Motzkin Paths and Motzkin Polynomials of Γ_1- Non Deranged Permutations

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***Abstract*— Aminu et al (2016) defined a Γ_1-non deranged permutation group which is a subgroup of symmetric group of prime length. In this paper, we partition Γ_1-non deranged permutations ($\omega \in G_p^{\Gamma_1}$) into its ascent and descent blocks and then define a mapping $\Psi_{AI}: G_p^{\Gamma_1} \to \Omega_p$ which takes permutation form Γ_1-non deranged permutation group $G_p^{\Gamma_1}$ to weighted Motzkin path in Ω_p with respect to both ascent and decent blocks. We then investigate the motzkin polynomial of $(G_p^{\Gamma_1})$. An enumeration was formulated for generating Motzkin polynomial of $(G_p^{\Gamma_1})$.**

***Keywords:* Γ_1 non-deranged permutations Motzkin path, Motzkin polynomial**

I. INTRODUCTION

Lattice path has a long history in studying the combinatorial properties of Motzkin numbers (see. [5],[15],[16],[17],[21],[20],[18]). A lattice path of size n starting from $(0,0)$ and ending $(0,n)$ and whose permitted steps are the up step $(1,1)$, the down step $(-1,1)$, and the horizontal step $(1,0)$, and never passed below the $x-$axis is called a Motzkin path of size n. The number of possible Motzkin paths for any size $n \in \mathbb{N}$ can be count using the Motzkin path

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,$$

where

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The Motzkin number had been used in several researches on Motzkin path (see. [1],[6],[21],[19],[18]). Several researchers defined bijection which takes a permutation to a weighted Motzkin path (see. [15],[7],[8]).This research will study the Motzkin path and Motzkin polynomial of the elements of Γ_1-non deranged permutation group (see. [2],[10],[9],[11,[12],[13],[14]]). The research will be conducted in two relevant combinatorial process. Firstly, the permutations will be represent into its ascent blocks. Then the permutations will be represent in to its descent blocks. Finally, Motzkin polynomial will be construct for both processes.

II. PRELIMINARIES

Definition 2.1 [18]

The Γ_1-non deranged permutation group $(G_p^{\Gamma_1})$ is a permutation group developed from a special class of permutations generated by a modulo p function that exhibits the following properties:

i. Each element of the group has the following form
$$\begin{pmatrix} 1 & 2 & 3 & \cdots & p \\ 1 & (1+i)mp & (1+2i)mp & \cdots & (1+(p-1)i)mp \end{pmatrix}$$
for $1 \le i \le p, p \ge 5, p\ a\ prime$.

ii. The length of each $\omega_i \in G_p^{\Gamma_1}$ is p, and the order of $G_p^{\Gamma_1}$ is $p-1$, for $p \ge 5$, p is prime.

iii. The Γ_1-non deranged permutation group is abelian.

Definition 2.2

An ascent (resp. descent) block of a permutation $\pi = a_1 a_2 \cdots a_n$ is the subword obtained by putting dashes between a_i and a_{i+1} whenever $a_i > a_{i+1}$ (resp. $a_i < a_{i+1}$).

Definition 2.3

$Run_a(\pi)$ (resp. $Run_d(\pi)$) is the number of ascent blocks (resp. descent blocks) in ω.

Definition 2.4

A proper ascent (resp. descent) block is an ascent (resp. descent) block that has more than one letter. An ascent (resp. descent) block that is not proper is called improper.

Definition 2.5

The outsider of π denoted by $Out(\pi)$ is the set of all letters in improper blocks. It is cardinality is denoted by $out(\pi)$

Definition 2.5

The opener of a permutation π denoted by $O(\pi)$ is the set of all minimal letters in proper blocks. Its cardinality is denoted by $o(\pi)$.

Definition 2.6

The closer of a permutation π donated by $C(\omega_i)$ is the set of all maximal letters in proper blocks. Its cardinality is denoted by $c(\pi)$.

Definition 2.7

The insider of a permutation π denoted by $Ins(\omega_i)$ is the set of all non-extremal elements in proper blocks. Its cardinality is denoted by $ins(\pi)$.

