;23:375–89.

[4]   Lucy LB. A numerical approach to the testing of the fission hypothesis. Astro J 1977;8:1013–24.

[5]   Belytschko T, Lu YY, Gu L. Element-free Galerkin methods. Int. J. Numer Meth Eng 1994;37:229–54.

[6]   Liu GR, Jun S, Zhang YF. Reproducing Kernel particle methods. Int J Numer Meth Eng 1995;20:1081–106.

[7]   Liu GR, Gu YT. A point interpolation method for two-dimensional solid. Int J Numer Meth Eng 2001;50:937–51.

[8]   Atluri SN, Zhu T., A new meshless local Petrov-Galerkin approach in the computational mechanics, Comput. Mech., 1998; pp. 22:117–127.

[9]   Singh, K.M. and Tanaka, M.,' On exponential variable transformation based boundary element formulation for advection-diffusion problems', Engineering Analysis with Boundary Element, Vol. 24, pp. 225-235, 2000.

[10]   Isenberg, J. and Gutfinger, C.,' Heat transfer to a draining film', Int. J. Heat Transfer, Vol. 16, pp. 505-512, 1972.

[11]   Parlarge, J.Y.,' Water transport in soils', Ann. Rev. Fluids Mech., Vol. 2, pp. 77-102, 1980.

[12]   Fattah, Q.N. and Hoopes, J.A.,' Dispersion in anistropic homogenoeous porous media', J. Hydraul. Eng., Vol. 111, pp. 810-827, 1985.

[13]   Salmon, J.R., Ligett, J.A. and Gallager, R.H.,' Dispersion analysis in homogeneous lakes', Int. J. Numer. Meth. Eng., Vol. 15, pp. 1627-1642, 1980.

[14]   Zlatev, Z., Berkowicz, R. and Prahm, L.P.,' Implementation of a variable stepsize variable formula in the time-integration part of a code for treatment of long-range transport of air pollutants', J. Comp. Phys., Vol. 55, pp. 278-301, 1984.

[15]   Guvanasen, V. and Volker, R.E.,' Numerical solution for solute transport in unconfined aquifers', Int. J. Numer. Meth. Fluids, Vol. 3, pp. 103-123, 1983.

[16]   Donea, J., Giulinai, S., Laval, H. and Quartapelle,' Time accurate solution of advection-diffusion problems by finite elements', Computer Methods in Applied Mechanics and Engineering, Vol. 45, pp. 123-145, 1984.

[17]   Yu, C.C., Heinrich, J.C.,' Petrov-Galerkin methods for the time-dependent convective transport equation', Int. J. Numer. Meth. Eng., Vol. 23, pp. 883-901, 1986.

[18]   S. G. Ahmed, A collocation method using new combined radial basis functions of thin plate and multiquadraic types, Engineering Analysis with Boundary Elements, vol. 30, pp. 697–701, (2006)

[19]   S. G. Ahmed and M. L. Mekey, A collocation and Cartesian grid methods using new radial basis function to solve class of partial differential equations, International Journal of Computer Mathematics, 2009, pp. 1–14, iFirst.

[20]   Fornefett, M.; Rohr, K.; Stiehl, H.S.: "Radial Basis Functions with Compact Support for Elastic Registration of Medical Images", Image and Vision Computing, vol. 19:1-2, pp. 87-96, (2001).

[21]   Wright G., Fornberg B., scattered node compact finite difference type formulas generated from radial basis functions.J.comput.phys., vol. 212, pp. 99-123, (2006).

[22]   Boztosum I., Charafi A., Zerroukat M. and Djidjeli K., Thin plate spline radial basis function scheme for Advection-diffusion problems, Electronic Journal of Boundary Elements, vol. 2, pp. 267-282, (2002).