Geometric Methods in the Theory of Ordinary Differential Equations: A Review

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Abstract— **Differential equations containing unknown functions, their derivatives of various orders, and independent variables. The theory of differential equations arose at the end of the 17th century in response to the needs of mechanics and other natural sciences, essentially at the same time as the integral calculus and the differential calculus. Differential geometry is a mathematical discipline that uses the methods of differential and integral calculus, as well as linear and multilinear algebra, to study problems in geometry. The theory of plane and space curves and of surfaces in the three-dimensional Euclidean space formed the basis for its initial development in the eighteenth and nineteenth century. Since the late nineteenth century, differential geometry has grown into a field concerned more generally with geometric structures on differentiable manifolds.**

Keywords: Implicit Function, Perturbation theory, Bifurcation Theory.

I. INTRODUCTION

An ordinary differential equation (frequently called an "ODE," "diff eq," or "diffy Q") is an equality involving a function and its derivatives. An ODE of order n is an equation of the form

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \qquad (1)$$

where y is a function of x, $y' = dy/dx$ is the first derivative with respect to x, and $y^{(n)} = d^n y / d x^n$ is the nth derivative with respect to x.

No homogeneous ordinary differential equations can be solved if the general solution to the homogenous version is known, in which case the undetermined or variation of parameters can be used to find the particular solution.

Many ordinary differential equations can be solved exactly in *Mathematica* using DSolve[*eqn*, *y*, *x*], and numerically using NDSolve[*eqn*, *y*, {*x*, *xmin*,*xmax*}].

An ODE of order n is said to be linear if it is of the form

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \ldots + a_1(x)\, y' + a_0(x)\, y = Q \qquad (2)$$

A linear ODE where $Q(x) = 0$ is said to be homogeneous. Confusingly, an ODE of the form

$$y' = f\left(\frac{y}{x}\right) \qquad (3)$$

is also sometimes called "homogeneous."

In general, an nth-order ODE has n linearly independent solutions. Furthermore, any linear combination of linearly independent functions solutions is also a solution.

Simple theories exist for first-order (integrating factor) and second-order (Sturm-Liouville theory) ordinary differential equations, and arbitrary ODEs with linear constant coefficients can be solved when they are of certain factorable forms. Integral transforms such as the Laplace transform can also be used to solve classes of linear ODEs. Morse and Feshbach (1953, pp. 667-674) give canonical forms and solutions for second-order ordinary differential equations.

While there are many general techniques for analytically solving classes of ODEs, the only practical solution technique for complicated equations is to use numerical methods (Milne 1970, Jeffreys and Jeffreys 1988). The most popular of these is the Runge-Kutta method, but many others have been developed, including the collocation method and Galerkin method. A vast amount of research and huge numbers of publications have been devoted to the numerical solution of differential equations, both ordinary and partial (PDEs) as a result of their importance in fields as diverse as physics, engineering, economics, and electronics.

The solutions to an ODE satisfy existence and uniqueness properties. These can be formally established by Picard's existence theorem for certain classes of ODEs. Let a system of first-order ODE be given by

$$\frac{d x_i}{d t} = f_i(x_1, \ldots, x_n, t), \qquad (4)$$

for $i = 1$, ..., n and let the functions $f_i(x_1, \ldots, x_n, t)$, where $i = 1$, ..., n, all be defined in a domain D of the $(n+1)$-dimensional space of the variables x_1, ..., x_n, t. Let these functions be continuous in D and have continuous first partial derivatives $\partial f_i / \partial x_j$ for $i = 1$, ..., n and $j = 1$, ..., n in D. Let $(x_1^0, \ldots, x_n^0)$ be in D. Then there exists a solution of (4) given by

$$x_1 = x_1(t), \ldots, x_n = x_n(t) \quad (5)$$

for $t_0 - \delta < t < t_0 + \delta$ (where $\delta > 0$) satisfying the initial conditions

