# Inversion and Major index on $\Gamma_1$ non-deranged Permutations

A.I. Garba, Ibrahim M.*

*Department of Mathematics, Usmanu Danfodiyo, University Sokoto, P.M.B. 2346 Sokoto, Nigeria*

**Corresponding Author*

***Abstract*- Some further theoretic properties of the scheme called $\Gamma_1$ non-deranged permutation Group, especially in relation to inversion and major index were identified and studied in this paper. This was done first through some computations on this scheme using prime numbers $p \geq 5$. inversion number and major index are not equidistributed also the difference between sum of the major index and sum of the Inversion number is equal to the sum of descent number in $\Gamma_1$ non-deranged permutations, a recursion formula for generating the inversion number, and major index was developed and used these numbers to identify theoretic consequences.**

**Keywords*:* inversion numbers, major index, descent number, ascent number, $\Gamma_1$ non-deranged permutations.**

## I. INTRODUCTION

A Permutation $f$ of the $\Gamma_1$-non deranged permutations presents as inversion, is a pair $(i, j)$ such that $i < j$ and $f(i) > f(j)$ and major index $maj(f) = \sum_{i \in Des(f)} i$ such that $Des(f) = \{i : f(i) > f(i+1)\}$.Permutation statistics were first introduced by [10] and then extensively studied by [3].in the last decades much progress has made, both in the discovery and the study of new statistics, and in extending these to other type of permutations such as words and restricted permutation. The concept of derangements in permutation groups (that is permutations without a fix element) has proportion in the underlying symmetric group $S_n$. [4] used concept to develop a scheme for prime numbers $P \leq 5$ and $\Omega \subseteq N$ which generate the cycles of permutations (derangements) using $\omega_i = ((1)(1+i)_{mp}(1+2i)_{mp}...(1+(p-1)i)_{mp})$ to determine the arrangements. It is difficult for a set of derangements to be a permutation group because of the absence of the natural identity element (a non derangement),The construction of the generated set of permutations from the work of [4] as a permutation group was done by [11] .They achieved this by embedding an identity element into the generated set of permutation(strictly derangements) with the natural permutation composition as the binary operation (the group was denoted as $G_p$)

With no doubt, patterns in permutations have been well studied for over a century. As seem to be the case, these patterns were studied on permutations arbitrary. The symmetric group $S_n$ is the set of all permutations of a set $\Gamma$ of cardinality n. There are several types of other smaller permutation groups (subgroup of $S_n$) of set $\Gamma$,a notable one among them is the alternating group $A_n$.,Afterwards, [6] studied the representation of $\Gamma_1$-non deranged permutation group $G_P{}^{\Gamma_1}$ via group character, hence established that the character of every $\omega_i \in G_p{}^{\Gamma_1}$ is never zero. Also the non standard Young tableaux of $\Gamma_1$-non deranged permutation group $G_P{}^{\Gamma_1}$ has been studied by [5], they established that the Young tableaux of this permutation group is non standard. [1] studied pattern popularity in $\Gamma_1$-non deranged permutations they establish algebraically that pattern $\tau_1$ is the most popular and pattern $\tau_3, \tau_4$ *and* $\tau_5$ are equipopular in $G_P{}^{\Gamma_1}$ they further provided efficient algorithms and some results on popularity of patterns of length-3 in $G_P{}^{\Gamma_1}$.[2] studied Fuzzy on $\Gamma_1$-non deranged permutation group $G_P{}^{\Gamma_1}$ and discover that it is a one sided fuzzy ideal ( only right fuzzy but not left ) also the $\alpha-$level cut of $f$ coincides with $G_P{}^{\Gamma_1}$ if $\alpha = \frac{1}{p}$.

[7] studied ascent on $\Gamma_1$-non deranged permutation group $G_P{}^{\Gamma_1}$ and discover that the union of ascent of all $\Gamma_1$-non derangement is equal to identity also observed that the difference between $Asc(\omega_i)$ and $Asc(\omega_{p-1})$ is one. [8] provide very useful theoretical properties of $\Gamma_1$-non deranged permutation s in relation to excedance and shown that the excedance set of all $\omega_i$ in $G_P{}^{\Gamma_1}$ such that $\omega_i \neq e$ is

$\frac{1}{2}(p-1)$. More recently [9] established that the intersection of descent set of all $\Gamma_1$-non derangement is empty, also observed that the descent number is strictly less than ascent number by $p-1$. Hence we will in this paper show that inversion number and major index are not equidistributed and also show that the difference between sum of the major index and sum of the inversion number is equal to the sum of descent number, a recursion formula for generating the inversion number, and major index was developed and used these numbers to identify theoretic consequences.

