Exact Analysis of GMSK for Large Design Parameter

Shailendra Jain[#] , Anurag Saraf [*], M. Tiwari [*]

[#]*Department of Physics and Electronics, Jaipur Institute of Technology Group of Institutions, Jaipur, India.*

**Department of Physics and Electronics, Dr. H. S. Gour University Sagar 470003 India.*

Abstract **- The Exact analysis of Gaussian Minimum Shift Keying (GMSK) signal for different design parameters is being presented. Binary signals are approximated to the three consecutive binary pulses for a particular interval .The expression for power spectral density is derived and analyzed for different BT and for different three bit sequences. The expression for minimum Euclidean distance is derived and hence, the probability of bit error for BT=0.3, 0.5 and 0.7 is computed. These studies are useful in practical systems where large BT is needed. The system designers get many choices of reasonable trades off between spectral efficiency and performance loss. Due to large BT and various choices of BT and N, the error performance degradation can be minimized without affecting the spectral efficiency. A simple design of receiver is possible.**

I. INTRODUCTION

Continuous Phase Sift Keying (CPM) is employed in which the phase of the carrier is constrained to be continuous and hence, it has memory. CPFSK signal is the special case of continuous phase modulated signal (CPM). One of the CPM signal is the Gaussian minimum shift keying (GMSK).GMSK is a binary partial response continuous phase modulation [1], [2] .This modulation technique was first employed for digital mobile radio telephony by Murota and Hirade [3] in 1981. Due to its excellent power and bandwidth efficiency, it is very suitable for wireless communication. In GMSK signaling a trade off among bandwidth efficiency and receiver complexity can be achieved by appropriately configuring the product of Gaussian pre–modulation frequency pulse bandwidth and symbol period (BT). The bandwidth efficiency of signals and the receiver complexity increases for the lower BT [4] and there exists performance degradation and large adjacent channel interference. In practical application, the pulses are usually truncated to some specified fixed duration .For global system mobile communication (GSM) and cellular Digital Packet Data (CDPD), BT=0.3 and 0.5 are used respectively [5] – [7]. In practical application the pulse response is limited to only two nearest neighbor i.e. three consecutive bits at a time .This approximation was employed by Haspeslagh and others [8]. For BT<0.3, there is great interference with second nearest neighbors and becomes very complex to achieve reliable detection. So, practical systems require large BT. The three consecutive bits for pulse response and its application to the study of GMSK frequency and phase trajectories is employed by Linz and A. Hendrickson [9]. Gaussian minimum shift keying over an Additive White Gaussian Noise (AWGN) channel has been extensively studied by different authors [10], [11] and differential versus limiter discriminator detector [12].GMSK modulation and detection have been recently discussed by authors [13], [14].

All the above studies are concentrated to lower BT≤0.3. In the present paper, analytical expression for power spectral density for GMSK modulation is derived and the variation of spectral density with normalized frequency for design parameter (BT=0.3, 0.7) have been studied. The expression for minimum Euclidian distance (d_{min}) for GMSK is derived and studied for different number of bit observation (decision) intervals (N) as a parameter for three bits sequences. The variation of d_{min} with parameter BT for different N has been studied. Probability of bit error on additive white Gaussian channel for different parameters BT and N have been computed and studied. Paper is divided into five sections. The organization of paper is as follows. 2nd section contains basic modulation and power spectral density of GMSK. Probability of error for GMSK with additive white Gaussian noise (AWGN) is being discussed for optimum receivers in section 3rd. In section 4th, results have been discussed. Finally, conclusion is presented in section 5th.

Fig. 1 Basic circuit diagram of GMSK modulator

II BASICS OF GMSK MODULATION

The block diagram of basic GMSK modulation circuit is shown in Fig. 1.