$$x_1(t_0) = x_1^0, \ldots, x_n(t_0) = x_n^0. \quad (6)$$

Furthermore, the solution is unique, so that if

$$x_1 = x_1^*(t), \ldots, x_n = x_n^*(t) \quad (7)$$

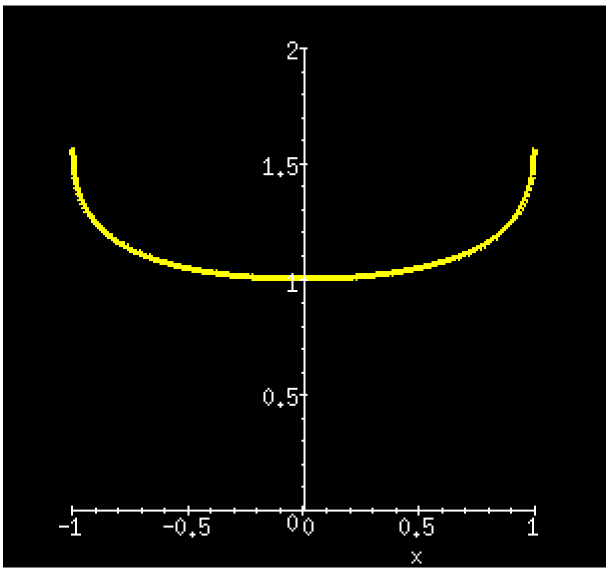
is a second solution of (◇) for $t_0 - \delta < t < t_0 + \delta$ satisfying (◇), then $x_i(t) \equiv x_i^*(t)$ for $t_0 - \delta < t < t_0 + \delta$. Because every nth-order ODE can be expressed as a system of n first-order ODEs, this theorem also applies to the single n th-order ODE.

II. IMPLICIT FUNCTION

A function which is not defined explicitly, but rather is defined in terms of an algebraic relationship (which can not, in general, be "solved" for the function in question). For example, the eccentric anomaly E of a body orbiting on an ellipse with eccentricity e is defined implicitly in terms of the mean anomaly M by Kepler's equation

$$M = E - e \sin E.$$

There are two ways to define functions, *implicitly* and *explicitly*. Most of the equations we have dealt with have been explicit equations, such as *y* = 2*x*-3, so that we can write *y* = *f*(*x*) where *f*(*x*) = 2*x*-3. But the equation 2*x*-*y* = 3 describes the same function. This second equation is an implicit definition of *y* as a function of *x*. As there is no real distinction between the appearance of *x* or *y* in the second form, this equation is also an implicit definition of *x* as a function of *y*.



The use of inverse trigonometric functions allow this to be solve for *y* as an explicit function of *x* and graphed, as shown. However, this function serves as a good example of implicit differentiation:

$$1 = \cos(x\,y)\left(y + x\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{1}{x\cos(x\,y)} - \frac{y}{x}.$$

Definition and Methods of Solution

An equation of type

$$F(x, y, y') = 0,$$

where *F* is a continuous function, is called the *first order implicit differental equation*. If this equation can be solved for *y'*, we get one or several *explicit differental equations* of type

$$y' = f(x, y),$$

that can be solved by methods covered in other sections.

Further we suppose that the differential equation can not be solved in the explicit form, so we should use different methods. The main techniques for solving an implicit differential equation is the *method of introducing a parameter*. Below we show how this method works to find the general solution for some most important particular cases of implicit differential equations.

Here we note that the general solution may not cover all possible solutions of a differential equation. Besides the general solution, the differential equation may also have so-called *singular solutions*. We consider this in more detail on the pageSingular Solutions of Differential Equations.

Case 1. Implicit Differential Equation of Type x=f(y,y').

In this case, the variable *x* is expressed explicitly through *y* and the derivative *y'*. Introduce the parameter $p = y' = \frac{dy}{dx}$. Differentiate the equation *x* = *f*(*y*,*y'*) with respect to *y*. This produces:

$$\frac{dx}{dy} = \frac{d}{dy}\left[f(y, p)\right] = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}.$$

Since $\frac{dx}{dy} = \frac{1}{p}$, the last expression can be written as follows:

$$\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\frac{dp}{dy}.$$

We obtain an explicit differential equation such that its general solution is given by the function

$$g(y,p,C) = 0,$$

where C is a constant.

Thus, the general solution of the original implicit differential equation is defined in the parametric form by the system of two algebraic equations:

$$\begin{cases} g(y,p,C) = 0 \\ x = f(y,p) \end{cases}.$$

If the parameter p can be eliminated from the system, the general solution is given in the explicit form $x = f(y,C)$.