## II. PRELIMINARIES

*Definition 2.1*

Let $\Gamma$ be a non empty set of prime cardinality greater or equal to 5 such that $\Gamma \subset \square$. A

bijection $\omega$ on $\Gamma$ of the form

$$\omega_i = \begin{pmatrix} 1 & 2 & 3 & . & . & . & p \\ 1 & (1+i)_{mp} & (1+2i)_{mp} & & & & (1+(p-1)i)_{mp} \end{pmatrix}$$

is called a $\Gamma_1$-non deranged permutation. We denoted $G_p$ to be the set of all $\Gamma_1$-non deranged permutations.

*Definition 2.2*

The pair $G_p$ and the natural permutation composition forms a group which is denoted as $G_P^{\Gamma_1}$.This is a special permutation group which fixes the first element of $\Gamma$.

*Definition 2.3*

A descent of permutation $f = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ f(1) & f(2) & f(3) & . & . & . & f(n) \end{pmatrix}$ is any positive $i < n$ (where $i$ and $n$ are positive integers and the current value is greater than the next), that is $i$ is an descent of a permutation $f$ if $f(i) > f(i+1)$.The descent set of $f$, denoted as $Des(f)$, is given by $Des(f) = \{i : f(i) > f(i+1)\}$.The descent number of $f$, denoted as $des(f)$, is defined as the number of descent and is given by $des(f) = |Des(f)|$.

*Definition 2.4*

An ascent of permutation $f = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ f(1) & f(2) & f(3) & . & . & . & f(n) \end{pmatrix}$ is any positive $i < n$ (where $i$ and $n$ are positive integers and the current value is less than the next), that is $i$ is an ascent of a permutation $f$ if $f(i) < f(i+1)$.The ascent set of $f$, denoted as $Asc(f)$, is given by $Asc(f) = \{i : f(i) > f(i+1)\}$.The ascent number of $f$, denoted as $asc(f)$, is defined as the number of ascent and is given by $asc(f) = |Asc(f)|$.

*Definition 2.5*

An inversion of permutation $f = \begin{pmatrix} 1 & 2 & 3 & . & . & . & n \\ f(1) & f(2) & f(3) & . & . & . & f(n) \end{pmatrix}$ is a pair $(i, j)$ such that $i < j$ and $f(i) > f(j)$.The inversion set of $f$, denoted as $Inv(f)$, is given by

$$Inv(f) = \{(i, j) : 1 \le i < j \le n \text{ and } f(i) > f(j)\},$$

the inversion number of $f$, denoted by $inv(f) = |Inv(f)|$.

*Definition 2.6*

The major index of $f$ denoted by $maj(f)$ is the sum of the descent set of the permutation $f$ that is $maj(f) = \sum_{i \in Des(f)} i$

## III. MAIN RESULTS

In this section, we present some Inversion number and Major index results of subgroup $G_P^{\Gamma_1}$ of $S_p$ (Symmetry group of prime order with $p \ge 5$).

*Proposition 3.1*

Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, Then

$$\sum_{i=1}^{p-1} asc(\omega_i) - \sum_{i=1}^{p-1} des(\omega_i) = p-1$$

*Proof*

it is immediate that the $\sum_{i=1}^{p-1} asc(\omega_i) = \frac{p(p-1)}{2}$, also $\sum_{i=1}^{p-1} des(\omega_i) = \frac{(p-1)(p-2)}{2}$. Therefore , the difference between $\sum_{i=1}^{p-1} asc(\omega_i)$ and $\sum_{i=1}^{p-1} des(\omega_i)$ can be obtained by

$$\sum_{i=1}^{p-1} asc(\omega_i) - \sum_{i=1}^{p-1} des(\omega_i) = \frac{p(p-1)}{2} - \frac{(p-1)(p-2)}{2}$$

$$= \frac{p(p-1)-(p-1)(p-2)}{2}$$

$$= \frac{(p-1)(p-p+2)}{2}$$

$$= \frac{2(p-1)}{2}$$

$$= p-1$$

*Proposition 3.2*

Let $G_P{}^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, Then the sum of inversion number is

$$\sum_{i=1}^{p-1} asc(\omega_i) = \frac{1}{2} p(p-1)$$

*Proof*

If $S_p$ is Symmetry group of degree $p$ and $G_p{}^{\Gamma_1} \subset S_p$, then the total of position of ascent in $S_p$ is given by

