Some Commutativity Results on Prime-Near Rings

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Abstract: **- In this paper, we prove that: Let G_1be a prime near-ring and G is a $(\theta, \emptyset)$-generalized derivation associated with $(\theta, \emptyset)$-derivation, δ, if$[a, b]G(c) = 0$ and $\delta \neq 0$ for all $a, b, c \in G_1$, then a $a \in Z(G_1)$. Furthermore, we obtain some new results on prime near-rings G_1 with some examples in order to justify the conditions.**

Keywords**: - Prime near-rings, generalized derivations, two-sided $(\theta, \emptyset)$- derivation, two-sided $(\theta, \emptyset)$-generalized derivation.**

I. INTRODUCTION

The conceptualization of rings started in the 1870s and completed in the 1920s. Key contributors include Dedekind, Hilbert, Frankel, and Noether. Rings were first formalized as a generalization of Dedekind domains that occur in number theory, polynomial rings and rings of invariants that occur in algebraic geometry [6]. Afterwards, they also proved to be useful in other branches of mathematics such as geometry and mathematical analysis.

Near-rings are one of the generalized structures of rings. The study and research on near-rings is very systematic and continuous. More precisely what makes it difference from a ring is the commutativity of addition is not necessarily required and just one of the distributive laws is postulated. Many authors established several commutativity theorems for prime and semi-prime rings admitting a suitably constrained generalized derivation. [1], they investigate commutativity of prime rings R with involution _ of the second kind in which generalized derivations satisfy certain algebraic identities and prove some well-known results on commutativity of prime rings. In addition, they provide an example to show that the restriction imposed on the involution is not superuous.

Several authors [1], [7], [2], [4], for their recent papers on derivations and commutativity of prime ringswith involution they drive many interesting results concerning commutativity theorems on prime and semiprime rings. In the paper of [5], initiated the study of 3-prime near-rings with generalized derivations. An additive mapping $G: G_1 \to G_1$ is said to be a generalized derivation on G_1 if there exists a derivation $\delta: G_{1 \to} G_1$ such that$G(xy) = G(x)y + x\delta(y)$ holds for all $x, y \in G_1$.

In the paper of [3], it was proved that let K be a semiprime ring and $T: K \to K$ an additive mappingsuch that $T(x^2) = T(x)x$ holds for all $x \in K$. Then T is a left centralizer of K. He also proved that Jordan centralizers and centralizers of K coincide.

Note that, every derivation on a near-ring G1 is a generalized derivation, but the converse statement does not hold in general. As usual, for all $x, y \in G_1, [x, y] = xy - yx$ and $x \circ y = xy + yx$ for the Lie products and Jordan products, respectively.

II. BASIC DEFINITIONS

2.1 Prime Ring

A ring M is called a prime ring if for any $a, b \in M$, $aMb = \{0\}$, implies that either $a = 0$ or $b = 0$.

2.2 Semiprime Ring

A ring M is called a semiprime ring if for any $a \in M$, $aMa = \{0\}$, implies that $a = 0$.

2.3 Remark

Every prime ring is a semiprime ring, but the converse in general is not true. In order to justifies the statement of this remark, consider the following Example

2.4 Example

TakeM $= Z_6$, then M is semiprime ring but is not prime. Because,$0 \neq 2 \in M$ and $0 \neq 3 \in M$ implies $2M3 = 0$.

2.5 Characteristic of the Ring

Let M be an arbitrary ring. If there exists a positive integer n such that $na = 0$, for all $a \in M$, then thesmallest positive integer with this property is called characteristic of the ring , denoted by $char(M)$. If no such positive integer exists (that is, $n = 0$is only integer for which $na = 0$, for all $a \in M$), then M is said to be of characteristic zero.

2.6 Near-Ring with Involution

Let N be a near-ring. Then an additive mapping $\sigma: N \to N$ is called an involution if $\sigma(xy) = \sigma(y)\sigma(x)$and $\sigma^2(x)$ for all $x, y \in N$. An additive mapping $d: N \to N$ is said to be a derivation if$d(xy) = d(x)y + xd(y)$ for all $, y \in N$. An additive mapping $F: N \to N$ is called a generalizedderivation if there exists a derivation $d: N \to N$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in N$.

2.7 Definition

An additive mapping $d: N \to N$ is called an $(\theta, \emptyset)$ $-$derivation if there exist functions $\propto, \beta: N \to N$such that $d(xy) = d(x) \propto (y) + \beta(x)d(y)$ for all $x, y \in N$. An additive mapping $d: N \to N$ is called a two-sided $\propto$-

derivation if d is an $(\propto,1)$-derivation as well as $(1,\propto)$-derivation.

2.8 Definition

An additive mapping $F: N \to N$ is called an $(\theta,\emptyset)$-generalized derivation if there exist functions

$\propto, \beta: N \to N$ and $(\theta,\emptyset)$-derivation d such that $F(xy) = F(x) \propto (y) + \beta(x)d(y)$ for all $x, y \in N$. An additive mapping $F: N \to N$ is called a two-sided $\propto$-generalized derivation if F is an $(\propto,1)$-generalized derivation as well as $(1,\propto)$-generalized derivation.