Case 2. Implicit Differential Equation of Type y=f(x,y').

Here we consider a similar case, when the variable y is an explicit function of x and y'. Introduce the parameter $p = y' = \frac{dy}{dx}$ and differentiate the equation $y = f(x,y')$ with respect to x. As a result, we have:

$$\frac{dy}{dx} = \frac{d}{dx}\left[f(x,p)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx}, \quad \text{or}$$

$$p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx}.$$

Solving the last differential equation, we get the algebraic equation $g(x,p,C) = 0$. Together with the original equation, they form the following system of equations:

$$\begin{cases} g(x,p,C) = 0 \\ y = f(x,p) \end{cases},$$

which is the general solution of the given differential equation in the parametric form. In some cases, when the parameter p can be eliminated from the system, the general solution can be written in the explicit form $y = f(x,C)$.

Case 3. Implicit Differential Equation of Type x=f(y').

Here the differential equation does not contain the variable y. Using the parameter $p = y' = \frac{dy}{dx}$, it's easy to constructthe general solution of the equation. Since $dy = pdx$ and

$$dx = d\left[f(p)\right] = \frac{\partial f}{\partial p}dp,$$

then

$$dy = p\frac{\partial f}{\partial p}dp.$$

Integrating the last equation gives the general solution in the parametric form:

$$\begin{cases} y = \int p\frac{\partial f}{\partial p}dp + C \\ x = f(p) \end{cases}.$$

Case 4. Implicit Differential Equation of Type y=f(y').

The equation of this kind does not contain the variable x and can be solved the similar way. Using the parameter $p = y' = \frac{dy}{dx}$, we can write $dx = {dy}/{p}$. Then it follows from the equation that

$$dx = \frac{dy}{p} = \frac{1}{p}\frac{df}{dp}dp.$$

Integrating the last expression gives the general solution of the original implicit equation in parametric form:

$$\begin{cases} x = \int \frac{1}{p}\frac{df}{dp}dp \\ y = f(p) \end{cases}.$$

III. FIRST-ORDER PARTIAL DIFFERENTIAL EQUATION

In mathematics, a **first-order partial differential equation** is a partial differential equation that involves only first derivatives of the unknown function of n variables. The equation takes the form

$$F(x_1, \ldots, x_n, u, u_{x_1}, \ldots u_{x_n}) = 0.$$

Such equations arise in the construction of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems, and in simple models for gas dynamics whose solution involves the method of characteristics. If a family of solutions of a single first-order partial differential equation can be found, then additional solutions may be obtained by forming envelopes of solutions in that family. In a related procedure, general solutions may be obtained by integrating families of ordinary differential equations.

The notation is relatively simple in two space dimensions, but the main ideas generalize to higher dimensions. A general first-order partial differential equation has the form

$$F(x,y,u,p,q) = 0,$$

where

$$p = u_x, \quad q = u_y.$$

A **complete integral** of this equation is a solution $\varphi(x,y,u)$ that depends upon two parameters a and b. (There are n parameters required in the n-dimensional case.) An envelope of such solutions is obtained by choosing an arbitrary function w, setting $b=w(a)$, and determining $A(x,y,u)$ by requiring that the total derivative

$$\frac{d\varphi}{da} = \varphi_a(x,y,u,A,w(A)) + w'(A)\varphi_b(x,y,u,A,w(A)) = 0.$$

In that case, a solution u_w is also given by

$$u_w = \phi(x,y,u,A,w(A))$$

Each choice of the function *w* leads to a solution of the PDE. A similar process led to the construction of the light cone as a characteristic surface for the wave equation.

