Subdegrees and Suborbital Graphs of the Symmetry Group of a Tetrahedron Acting on Its Edges

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***Abstract*: In this research paper, we compute the rank, subdegree and suborbital graphs of the action of the symmetry group of a tetrahedron acting on its edges. The rank of the symmetry groupacting on the edges of a tetrahedron is 4. The main focus will be on the subdegrees of the suborbitals. Also, the suborbital graphs corresponding to the suborbitals of the action of the symmetry groups are constructed. Moreover, the theoretical properties of the graphs are discussed. When the symmetry groupacts on the edges of a tetrahedron, the suborbital graphs Γ_1 and $\Gamma_2$corresponding to the non-trivial suborbits $\Delta_1$and Δ_2, are directed and connected. The graph Γ_3, corresponding to the non-trivial suborbits Δ_3, is undirected and disconnected.**

***Keywords:* Rank, Subdegree, Suborbital graphs**

I. INTRODUCTION

Let H be a transitive permutation group acting on a set Y. Then H acts on Y ×Y by h(x,y) = (hx,hy), h ∈ H, x,y∈ Y.If O ⊆ Y ×Y is a H-orbit, then for a fixed x ∈ Y,Δ = {y ∈ Y: (x, y) ∈ O}is a H_x-orbit. On the other hand, if Δ ⊆ Y is a H_x -orbit, then O = {(hx,hy)|h ∈ H, y ∈ Δ} is a H-orbit on Y × Y. We say Δ corresponds to O. The H_x -orbits on Y are called suborbits and H-orbits on Y × Y are called suborbitals.

The orbit containing (x,y) is denoted by O(x,y). From O(x,y), we can form a suborbital graph Γ(x,y); its vertices are the elements of Y, and there is a directed edge from γ to δ if (γ,δ) ∈ O(x,y). Clearly, O(y,x) is also a suborbital, and it is either equal or disjoint from O(x,y). In the former case. Γ(y,x) = Γ(x,y) and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have undirected graph which we call self-paired. In the latter case, Γ(y,x) is just Γ(x,y) with arrows reversed, and we call Γ(x,y) and Γ(y,x) paired suborbital graphs. Also, if $x = y$, then O(x, x) = {(x,x)|x ∈Y} is the diagonal of Y × Y. The corresponding suborbital graph H(x, x), called the trivial suborbital graph, is self-paired and consists of a loop based at each vertex x ∈Y. These ideas were first introduced by Sims [5] and further discussed in papers by Neumann [4], books written by Tsuzuku [6], and Biggs and White [2], who focused on applications in finite groups.

II. NOTATIONS AND PRELIMINARY RESULTS

Notation 2.1

Throughout this paper, ⁊ denotes the symmetry group of a tetrahedron, Γ is the suborbital graph corresponding to the suborbit Δ and O is the suborbital of G on X×X.

Definition 2.2

A permutation group G acting on a set X is said to be transitive if for all x, y ∈ X there exists an element g ∈ G such that gx = y. Alternatively, if the action of the group G on the set X has one orbit, then it is said that G acts transitively on X.

Definition 2.3

Let H be transitive on a set Y and let Hy be the stabilizer in H of a point y ∈ Y. The orbits $\Delta_0 = \{y\}, \Delta_1, \cdots, \Delta_{r-1}$ of Hy on Y are called the suborbits of H. The rank of H in this case is r.

Definition 2.4

The sizes $n_i = |\Delta_i|$, (i = 0, 1, 2, …., r -1) often called the 'lengths' of the suborbits are known as subdegrees of H.

Theorem 2.5 (Benson and Grove [1])

Let G be a finite group acting on a finite set X, then the number of G- orbits is

$\frac{1}{|G|}\sum_{h\varepsilon G}|fix(h)|$, where Fix(h) = {y϶X: $hy = y$}.

Theorem 2.6 (Sims [5])

Let H be transitive on Y. Then H is primitive if and only if each suborbital graph Γ_x (x = 1, 2, …, r-1) is connected.

Theorem 2.7 (Cameron [3])

Let G be a transitive group on X. The number of self-paired orbits is $\frac{1}{|G|}\sum_{g\in G}|Fix(g2)|$.