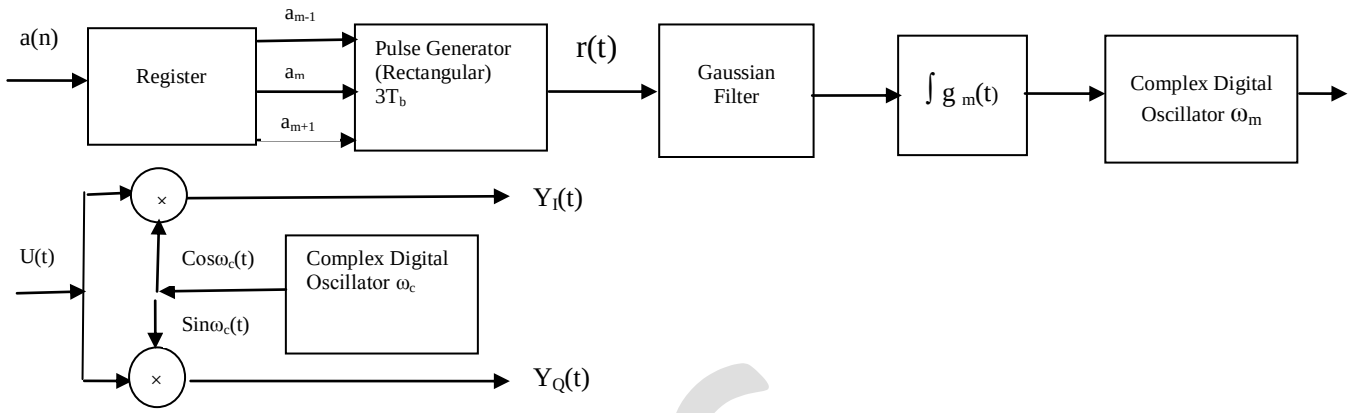


Fig. 1 Basic circuit diagram of GMSK modulator

Firstly, three consecutive non return to zero (NRZ) binary data obtained from register during the interval $[mT_b, (m+1)T_b]$ is allowed to pass through pulse shaper P(t) which generates the combination of rectangular pulses r(t) of unit amplitude of duration $3T_b$.These pulses are allowed to pass through Gaussian filter and the response of the filter produces pulse response $g_m(t)$ which constitutes the frequency shaping pulse of the GMSK modulator. After Integration of $g_m(t)$, the signal goes to complex digital oscillator [15] which produces the complex phase function U(t) of binary data. The modulation of phase with carrier of frequency ω_c is achieved through other complex digital oscillator having frequency ω_c to generate in phase component $Y_I(t)$ and quadrature component $Y_Q(t)$ of GMSK modulated signals. Following the method [8], [9], we can write,

$$g_m(t) = a_{m-1}[1-\text{erf}(\beta t)] + a_m[\text{erf}(\beta t) - \text{erf}(\beta(t-T_b))] + a_{m+1}[1+\text{erf}(\beta(t-T_b))] \quad \ldots\ldots\ldots\ldots (1)$$

Where

$$\beta = \sqrt{\frac{2}{\ln(2)}}\pi f_b x\,(BT) \qquad \text{for } B = \frac{B_{3dB}}{2\pi}$$

B_{3dB} is 3dB bandwidth and f_b is the bit frequency

$T=T_b$ is the bit period

erf ( ) is the error function.

Here, $\text{erf}(\beta T_b) \approx 1$ and

$$\beta T_b = 2\sqrt{2\pi}\ (BT)$$

For our convenience, amplitude is considered to be unity. The modulated signal Y(t) is given by

$$Y(t) = \cos(\omega_c t)\cos\left(2\pi f_m \int_0^t g_m(\tau)\,d\tau\right) - \sin(\omega_c t)\sin\left(2\pi f_m \int_0^t g_m(\tau)\,d\tau\right)$$

$$= U_I \cos(\omega_c t) - U_Q \sin(\omega_c t)$$

$$= Y_I + Y_Q \quad \ldots\ldots\ldots\ldots (2)$$

Where $U(t) = \exp\left(j2\pi f_m \int_0^t g_m(\tau)\,d\tau\right)$

$$Y_I = \text{Re}\left\{\exp j\left(\omega_c t + \omega_m \int_0^t g_m(\tau)\,d\tau\right)\right\} \quad \ldots\ldots\ldots\ldots (3)$$

$$Y_Q = I_m\left\{\exp j\left(\omega_c t + \omega_m \int_0^t g_m(\tau)\,d\tau\right)\right\} \quad \ldots\ldots\ldots\ldots (4)$$