If a complete integral is not available, solutions may still be obtained by solving a system of ordinary equations. To obtain this system, first note that the PDE determines a cone (analogous to the light cone) at each point: if the PDE is linear in the derivatives of *u* (it is quasi-linear), then the cone degenerates into a line. In the general case, the pairs (*p*,*q*) that satisfy the equation determine a family of planes at a given point:

$$u - u_0 = p(x - x_0) + q(y - y_0),$$

where

$$F(x_0, y_0, u_0, p, q) = 0.$$

The envelope of these planes is a cone, or a line if the PDE is quasi-linear. The condition for an envelope is

$$F_p\,dp + F_q\,dq = 0,$$

where F is evaluated at (x_0, y_0, u_0, p, q), and *dp* and *dq* are increments of *p* and *q* that satisfy *F*=0. Hence the generator of the cone is a line with direction

$$dx : dy : du = F_p : F_q : (pF_p + qF_q).$$

This direction corresponds to the light rays for the wave equation. To integrate differential equations along these directions, we require increments for *p* and *q* along the ray. This can be obtained by differentiating the PDE:

$$F_x + F_u p + F_p p_x + F_q p_y = 0,$$

$$F_y + F_u q + F_p q_x + F_q q_y = 0,$$

Therefore the ray direction in (x, y, u, p, q) space is

$$dx : dy : du : dp : dq = F_p : F_q : (pF_p + qF_q) : (-F_x - F_u p) : (-F_y - F_u q).$$

The integration of these equations leads to a ray conoid at each point (x_0, y_0, u_0). General solutions of the PDE can then be obtained from envelopes of such conoids.

IV. PERTURBATION THEORY

This section develops, in simplified terms, the general theory for the perturbative solution to a differential equation to the first order. To keep the exposition simple, a crucial assumption is made: that the solutions to the unperturbed system are not *degenerate*, so that the perturbation series can be inverted. There are ways of dealing with the degenerate (or *singular*) case; these require extra care.

Suppose one wants to solve a differential equation of the form

$$Dg(x) = \lambda g(x)$$

where *D* is some specific differential operator, and *λ* is an eigenvalue. Many problems involving ordinary or partial differential equations can be cast in this form. It is presumed that the differential operator can be written in the form

$$D = D^{(0)} + \varepsilon D^{(1)}$$

where *ε* is presumed to be small, and that furthermore, the complete set of solutions for $D^{(0)}$ are known. That is, one has a set of solutions $f_n^{(0)}(x)$, labelled by some arbitrary index *n*, such that

$$D^{(0)} f_n^{(0)}(x) = \lambda_n^{(0)} f_n^{(0)}(x).$$

Furthermore, one assumes that the set of solutions $\{f_n^{(0)}(x)\}$ form an orthonormal set:

$$\int f_m^{(0)}(x) f_n^{(0)}(x)\,dx = \delta_{mn}$$

with δ_{mn} the Kronecker delta function.

To zeroth order, one expects that the solutions *g*(*x*) are then somehow "close" to one of the unperturbed solutions $f_n^{(0)}(x)$. That is,

$$g(x) = f_n^{(0)}(x) + \mathcal{O}(\varepsilon)$$

and

$$\lambda = \lambda_n^{(0)} + \mathcal{O}(\varepsilon).$$

where $\mathcal{O}$ denotes the relative size, in big-O notation, of the perturbation. To solve this problem, one assumes that the solution *g*(*x*) can be written as a linear combination of the $f_n^{(0)}(x)$:

$$g(x) = \sum_m c_m f_m^{(0)}(x)$$

with all of the constants $c_m = \mathcal{O}(\varepsilon)$ except for *n*, where $c_n = \mathcal{O}(1)$. Substituting this last expansion into the differential equation, taking the inner product of the result with $f_n^{(0)}(x)$, and making use of orthogonality, one obtains

$$c_n\lambda_n^{(0)} + \varepsilon \sum_m c_m \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x)\, dx = \lambda c_n$$

This can be trivially rewritten as a simple linear algebra problem of finding the eigenvalue of a matrix, where

$$\sum_m A_{nm} c_m = \lambda c_n$$

where the matrix elements A_{nm} are given by

$$A_{nm} = \delta_{nm}\lambda_n^{(0)} + \varepsilon \int f_n^{(0)}(x) D^{(1)} f_m^{(0)}(x)\, dx$$

Rather than solving this full matrix equation, one notes that, of all the c_m in the linear equation, only one, namely c_n, is not small. Thus, to the first order in ε, the linear equation may be solved trivially as

$$\lambda = \lambda_n^{(0)} + \varepsilon \int f_n^{(0)}(x) D^{(1)} f_n^{(0)}(x)\, dx$$

since all of the other terms in the linear equation are of order $\mathcal{O}(\varepsilon^2)$. The above gives the solution of the eigenvalue to first order in perturbation theory.