III. MAIN RESULTS

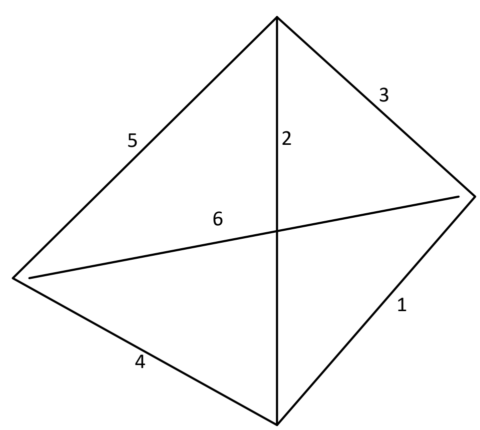


Figure 1: A tetrahedron with labelled edges

3.1 Group of Symmetries of a Tetrahedron

Suppose that a tetrahedron is situated such that its center is at the origin in $\mathbb{R}^3$. The subgroup of the rotations in $\mathbb{R}^3$ which leave the tetrahedron invariant is denoted by ℸ. The elements of ℸ include:

I. Two rotations through angles of 120° and 240° about each of the four axes joining the vertices to the center of opposite faces.
II. A rotation through the 180° angle about each of the three axes joining the midpoints of opposite edges.
III. The identity.

Thus, $|ℸ| = 4.2 + 3.1 + 1 = 12$. The group of symmetries of a tetrahedron is isomorphic to A_4 (See [1] for more details).

3.2 Cycle Type, Stabilizer and Transitivity of A_4 acting on the edges of a Tetrahedron

Table 1: Cycle Type ofℸActing On the Edges of a Tetrahedron

Rotations	Number of Permutations	Cycle Type
Rotations about each of the four axes joining the vertices with centers of opposite faces through120° and 240°	8	(0,0,2,0,0,0)
Rotation through the 180° angle about each of the three axes joining the midpoints of opposite edges	3	(2,2,0,0,0,0)
The Identity	1	(6,0,0,0,0,0)

The identity rotation stabilizes all the edges. Moreover, the rotation through 180° angle about each of the three axes joining the midpoints of opposite edges stabilizes opposite edges of a tetrahedron. Consequently, the stabilizer of the action of A_4 acting on the edges of a tetrahedron is {I, 180°}, which is a cyclic group of order 2.

From Table 1, we can compute the transitivity by first obtaining |Fix (g)| for each g $\in A_4$ then using the Cauchy-Frobenius lemma, we find the number of orbits when A_4 acts on the edges of the tetrahedron. The equation is given by

$$\frac{1}{12}(8.0 + 3.2 + 1.6) = 1$$

Therefore, from Definition 2.2, A_4 acts transitively on the edges of a tetrahedron.

3.3 Rank and Subdegrees of ℸacting on the edges of a Tetrahedron

Using Definition 2.3and the Cauchy-Frobenius lemma we can compute the rank of the action ofthe symmetry group on the edges of a tetrahedron whereby; $\frac{1}{|G|}\sum_{g\varepsilon G}|fix(g)|$, where Fix(g) ={x$\epsilon$X: $gx = x$} is $\frac{1}{|\{I,180°\}|}\sum_{g\varepsilon\{I,180°\}}|fix(g)|$, where Fix(g) = {x$\epsilon$X: $gx = x$} and X is the symmetry group of a tetrahedron acting on its edges, ℸ. Hence, we have the rank as; $\frac{1}{2}(6+2) = \frac{8}{2} = 4$

Where 6 is the number of edges fixed by the identity and 2 is the number of edges fixed by the rotation of 180° angle about the axes joining the midpoints of opposite edges.

Using the rank obtained, the stabilizer and Definition 2.4,we can obtain the corresponding subdegrees of the action of ℸon the edges of a tetrahedron. From Figure 1with the axis of rotation joining the midpoint of edges labelled 1 and 5, under the 180° angle rotation we obtain the following permutation (1) (5) (3 4) (2 4) whereby, the subdegrees are 1, 1, 2, 2 to the corresponding suborbits Δ_0 = {1}, Δ_1= {5}, Δ_2 = {3, 4}, Δ_3 = {2, 4} respectively.

We can change the axis to start from other opposite edges, for example 3 and 4, but the subdegrees will remain the same since we already showed the action is transitive with one orbit.

3.4 Suborbital graphs ofℸacting on the edges of a Tetrahedron

After computing the subdegrees, we proceed to analyze the suborbits whereby, if G acts on a set X then G acts on X×X = {(a,b)| a, b ϵ X} by g(a, b) = (ga, gb) and orb_G (a, b) is the orbital or suborbit of G containing (a, b), denoted by O_i. In our case, G denotes the symmetry group of a tetrahedron acting on its edges and X is the set of edges, as labeled in Figure 1,{1, 2, 3, 4, 5, 6}. For example, for 180° ϵ G such that 180° (1, 2) = {(1, 6), (5, 2), (5, 6)}. The rank is equal to the total number of suborbital graphs, so we need to find 4 orbitals. In this case we have that,