1) Power spectral density

The complex phase function [9] is given as

$$U(t) = \sum_{n=-\infty}^{+\infty} \exp\{j[\varphi_{l(n)}(t-nT)]\}\,P(t'-nT) \quad \ldots\ldots\ldots\ldots (5)$$

Where $\varphi_{l(n)}(t)$ is phase trajectory and given by.

$$\varphi_{l(n)}(t) = 2\pi f_m \int_0^t f_{l(n)}(\tau)\,d\tau \quad \ldots\ldots\ldots\ldots (6)$$

The correlation function can be written as

$$\langle u(t+\tau); u(t)\rangle = \tfrac{1}{2}[u(t+\tau)\,u^*(t)]$$

$$= \tfrac{1}{2} E\left[\sum_{n=-\infty}^{+\infty} \exp\{j(\varphi_{l(n)}(t+\tau-nT).P(t'+\tau-nT) - \varphi_{l(n)}(t-nT)\,P(t'-nT))\}\right] \quad \ldots\ldots\ldots (7)$$

The average correlation is defined as

$$\overline{\langle u(t+\tau); u(t)\rangle} = \frac{1}{T}\int_{-T/2}^{+T/2} \langle u(t+\tau);\ u(t)\rangle \quad \ldots (8)$$

With the help of

$$\sum_n f_{l(n)}(t-nT)\,P(t'-nT) = g_m(t) \quad \ldots\ldots\ldots\ldots (9)$$

and using equations (6), (7) in eqn. (8), the average correlation can be written as

$$\overline{\langle u(t+\tau); u(t)\rangle} = \frac{1}{T}\int_{-T/2}^{+T/2} [\exp(j2\pi f_m \int g_m(\tau)\,d\tau)]\,dt$$

..... (10)

Substituting equation (1) into equation (10) and solving, we get

$$\overline{< u(t+\tau); u(t) >} = \frac{1}{T} [\exp j2\pi f_m [a_{m-1}\{I_1(\beta,\tau) - I_1(\beta, T+\tau) + I_1(\beta, T) - I_1(\beta, 0)\} + a_m\{I_1(\beta, \tau) + I_1(\beta, \tau - T) + I_1(\beta, 0) + I_1(\beta, 0)\, T\} + a_{m+1}\{I_1(\beta,\tau) - I_1(\beta, \tau - T) - I_1(\beta, T)\}] - \tau]$$ ..... (11)

Where $\int \text{erf}\ (\beta x)\, dx = I_1(\beta, x)$ [16]

$$I_1(\beta, 0) = \frac{1}{\beta\sqrt{\pi}}$$

$$I_1(\beta, -x) = I_1(\beta, x)$$

when erf $(\beta x) = I$, $I_1(\beta, x) \approx x$

Now, taking the Fourier Transform of $\overline{< u(t+\tau); u(t) >}$ and solving for power Spectral density, we get

$$|G(f)|^2 = T$$

$$\left[\frac{Sin\left\{ \pi fT - 2\pi f_m T\left\{ a_{m-1}\left(-\frac{1}{2\beta\sqrt{\pi}}\right) + a_m\left(\frac{3}{2\beta\sqrt{\pi}}\right) + a_{m+1}\left(1 - \frac{1}{2\beta\sqrt{\pi}}\right) - \frac{1}{2}\right\}\right\}}{\left\{ \pi fT - 2\pi f_m T\left\{ a_{m-1}\left(-\frac{1}{2\beta\sqrt{\pi}}\right) + a_m\left(\frac{3}{2\beta\sqrt{\pi}}\right) + a_{m+1}\left(1 - \frac{1}{2\beta\sqrt{\pi}}\right) - \frac{1}{2}\right\}\right\}} \right.$$

$$+ \left. \frac{Sin\left\{ \pi fT + 2\pi f_m T\left\{ a_{m-1}\left(\frac{2}{\beta\sqrt{\pi}}\right) + a_m\left(2 + \frac{1}{2\beta\sqrt{\pi}}\right) + a_{m+1}\left(\frac{1}{2\beta\sqrt{\pi}}\right) + \frac{1}{2}\right\}\right\}}{\left\{ \pi fT + 2\pi f_m T\left\{ a_{m-1}\left(\frac{2}{\beta\sqrt{\pi}}\right) + a_m\left(2 + \frac{1}{2\beta\sqrt{\pi}}\right) + a_{m+1}\left(\frac{1}{2\beta\sqrt{\pi}}\right) + \frac{1}{2}\right\}\right\}} \right]^2$$