2.9 Definition

Let G_1 be a near-ring and δ is a two sided ζ-derivation of G_1. An additive mapping $G: G_1 \to G_1$ is called generalized two sided ζ-derivation associated with δ if it satisfies $G(xy) = G(x)\zeta(y) + x\delta(y) = G(x)y + \zeta(x)\delta(y)$.

Simply, we prefer to call G a generalized two sided ζ-derivation without mentioning δ. For $\zeta = I_{G_1}$, a two-sided ζ-derivation is of course the usual derivation, thereby G must be a generalized derivation associated with δ.

III. RESULTS

The results that we shall prove in this section are very much surprising, interesting, and certainly more exciting. With this preamble over we are ready to continue.

3.1 Lemma (TAY 1)

Let G_1 be a prime near-ring and δ is a $(\theta,\emptyset)$-derivation where $\propto, \beta$ are homomorphism in G_1, if $k\delta a = 0$ and $\delta \neq 0$ for all $a \in G_1$ and $k \in G_1$, then $k = 0$.

Proof

Let $k\delta a = 0$ for all $a \in G_1$ and $K \in G_1$, then

$k\delta(ab) = 0$ for all $a, b \in G_1$ (3.1)

$k\big(\delta(a)\theta(b) + \emptyset(a)\delta(b)\big) = 0$ (3.2)

$k\delta(a)\theta(b) + k\emptyset(a)\delta(b) = 0$ for all $a, b \in G_1$ (3.3)

which implies that

$k\emptyset(a)\delta(b) = 0$ all $a, b \in G_1$.

Since $\emptyset$ is a homomorphism in G_1, it follows that

$kG_1\delta(b) = \{0\}$ for all $a, b \in G_1$.

But since G_1 is a prime near-ring and $\delta \neq 0$, therefore, k must be equal zero. Hence the proved.

3.2 Theorem (TAY 2)

Let G_1 be a prime near-ring and G is a $(\theta,\emptyset)$-generalized derivation associated with $(\theta,\emptyset)$-derivation, δ, if $[a,b]G(c) = 0$ and $\delta \neq 0$ for all $a, b \in G_1$, then $a \in Z(G_1)$. That is G_1 is commutative ring.

Proof

Suppose that

$[a,b]G(c) = 0$ (3.5)

for all $a, b, c \in G_1$, then by replacing c with cd in Equation (3.5) where d is in G_1, and using the definition of derivation, we have

$$[a,b]G(cd) = 0$$

$$[a,b]\big(G(c)\theta(d) + \emptyset(c)\delta(d)\big) = 0$$

that is

$$[a,b]G(c)\theta(d) + [a,b]\emptyset(c)\delta(d) = 0.$$

This implies

$[a,b]\emptyset(c)\delta(d) = 0$ for all $a, b, c, d \in G_1$

Since $\emptyset$ is a homomorphism, then

$[a,b]G_1\delta(d)$ for all $a, b, d \in G_1$.

By primeness of G_1, it implies that either $[a,b] = 0$ or $\delta = 0$. But since δ is a non zero $(\theta,\emptyset)$-derivation, it follows that $[a,b] = 0$. That is $a \in Z(G_1)$, for all $b \in G_1$.

3.3 Theorem (TAY 3)

Let G_1 be a prime near-ring and G is a generalized $(\theta,\emptyset)$-derivation of G_1 associated with (ζ,ζ)-derivation δ and $\delta\zeta = \zeta\delta$. If $G([a,b]) = 0$ for all $a, b \in G_1$ and $\delta \neq 0$, then G_1 is a commutative ring.

Proof

To prove that G_1 is a commutative ring, we use the hypothesis and $G([a,b]) = 0$ for all $a, b \in G_1$.

By replacing b by ba in the above condition, we get

$G([a,ba]) = 0$ for all $a, b \in G_1$. (3.6)

But since $[a,ba] = [a,b]a$, then Equation (3.6) becomes

$G([a,ba]) = G([a,b])\zeta(a) + \zeta([a,b])\delta(a) = 0$ for all $a, b \in G_1$.

This implies that

$\zeta([a,b])\delta(a) = 0$ for all $a, b \in G_1$. (3.7)

Since $[a,b] = ab - ba$, the Equation (3.7), becomes

$\zeta(ab - ba)\delta(a) = 0$ for all $a, b \in G_1$.

On simplify, we get

$\zeta(ab)\delta(a) - \zeta(ba)\delta(a) = 0$ for all $a, b \in G_1$,

that is

$\zeta(a)\zeta(b)\delta(a) - \zeta(b)\zeta(a)\delta(a)$ for all $a, b \in G_1$. (3.8)

Substitute b by cb in Equation (3.8), we get

$\zeta(a)\zeta(cb)\delta(a) - \zeta(cb)\zeta(a)\delta(a)$ for all $a, b, c \in G_1$

$\zeta(a)\zeta(c)\zeta(b)\delta(a) - \zeta(c)\zeta(b)\zeta(a)\delta(a)$ for all $a, b, c \in G_1$.