Definition 2.8

A Motzkin path is a word $c = c_1c_2\cdots c_n$ whose members are the alphabet $\{u,d,h\}$ such that for each i the level h_i of the $i-th$ step is defined by

$$h_i = |\{j \mid j < i, c_j = u\}| - |\{j \mid j < i, c_j = d\}|,$$

where h_i is non negative, and equal to zero if $i = n$; and u is a northeast step, d is a southeast step, and h is the east steps respectively.

Definition 2.9

An elevated Motzkin path is a Motzkin path which never touches the $x-$ axis except initially and finally.

Definition 2.10

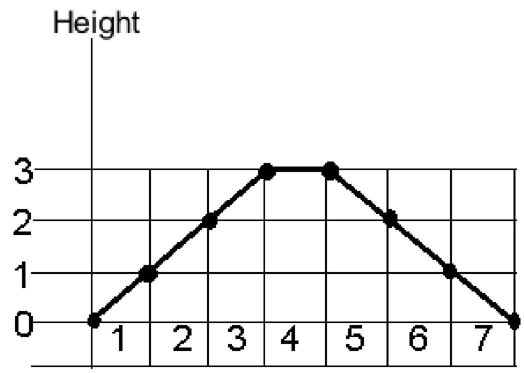
Let $\phi : G_p^{\Gamma_1} \to \Gamma_p$, $\Phi : \Gamma_p \to \Omega_p$ and $\Psi = \Phi \circ \phi$. For any $\omega \in G_p^{\Gamma_1}$, define the Motzkin path of ω by $\phi(\omega) = c_1c_2\cdots c_p$, where for each $i \in [p]$ define c_i as follows:

$$c_i = \begin{cases} h \text{ if } i \in Out(\omega) \cup Ins(\omega) \\ u \text{ if } i \in O(\omega) \\ d \text{ if } i \in C(\omega) \end{cases}.$$

Now define $\Phi(\phi(\omega))$ (weight of Motzkin path) by redefining $h, u,$ and d as h_j^k, u_j^k and d_j^k respectively, setting j as the height of the step c_i and k as the number of elements of the steps of same height, and finally replacing $u_j^k d_j^k$ by u_j^k.

Example 2.1

Given $\omega = 147-36-25$. Then the $O(\omega) = \{1,2,3\}$, $C(\omega) = \{5,6,7\}$, $Out(\omega) = \varnothing$ and $Ins(\omega) = \{4\}$. Hence $\phi(\omega) = uuuhddd$.

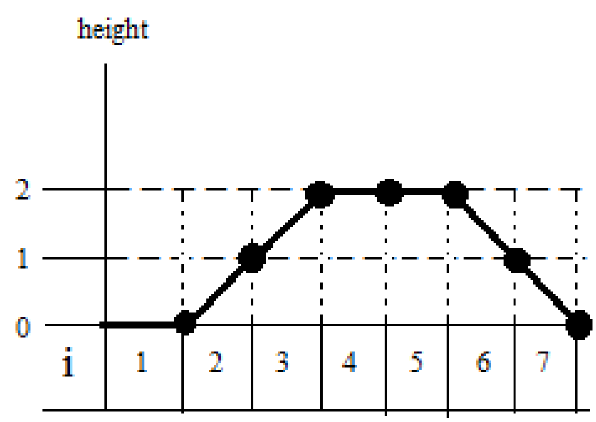


. and

$\Psi(\omega) = \Phi \circ \phi(\omega) = u_1u_2u_3h_3$.

Example 2.2

Given $\omega = 1-4-73-62-5$. *Then the* $O(\omega) = \{2,3\}$, $C(\omega) = \{6,7\}$ $Out(\omega) = \{1,4,5\}$ $Ins(\omega) = \varnothing$. *Hence* $\phi(\omega) = huuhhdd$.



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Hence $\Psi(\omega) = h_0u_1u_2h_2^2$.

Definition 2.11

Motzkin polynomial P is the sum of the weight of Motzkin path of all elements of $G_p^{\Gamma_1}$.

$$P_p(\underline{u},\underline{h}) = \sum_{\omega \in G_p^{\Gamma_1}} \Psi_{AI}(\omega).$$

Definition 2. 12

The parameter of a group G of order n, denoted as $Par\ (G)$, is defined by

$$\text{Par(G)} = \sum_{i=1}^{n} H(\psi_{AI}(\omega))$$

III. MAIN RESULTS

Proposition 3.1

Let $\omega_i \in G_p^{\Gamma_1}$. Then the

$$Run(\omega_i) = i$$

Proof

By induction for $j=1$ $Run(\omega_j) = j$, $Run(\omega_{j+1}) = j+1,$ for $k \geq 0$ and $j+k < \frac{p-1}{2}$, $Run(\omega_{j+k}) = j+k$. Setting $i = j+k,$ we have $Run(\omega_i) = i$.