……(12)

III. OPTIMUM RECEIVER

The basic block diagram of optimum receiver for GMSK is depicted in Fig. 2. The modulated in-phase signal $Y_I(t)$ and quardature-phase signal $Y_Q(t)$ are added separately with in-phase noise component $n_c \cos\omega_c t$ and qurdature-phase noise component $n_s \sin\omega_s t$ and then multiplied by carrier signals from complex digital oscillator (ω_c).

Fig. 2 Basic circuit diagram of GMSK detector

The output signal from multiplier are filtered using matching filters and then sampled at t=T and added to get the signal A(T). Signal A(T) is allowed to pass through maximum likelihood sequence (MLS) detector where the minimum Euclidian distance related to phase difference between two paths from phase trellis [9] is achieved. Thus, the phase of modulated signal is detected which contains the modulating signal g (t). The output of detector goes to discriminator which is combination of differentiator and envelope detector, reproduces the message signal and finally, the signal is filtered by BLPF to minimize the noise. Thus, we recover the signal g (t).

Let us consider the two signals $S_i(t)$ and $S_j(t)$ corresponding to two phase trajectories

$$\varphi_i(t) = 2\pi f_m \int_0^t f_i(\tau)\, d\tau + \varphi_i(0) \quad \text{and}$$

$$\varphi_j(t) = 2\pi f_m \int_0^t f_j(\tau)\, d\tau + \varphi_j(0)$$

Where $\varphi_i(0)$ and $\varphi_j(0)$ are initial phases.

Here, $S_i(t) = \cos(\omega_c t + 2\pi f_m \int_0^t g_i(\tau)\, d\tau)$ ---------- (13)

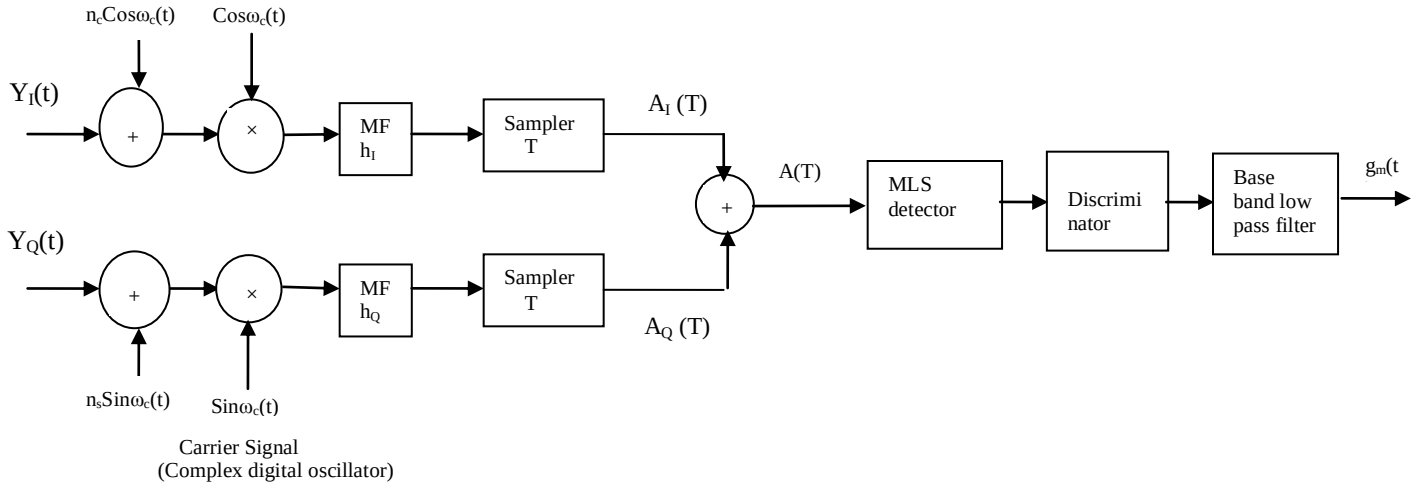
$Sj(t) = \cos(\omega_c t + 2\pi f_m \int_0^t g_j(\tau)\, d\tau)$ ---------- (14)