Since G_1 is a near-ring and ζ is an automorphism of G_1 and associativity hold in G_1, then we write

$\zeta(a)\zeta(c)\zeta(b)\delta(a) - \zeta(c)\zeta(a)\zeta(b)\delta(a)$ for all $a, b, c \in G_1$.

$(\zeta(a)\zeta(c) - \zeta(c)\zeta(a))\zeta(b)\delta(a) = 0$ for all $a, b, c \in G_1$

$(\zeta(ac) - \zeta(ca))\zeta(b)\delta(a) = 0$ for all $a, b, c \in G_1$.

By definition of lie product, we write

$\zeta([a, c])\zeta(b)\delta(a) = 0$ for all $a, b, c \in G_1$.

But please observe, since ζ is an automorphism of G_1, then we get

$\zeta([a, c])G_1(b)\delta(a) = \{0\}$ for all $a, b, c \in G_1$.

Using the primeness of G_1, we obtain

$\zeta([a, c]) = 0$ or $\delta(a) = 0$,

this contradict the assumption that $\delta \neq 0$.

This implies that

$\zeta([a, c]) = 0$ for all $a, b \in G_1$.

Since ζ is an automorphism of G_1, then we get that $a \in Z(G_1)$, for all $a \in G_1$. In other way, $\delta(a) \in Z(G_1)$. This show that $(G_1, +)$ is an abelian. Indeed, for all $b \in G_1$, we have

$ab = ba$. (3.9)

Now since $ab = ba$, it follows that

$\delta(ab) = \delta(ba)$ for all $a, b \in G_1$.

Using the definition of (ζ, ζ)-derivation, we obtain

$\delta(a)\zeta(b) + \zeta(a)\delta(b) = \delta(b)\zeta(a) + \zeta(b)\delta(a)$ for all $a, b \in G_1$. (3.10)

Since $a \in Z(G_1)$, implies that

$$\delta(a)\zeta(b) = \zeta(b)\delta(a) b \in G_1.$$

Therefore, $\delta(a)b = b\delta(a)$ for all $b \in G_1$.

This show that

$$\delta(a) \in Z(G_1) a \in G_1$$

which conclude that G_1 is a commutative ring.

3.4 Example

Let M be a commutative ring and

$G_1 = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in M\right\}$.

If we define $\delta: G_1 \to G_1$ by

$$\delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

Then δ is a non zero derivation on G_1. Furthermore, if $A_1 = \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0 & b_2 \\ 0 & 0 \end{pmatrix}$ and set

$$G\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b + c \\ 0 & 0 \end{pmatrix},$$

then G is a generalized derivation on G_1 which satisfies the condition

$$G([A_1, A_2]) = [A_1, A_2]$$

for all $A_1, A_2 \in G_1$, where G_1 is non-commutative ring.

3.5 Example

Let R be a commutative ring and

$$G_1 = \left\{\begin{pmatrix} 0 & t^2 \\ 0 & f^2 \end{pmatrix} \middle| t, f \in R\right\}.$$

An endomorphism $\delta: G_1 \to G_1$ define by

$\delta\begin{pmatrix} 0 & t^2 \\ 0 & f^2 \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}$, is a non zero derivation on G_1. If $B = \begin{pmatrix} 0 & z^2 \\ 0 & 0 \end{pmatrix}$, where $z \neq 0$, then

$$BG_1B = 0$$

which show that G_1 is not prime. Also, δ satisfies the condition

$$\delta([B, C]) = [B, C]$$

for all $B, C \in G_1$ where G_1 is a non commutative ring.

IV. CONCLUSION

This paper discuss the prime near-rings with generalized derivations, and we prove that a prime near-ring that admits a nonzero derivation satisfying certain algebraic (or differential) identities is a commutative ring. Finally, some research questions are presented.

V. RESEARCH QUESTIONS

This paper disclosed with some open problems:

- Let δ be a derivation of prime near-ring G_1 and N be a set of natural numbers. Then, can we say G_1 is a commutative ring if there exist $\alpha, \beta \in N$ such that

$$\delta\left(((t \circ h) \circ f)\right) = f^{\alpha}((t \circ h) \circ f)f^{\beta}$$

or

$$\delta\left(((t \circ h) \circ f)\right) = -f^{\alpha}((t \circ h) \circ f)f^{\beta}$$

for all $t, f, h \in G_1$.

- Let G_1 be a prime near-ring with involution σ and α, β be any positive integers. Does the condition

$$\delta([a, \sigma(a)]^{\alpha}) - ([a, \sigma(a)]^{\beta}) \in Z(G_1)$$

for all $a \in G_1$ and derivation δ, implies that G_1 is commutative ?

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