$$\frac{1}{2} p(p-1)p! \qquad (1)$$

If $S_p = G_p{}^{\Gamma_1}$,then equation (1) becomes

$$\frac{1}{2} p(p-1)(p-1) \qquad (2)$$

By $\Gamma_1$-non derangement (fixed at 1) of $G_P{}^{\Gamma_1}$,there will always be an ascent at position 1 with becomes

$$\frac{1}{2} p(p-1)(p-1) + (p-1) \qquad (3)$$

Simplifying equation above, we have the sum of ascents number to be

$$\sum_{i=1}^{p-1} asc(\omega_i) = \frac{1}{2} p(p-1)$$

*Theorem 3.3*

Let $G_P{}^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, Then the sum of descent number is

$$\sum_{i=1}^{p-1} des(\omega_i) = \frac{1}{2}(p-1)(p-2)$$

*Proof*

The order of $G_P{}^{\Gamma_1}$ and the number of positions with ascent or descent are both $p-1$.We know that total number of ascent is greater than the total number of descent by $p-1$ in $G_P{}^{\Gamma_1}$ from the proposition (3.1) and proposition (3.2).Then

$$\sum_{i=1}^{p-1} des(\omega_i) = \frac{1}{2}(p-1) - (p-1)$$

$$= \frac{1}{2}(p-1)(p-2)$$

*Theorem 3.4*

Suppose that $G_P{}^{\Gamma_1}$ is $\Gamma_1$-non derangement permutations, Then the inversion number and major index are not equidistrubuted

$$\sum_{i=1}^{p-1} maj(\omega_i) \neq \sum_{i=1}^{p-1} inv(\omega_i)$$

*Proof*

We proceed by contradiction. Assume that $\sum_{i=1}^{p-1} maj(\omega_i) = \sum_{i=1}^{p-1} inv(\omega_i)$, then

$$0 = \left| \sum_{i=1}^{p-1} maj(\omega_i) - \sum_{i=1}^{p-1} inv(\omega_i) \right|.$$

Hence

$$0 = \left| \frac{p+1}{2} \sum_{i=1}^{p-1} des(\omega_i) - \frac{p-1}{2} \sum_{i=1}^{p-1} des(\omega_i) \right|$$

$$= \left( \frac{p+1}{2} - \frac{p-1}{2} \right) \sum_{i=1}^{p-1} des(\omega_i)$$

$$= \sum_{i=1}^{p-1} des(\omega_i)$$

$$= \frac{(p-1)(p-2)}{2} \geq 6 \text{, since } p \geq 5,$$

This contradicts our assumption. Hence $\sum_{i=1}^{p-1} maj(\omega_i) \neq \sum_{i=1}^{p-1} inv(\omega_i)$

*Proposition 3.5*

Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, Then the sum of inversion number is

$$\sum_{i=1}^{p-1} inv(\omega_i) = \frac{p-1}{2} \sum_{i=1}^{p-1} des(\omega_i)$$

*Proof*

We can observe that the $\sum_{i=1}^{p-1} inv(\omega_i)$ is $\frac{p-1}{2}$ multiple of $\sum_{i=1}^{p-1} des(\omega_i)$. Thus

$$\sum_{i=1}^{p-1} inv(\omega_i) = \frac{p-1}{2} \sum_{i=1}^{p-1} des(\omega_i)$$

*Proposition 3.6*

Let $G_P^{\Gamma_1}$ be a $\Gamma_1$-non derangement permutations, Then the sum of major index is

$$\sum_{i=1}^{p-1} maj(\omega_i) = \frac{p+1}{2} \sum_{i=1}^{p-1} des(\omega_i)$$

*Proof*

We can observe that the $\sum_{i=1}^{p-1} maj(\omega_i)$ is $\frac{p+1}{2}$ multiple of $\sum_{i=1}^{p-1} des(\omega_i)$. Thus

$$\sum_{i=1}^{p-1} maj(\omega_i) = \frac{p+1}{2} \sum_{i=1}^{p-1} des(\omega_i)$$

*Remark 3.7*

The sum of major index number is strictly greater than the sum of inversion number in $G_P^{\Gamma_1}$

*Proposition 3.7*

Suppose that $G_P^{\Gamma_1}$ is $\Gamma_1$-non derangement permutations, Then

$$\sum_{i=1}^{p-1} maj(\omega_i) - \sum_{i=1}^{p-1} inv(\omega_i) = \sum_{i=1}^{p-1} des(\omega_i)$$