Euclidean distance between these two signals can be written as

$$d_{ij}^2 = \int_0^{NT} (1 - \cos(2\pi f_m \{\int g_i(\tau)\, d\tau - \int g_j(\tau)\, d\tau\})\, d\tau]\, dt$$ ----------(15)

NT is the interval for N = 1,2,3

Substituting the values of $g_i(\tau)$ and $g_j(\tau)$ from equ. (1) into (15) and solving we get normalized Euclidean distance



$$d_{ij}^2 / T = \delta_{ij}^2 = N \frac{\sin \theta(T)}{\theta(T)} \quad \text{---------- (16)}$$

Where $T = 2E_b$ for unit amplitude

$$\theta(T) = 2\pi f_m[(a_{i-1} - a_{j-1}) \{ NT + \frac{1}{\beta\sqrt{\pi}} - I_1(\beta, NT) \} + (a_i - a_j)\{ I_1(\beta, NT) - \frac{1}{\beta\sqrt{\pi}} - I_1(\beta, (N-1)T) + I_1(\beta, -T)\} + (a_{i+1} - a_{j+1}) \{NT + I_1(\beta, (N-1)T) - I_1(\beta, -T)] \quad \text{------- (17)}$$

Where a_{i-1}, a_i, a_{i+1} and a_{j-1}, a_j, a_{j+1} are three consecutive bits of two signals $a_i - a_j$ shows the difference of two bits.

Probability of bit error rate is given by

$$Pe = \tfrac{1}{2} \operatorname{erfc} \left(\sqrt{\frac{\delta_{ij}^2 E}{2No}} \right) \quad \text{----------(18)}$$

Where $N_0 / 2$ is the power spectral density of the AWGN.
$SNR = E / N_0$, E is the energy of signal
Here $\delta_{ij}^2 = \delta_{min}^2$, since we have two signal S_i and S_j only.

IV. RESULTS AND DISCUSSION

Power spectrum for the sequence (0 0 1) (1 01), (1 1 0) and for BT =0.7 and h=0.5 is plotted in Fig. (3) as indicated by solid lines .The dotted lines and broken line indicate the spectrum of GMSK due to author [3], [9] respectively using simulation of pseudorandom data . The spectrum of MSK is also shown. The main lobe of our spectrum for the sequence (1 0 1), Linz and Hendrickson [9] and author [3] have the same bandwidth (fT=0.6) and the same power content (37.4dB) in main lobe in our spectrum and author [3]. The side lobes of spectrum for (1 0 1) are wider than the side lobes of others sequences. The width of Ist side lobe in our case is 0.5 whereas 0.3 in others [3], [9].

The normalized minimum Euclidean distance $d_{ij}^2 / 2E_b$ is evaluated by the equation (16) for bit sequence (001- 010) starting from bits 0 and 1 respectively for BT=0.3, 0.5 and 0.7 is presented in the table. The value of d_{min} depends on the sequences chosen from phase trajectories on the basis that in the one symbol interval, there is change of one bit out of three bits. The possible sequences are like 001 -010, 001 – 011, 110-101, 100- 111 etc. Here $a_{i-1} = 1$, $a_i = 0$, $a_{i+1} = 0$ for sequence (001) and similarly for (010); $a_{j-1} = 0$, $a_j = 1$ and $a_{j+1} = 0$. The machine computed result for d_{min} by Murota and Hirade [3] for frequency 16 KHz is also indicated in the table for comparison. The values of normalized Euclidian distance for BT =0.3 ,0.5, and 0.7 are smaller than the computerized values of author [3] but with the increase of BT it increases in both the cases and less than 2 for $B_bT = \infty$ (MSK) . The power degradation for BT=0.3, 0.5, and 0.7 is also shown in the same table and compared to the results author [3]. Our results indicate large E_b/N_0 degradation. The same results are found for the sequence (110 – 101). From the knowledge of δ_{ij}^2, the probability of error is calculated from eqn. (18) for BT= 0.3, 0.5 and 0.7 with N=2 for the sequence (001 – 010). The results so obtained are depicted in Fig. 4.The dotted line represents the results of author [3]. Probability of error for the sequence (110-101) for observation intervals N=1, 2, and 3 interval and design parameters BT=0.3, 0.5 and 0.7 is shown in Fig. 5. From the figure, we infer that the rate of decrease of probability of error increases with the increase of N. Probability of error also decreases rapidly with increase of N. For N=1, the upper curve shows Pe for BT=0.3 and lower for BT=0.7 and BT=0.3 in middle. But for N=2 the situation is just reverse i.e. the curve of P_e for N=0.7 is at upper, for N=0.5 in middle and for N=0.3 at lower. The difference in P_e for BT=0.3, 0.5 and 0.7 is small for both N=1 and N=2. The curve for BT=0.3 and 0.5 shows overlapping and slight variation in P_e for BT=0.7 is obtained at N=3. The gain in performance is achieved by increasing the observation interval.