Proposition 3.2

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a bijection. Then the weight of Motzkin path of $\omega_i \in G_p^{\Gamma_1}$ for $i \leq \frac{p-1}{2}$ is given by

$$\Psi_{AI}(\omega_i) = h_i^{p-2i} \prod_{k=1}^{i} u_k$$

Proof

For $i \leq \frac{p-1}{2}$, $o(\omega_i) = Run(\omega_i) = i$. Thus every $1 \leq c_i \leq i$ is an opener, and each u_{c_i} is unique and of different height. Let $c_i = k$, then the $u_k's$ are of heights $1 \leq k \leq i$. Therefore, the up steps are generated by $\prod_{k=1}^{i} u_k$.

For $i \leq \frac{p-1}{2}$, there are i openers and i closers and thus there are $p - i - i = p - 2i$ insiders since ω_i (for $i \leq \frac{p-1}{2}$) has no outsiders. The horizontal steps h_k are of the same height as u_k ($for\ k = 1$), and thus there are $p - 2i$ horizontal steps of height i (h_i^{p-2i}). Therefore

$$\Psi_{AI}(\omega_i) = \prod_{k=1}^{i} u_k h_i^{p-2i}.$$

Proposition 3.3

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a bijection and $\omega_i \in G_p^{\Gamma_1}$. Then the

$$\Psi_{AI}(\omega_i) = \Psi_{AI}(\omega_{p-i}).$$

Proof

For any $\omega_i = a_1 a_2 \cdots a_{p-1} a_p$, $\omega_{p-i} = a_1 a_p a_{p-1} \cdots a_2$ where $a_1 = 1$. Since 1 is fixed at the first position of each permutation, then the openers and closers (up and horizontal steps)in ω_i are respectively the opener and closers in ω_{p-i}. Hence

$$\Psi_{AI}(\omega_i) = \Psi_{AI}(\omega_{p-i}).$$

Proposition 3. 4

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a bijection. Then the Motzkin polynomial of size p is given by

$$P_p = P_p(\underline{u},\underline{h}) = 2\left(\sum_{i=1}^{\frac{p-1}{2}} \prod_{k=1}^{i} u_k h_i^{p-2i} \right).$$

Where $\underline{h} = (h_0, h_1, h_2, \ldots)$, $\underline{u} = (u_1, u_2, u_3, \ldots)$.

Proof

By Proposition 3.2 the $\Psi_{AI}(\omega_i) = h_i^{p-2i} \prod_{k=1}^{i} u_k$ for $i \leq \frac{p-1}{2}$ and from Proposition 3 the $\Psi_{AI}(\omega_i) = \Psi_{AI}(\omega_{p-i})$. Therefore since $P_p(\underline{u},\underline{h}) = \sum_{\omega \in G_p^{\Gamma_1}} \Psi_{AI}(\omega)$ it follows that

$$P_p = P_p(\underline{u},\underline{h}) = 2\left(\sum_{i=1}^{\frac{p-1}{2}} \prod_{k=1}^{i} u_k h_i^{p-2i} \right).$$

Proposition 3.5

Let $\omega_i \in G_p^{\Gamma_1}$ *and bijection* $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$. *Then the parameter of* $G_p^{\Gamma_1}$ *is given as*

$$Par(G_p^{\Gamma_1}) = \left(\frac{p^2-1}{4}\right).$$

Proof

By definition the $Par(G_p^{\Gamma_1}) = \sum_{i=1}^{p-1} o(\omega_i)$. Since $o(\omega_i) = i$ for $1 \le i \le \frac{p-1}{2}$ and $\Psi_{AI}(\omega_i) = \Psi_{AI}(\omega_{p-i})$. Therefore

$$Par(G_p^{\Gamma_1}) = 2\left(1+2+\cdots+\frac{p-1}{2}\right)$$

$$= 2\left(\frac{1}{2}\left(\frac{p-1}{2}\right)\left(\frac{p-1}{2}+1\right)\right)$$

$$= \left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)$$

$$= \left(\frac{p^2-1}{4}\right)$$

Proposition 3.6

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a bijection. setting $\underline{u} = \underline{h} = 1$, we have

$$P_p|_{\underline{u}=1,\underline{h}=1} \equiv P_p(\underline{1},\underline{1}) = P_p(1,1,\ldots,1,1,\ldots) = |G_p^{\Gamma_1}|.$$

Where $\underline{h} = (h_0, h_1, h_2, \ldots)$, $\underline{u} = (u_1, u_2, u_3, \ldots)$.