X×X = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

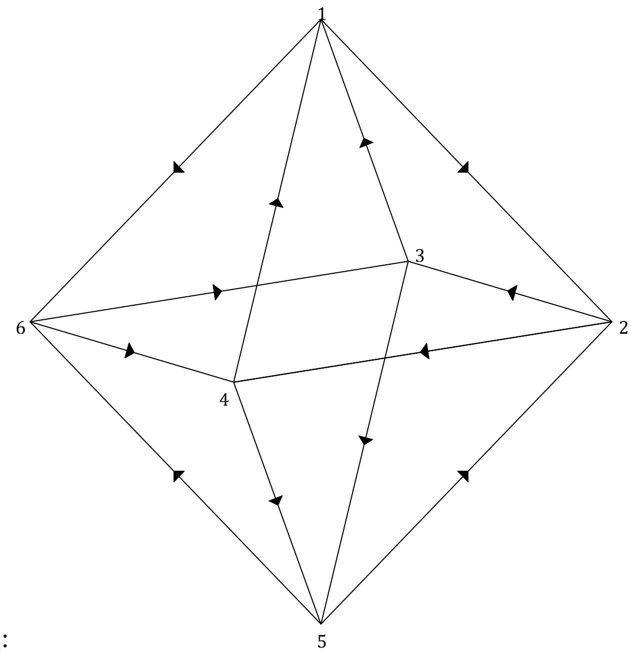
Orb_G (1, 1) = O_0 = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

Orb_G (1, 2) = O_1 = {(1, 2), (6, 3), (4, 5), (2, 3), (3, 1), (2, 4), (4, 1), (6, 4), (3, 5), (1, 6), (5, 2), (5, 6)}

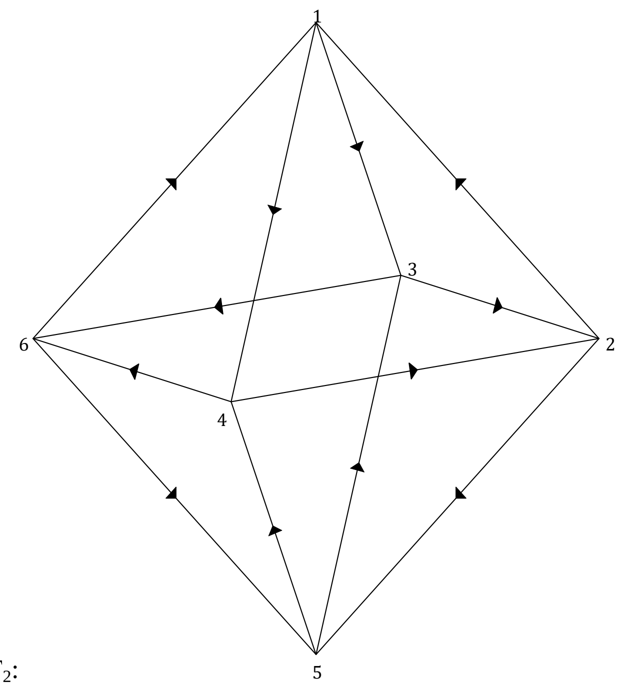
Orb_G (2, 1)= O_2 = {(2, 1), (3, 6), (5, 4), (3, 2), (1, 3), (4, 2), (1, 4), (4, 6), (5, 3), (6, 1), (2, 5), (6, 5)}

Orb_G (1, 5)= O_3 = {(1, 5), (6, 2), (4, 3), (2, 6), (3, 4), (2, 6), (4, 3), (6, 2), (3, 4), (1, 5), (5, 1), (5, 1)}

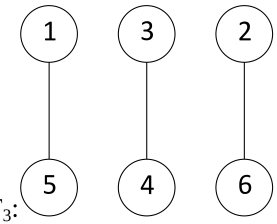
Next we construct the corresponding suborbital graphs. The suborbital graph corresponding to the suborbit Δ_0 is the trivial graph, Γ_0, and so we ignore it.



Γ_1:



Γ_2:



Γ_3:

3.5 Properties of the Suborbital Graphs

The graphs Γ_1 and Γ_2 are directed, connected and their girth is 3. The graph $\Gamma_3$is undirected, disconnected and has 3 connected components.

From Theorem 2.6,we find that the action of ℸ is imprimitive, as Γ_3 is disconnected.

We can also confirm the number of self-paired graphs, using Theorem 2.7, such that for the equation $\frac{1}{|G|}\sum_{g\in G}|Fix(\mathrm{g}2)|$, where G is the symmetry group acting on the edges of a tetrahedron and using Table 1, we have;

$$\frac{1}{12}(6.3+6.1)=\frac{24}{12}=2$$

Where the 180° angle rotation about the center of opposite edges becomes the identity rotationwhen squared hence, fixes 6 edges with 3 permutations. Also, the identity rotation fixes 6 edges with 1 permutation. We have the following self-paired graphs; Γ_0 and$\Gamma_{3.}$

IV. CONCLUSION

The tetrahedron is a unique solid full of interesting mathematical properties for research and so is the reason why it is the focus of this research.

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