*Proof*

it is immediate that the $\sum_{i=1}^{p-1} maj(\omega_i) = \frac{p+1}{2} \sum_{i=1}^{p-1} des(\omega_i)$ and also $\sum_{i=1}^{p-1} inv(\omega_i) = \frac{p-1}{2} \sum_{i=1}^{p-1} des(\omega_i)$. Therefore, the difference between $\sum_{i=1}^{p-1} maj(\omega_i)$ and $\sum_{i=1}^{p-1} inv(\omega_i)$ can be obtained by

$$\sum_{i=1}^{p-1} maj(\omega_i) - \sum_{i=1}^{p-1} inv(\omega_i) = \frac{p+1}{2} \sum_{i=1}^{p-1} des(\omega_i) - \frac{p-1}{2} \sum_{i=1}^{p-1} des(\omega_i)$$

$$= \sum_{i=1}^{p-1} des(\omega_i) \left( \frac{p+1}{2} - \frac{p-1}{2} \right)$$

$$= \sum_{i=1}^{p-1} des(\omega_i) \left( \frac{p+1-p+1}{2} \right)$$

$$= \sum_{i=1}^{p-1} des(\omega_i) \left( \frac{2}{2} \right)$$

$$= \sum_{i=1}^{p-1} des(\omega_i)$$

*Proposition 3.8*

Let $\omega_i$ and $\omega_{i+1} \in G_P^{\Gamma_1}$, then the

$$\left| maj(\omega_i) - maj(\omega_{i+1}) \right| = \frac{p+1}{2}$$

*Proof*

We know that for $\omega_i \in G_P^{\Gamma_1}$

$maj(\omega_i) = \frac{(p+1)(i-1)}{2}$, so $maj(\omega_{i+1}) = \frac{(p+1)i}{2}$.

Therefore

$$\left| maj(\omega_i) - maj(\omega_{i+1}) \right| = \left| \frac{(p+1)(i-1)}{2} - \frac{(p+1)i}{2} \right|$$

$$= \left| \frac{(p+1)(i-1) - (p+1)i}{2} \right|$$

$$= \left| \frac{-(p+1)}{2} \right|$$

$$= \frac{p+1}{2}$$

## IV. CONCLUSIONS

This paper has provided very useful theoretical properties of this scheme called $\Gamma_1$-non deranged permutations in relation to the inversion and major index. We have shown that inversion number and major index are not equidistributed, we also observed that the difference between sum of the major index and sum of the inversion number is equal to the sum of descent number, a recursion formula for generating the inversion number, and major index was also developed

## ACKNOWLEDGMENT

The Authors are very grateful to the referees for their valuable suggestion and opinions.

## REFERENCES

[1]. Aremu K.O., Ibrahim A.H., Buoro S. and Akinola F.A. (2017a). Pattern Popularity in $\Gamma_1$-non deranged permutations: An Algebraic and Algorithmic Approach.*Annals. Computer Science Series***15**(2) 115-122.

[2]. Aremu K.O.,Ejima O. and Abdullahi M.S.(2017b). On the Fuzzy $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$.*Asian Journal of Mathematics and Computer Research* 18(4) 152-157.

[3]. Euler L.,(1913) Institutiones Calculi differntialis in "*opera omnia series prime* " Volx, Teubner,Leipzig.

[4]. Garba A.I. and Ibrahim A.A.(2010), A New Method of Constructing a Variety of Finite Group Based on Some Succession Scheme. *International Journal of Physical Sciences* **2**(3) 23-26.

[5]. Garba A.I.,Ejima O.,Aremu K.O. and Hamisu U.(2017). Non standard Young tableaux of $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$.*Global Journal of Mathematical Analysis***5**(1) 21-23.

[6]. Ibrahim A.A,Ejima O. and Aremu K.O.(2016).On the Representation of $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ *Advance in Pure Mathematics*, **6:**608-614.

[7]. Ibrahim M., Ibrahim A.A, Garba A.I and Aremu K.O.(2017).Ascent on $\Gamma_1$-non deranged permutation group $G_P^{\Gamma_1}$ *International journal of science for global sustainability*, 4(2) 27-32.

[8]. Ibrahim M. and Garba A.I (2018). Exedance on $\Gamma_1$-non deranged permutations *proceedings of Annual National Conference of Mathematical Association of Nigeria (MAN)* , 197-201.

[9]. Ibrahim M. and Garba A.I (2019). Descent on $\Gamma_1$-non deranged permutation group *Journal of Mathematical Association of Nigeria ABACUS* , 46(1)12-18.

[10]. MacMahon P.A.(1915).*Combinatory Analysis* Vol. 1 and 2 Cambridge University Press (reprinted by Chesea,New York,1955)

[11]. Usman, A. and Ibrahim, A.A. (2011). A New Generating Function for Aunu Patterns: Application in Integer Group Modulo n. *Nigerian Journal of Basic and Apllied Sciences* , 9(1)1-4.