Proof

By Proposition 3.4 the

$$P_p = P_p(\underline{u},\underline{h}) = 2\left(\sum_{i=1}^{\frac{p-1}{2}} \prod_{k=1}^{i} u_k h_i^{p-2i}\right).$$

Setting setting $\underline{u} = \underline{h} = 1$ we have

$$P_p|_{\underline{u}=1,\underline{h}=1} \equiv P_p(\underline{1},\underline{1}) = P_p(1,1,\ldots,1,1,\ldots) = 2\left(\sum_{i=1}^{\frac{p-1}{2}} 1\right)$$

$$= 2\left(\frac{p-1}{2}\right) = p-1 = |G_p^{\Gamma_1}|.$$

Proposition 3.7

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a bijection. Then the weight of Motzkin path of $\omega_i \in G_p^{\Gamma_1}$ for $i \le \frac{p-1}{2}$ is given by

$$\Psi_{AI}(\omega_i) = h_0 \prod_{k=1}^{i-1} u_k h_i^{p-(2i-1)}$$

where $u_0 = 1$.

Proof

For each ω_i, there is one horizontal step of height zero (h_0) at c_1, and for $i \le \frac{p-1}{2}, o(\omega_i) = i-1$, thus every $2 \le c_i \le i$ is an opener, and each u_{c_i} is unique and of different height.

Let $c_i = k$, then the $u_k's$ are of height $i \le k \le i-1$. Therefore, the up steps are generated by $\prod_{k=1}^{i-1} u_k$.

For $i \le \frac{p-1}{2}$, there are $i-1$ openers and $i-1$ closers and thus there are $p-2(i-1)$ outsiders, since for ω_i ($i \le \frac{p-1}{2}$) there are no insiders.

Since c_i is insider of height zero (h_0), then there are $p-2(i-1)-1 = p-(2i-1)$ insiders (horizontal steps) of height as u_k (for $k = i$), generated by $h_i^{p-(2i-1)}$. Hence

$$\Psi_{AI}(\omega_i) = h_0 \prod_{k=1}^{i-1} u_k h_i^{p-(2i-1)}$$

Remark 2.

Aremu et al. [17] defined similar motzkin path with descent blocks and show that the Motzkin path of $\omega_i$is the same as the Motzkin path of $\omega_{p-des(\omega_i)}$. Hence from this we have that $\Psi_{AI}(\omega_i) = \Psi_{AI}(\omega_{p-des(\omega_i)})$.

Proposition 3.8

Let $\Psi_{AI} : G_p^{\Gamma_1} \to \Gamma_p$ be a mapping. Then the Motzkin polynomial of size p is given by

$$\mathrm{P}_p = \mathrm{P}_p(\underline{u},\underline{h}) = h_0\left(h_0^{p-1} + 2\sum_{i=1}^{\frac{p-1}{2}}(h_i^{p-(2i-1)}\prod_{k=0}^{i}u_k) + \prod_{j=1}^{\frac{p-1}{2}}u_j\right).$$

Where $\underline{h} = (h_0, h_1, h_2, \ldots)$, $\underline{u} = (u_1, u_2, u_3, \ldots)$, and $u_0 = 1$.

Proof

The result follows immediately by Propsoition 3.7 and Remark 2.

IV. CONCLUSIONS

The study provide and enumeration for finding the weight Motzkin path of $\Gamma_1 - non\ deranged\ permutations$ and the Moztkin polynomial of $\mathrm{G}_p^{\Gamma_1}$, for both ascent and descent blocks. The study also show that when each step in the Motzkin path of $\mathrm{G}_p^{\Gamma_1}$ is set o be equal to one, the Motzkin polynomial is the same as the order of $\mathrm{G}_p^{\Gamma_1}$.

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