The function $g(x)$ to first order is obtained through similar reasoning. Substituting

$$g(x) = f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x)$$

so that

$$\left(D^{(0)} + \varepsilon D^{(1)}\right)\left(f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x)\right) = \left(\lambda_n^{(0)} + \varepsilon\lambda_n^{(1)}\right)\left(f_n^{(0)}(x) + \varepsilon f_n^{(1)}(x)\right)$$

gives an equation for $f_n^{(1)}(x)$. It may be solved integrating with the partition of unity

$$\delta(x-y) = \sum_n f_n^{(0)}(x) f_n^{(0)}(y)$$

to give

$$f_n^{(1)}(x) = \sum_{m\,(\neq n)} \frac{f_m^{(0)}(x)}{\lambda_n^{(0)} - \lambda_m^{(0)}} \int f_m^{(0)}(y) D^{(1)} f_n^{(0)}(y)\, dy$$

which gives the exact solution to the perturbed differential equation to the first order in the perturbation ε.

Several important observations can be made about the form of this solution. First, the sum over functions with differences of eigenvalues in the denominator resembles theresolvent in Fredholm theory. This is no accident; the resolvent acts essentially as a kind of Green's function or propagator, passing the perturbation along. Higher-order perturbations resemble this form, with an additional sum over a resolvent appearing at each order.

The form of this solution is sufficient to illustrate the idea behind the small-divisor problem. If, for whatever reason, two eigenvalues are close so that difference $\lambda_n^{(0)} - \lambda_m^{(0)}$ become small, the corresponding term in the sum will become disproportionately large. In particular, if this happens in higher-order terms, the high-order perturbation may become as large or larger in magnitude than the first-order perturbation. Such a situation calls into question the validity of doing a perturbation to begin with. This can be understood to be a fairly catastrophic situation; it is frequently encountered in chaotic dynamical systems, and requires the development of techniques other than perturbation theory to solve the problem.

Curiously, the situation is not at all bad if two or more eigenvalues are exactly equal. This case is referred to as singular or degenerate perturbation theory. The degeneracy of eigenvalues indicates that the unperturbed system has some sort of symmetry, and that the generators of the symmetry commute with the unperturbed differential operator. Typically, the perturbing term does not possess the symmetry; one says the perturbation *lifts* or *breaks* the degeneracy. In this case, the perturbation can still be performed; however, one must be careful to work in a basis for the unperturbed states so that these map one-to-one to the perturbed states, rather than being a mixture.

V. LOCAL BIFURCATION THEORY

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is *non-hyperbolic* at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighbourhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local').

More technically, consider the continuous dynamical system described by the ODE

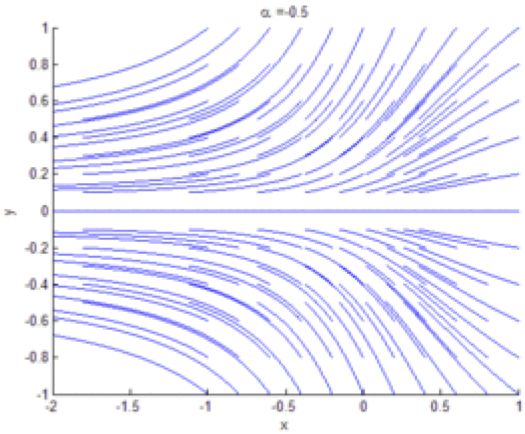
$$\dot{x} = f(x, \lambda) \quad f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$$

A local bifurcation occurs at (x_0, λ_0) if the Jacobian matrix $\mathrm{d}f_{x_0,\lambda_0}$ has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

For discrete dynamical systems, consider the system

$$x_{n+1} = f(x_n, \lambda) \,.$$

Then a local bifurcation occurs at (x_0, λ_0) if the matrix $\mathrm{d}f_{x_0,\lambda_0}$ has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation. If the eigenvalue is equal to −1, it is a period-doubling (or flip) bifurcation, and otherwise, it is a Hopf bifurcation.



VI. RESULT & CONCLUSION

We find that differential geometry has build into a field concerned normally with geometric structures on differentiable manifolds. A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. a crucial assumption is made: that the solutions to the unperturbed system are not degenerate, so that the perturbation series can be inverted.

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