All the numerical computations have been performed using MATLAB.

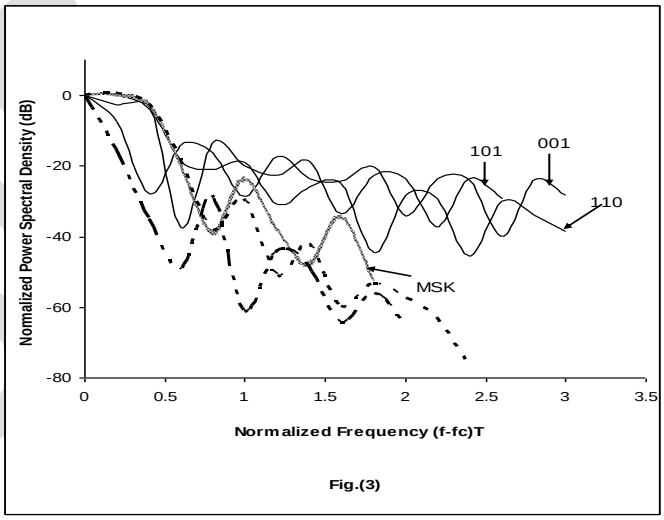


Fig. 3 Power spectral density for BT =0.7, h=0.5 for sequence (001), (101) and (110).

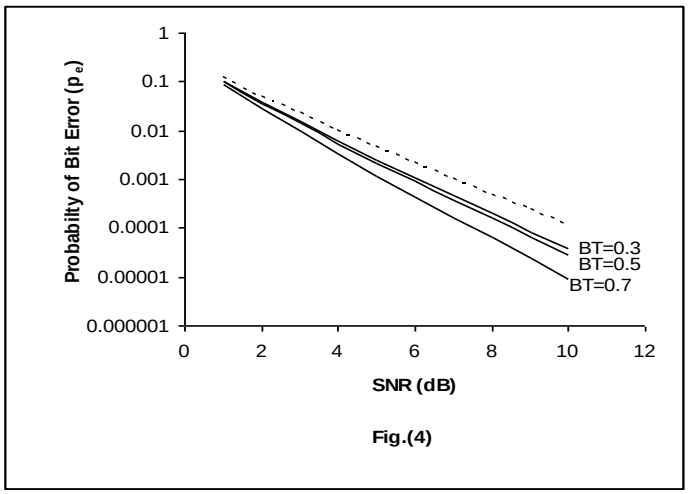


Fig. 4 Probability of bit error for BT= 0.3, 0.5 and 0.7 and sequence 001-010.

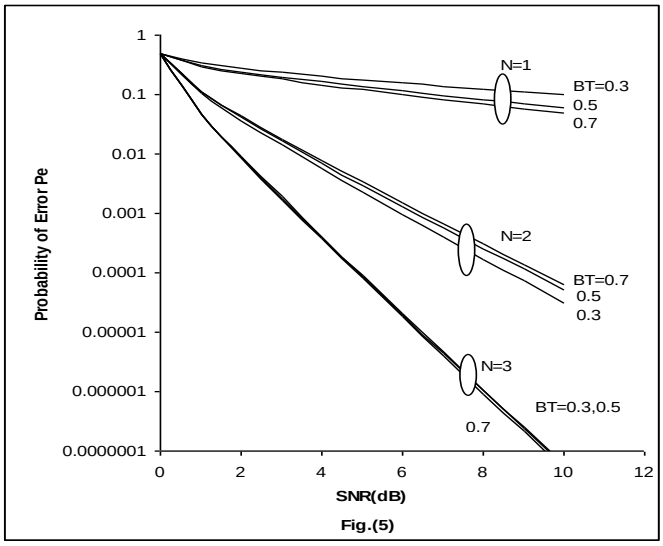


Fig. 5 Probability of bit error for observation intervals N=1, 2 and 3; BT=0.3, 0.5 and 0.7.

Table: Normalized minimum signal distance ($d_{min}^2/2E_b$) and degradation performance for different BT.

BT	**$d_{min}^2/2E_b$**	**$d_{min}^2/2E_b$ [3]**	**Degradation E_b/N_0 (dB)**	**Degradation E_b/N_0 (dB) [3]**
0.3	1.5714	1.81	1.047	0.46
0.5	1.6196	1.92	0.916	0.31
0.7	1.8387	1.97	0.365	0.19

V. CONCLUSION

Performance analysis of GMSK system through analytical method for high BT produces satisfactory results. The hardware implementation of GMSK modulator and detector is simple and straight like MSK .The complexity of receiver construction is minimized. The complex phase function and carrier signal have been generated using complex digital oscillator for frequency modulation in simple manner. The computation load is reduced much in the study of basic properties of GMSK without loss of its qualities. The application of GMSK study for BT =0.5 is useful in interplanetary mission for European Space Agency with transmitted bit rates higher than 2Mb /s and in cellular digital packet data . Finally, our system model provides versatility for the GMSK and CPM designer for the choice of BT, h and N.

REFERENCES

[1] Proakis J. G., (2001) Digital Communication: McGraw Hill, Inc., Fourth Edition.
[2] Anderson J, Aulin B. T. & Sundberg C-E, (1986) Digital Phase Modulation: New York: Plenum.
[3] Murota K. & Hirade K, IEEE Trans. Commun., COM 29 (1981) 1044-50.
[4] Korn, I., IEEE Trans. Commun., 39(1991) 94-101.
[5] Steele R., Williams J., Chandler D., Dehghan S. & Collard A., IEEE Commun. Mag. (1995) 102-108.
[6] MacKey D.J.C. , IEEE Trans. Inf. Theory IT 45 (1999) 399-431.
[7] Consultative Committee for Space Data Systems (CCSDS), Radio Frequency and Modulation Systems (CCSDS), 401(2001).
[8] Haspeslagh, J. et. al., IEEE J. Solid-State Circuits 25(1990) 1450-57.
[9] Linz A. & Hendrickson A., IEEE Trans. Circuits and System-II: Analog and Digital Signal Processing, 43 (996).
[10] Mathis H., IEEE Trans. Commun. 146 (1998) 428-30. [11] Benvenuto N., Salloum A., & Tomba L., IEEE Trans. Commun., 45 (1997) 761-64.
[11] Simon M.K. & Wang C.C., IEEE Trans. Commun., 31(1983) 1227-34.
[12] Gunther M.A. Sessler, Ricard Abello, Nick James, Roberto Madde & Enrico Vassallo, Proceeding of the IEEE 95 (2007) 2132-41.
[13] Rice M., Oliphant T., Haddadin O., & McIntire W., IEEE Trans. Aerospace & Electronic Systems 43 (2007) 1484- 95.
[14] Fliege N.J. & Wintermantel, J., IEEE Trans. Commun. 40 (1992).
[15] Gradshteyn I.S. & Ryzhik I.M., New York: Academic (1980) 633, 5.41.

CAPTIONS TO FIGURES

[1] Basic circuit diagram of GMSK modulator.
[2] Basic circuit diagram of GMSK detector.
[3] Power spectral density for BT =0.7, h=0.5 for sequence (001), (101) and (110).
[4] Probability of bit error for BT= 0.3, 0.5 and 0.7 and sequence 001-010.
[5] Probability of bit error for observation intervals N=1, 2 and 3; BT=0.3, 0.5 and 0.7.