On The Exponential-Gamma-X Mixed Distribution

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Abstarct- **Statistical distributions are very crucial in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. In this study we transform the Exponential-Gamma distribution using the technique that generates T-X family of distributions and we hope that this new T-X family of distributions will provide more channels for generalizing more flexible family of probability distributions.**

***Keywords*: Probability density function, Hazard function, Cumulative hazard function, generalized T-X, Exponential-Gamma-X family of distributions.**

I. INTRODUCTION

Statistical distributions are very crucial in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. In recent studies, new distributions developed in the literature seem to focus on more general and flexible distributions Alzaghal *et al.* in [1]. Using the technique that generates the *T -X* family, one can develop new distributions that may be very general and flexible or for fitting specific types of data distributions such as highly left-tailed (right-tailed, thin-tailed, or heavy-tailed) distribution as well as bimodal distributions in Alzaatreh*et al.*[2]

The interest in developing more flexible probability distributions have received so much attention over the years from various researchers such as Eugene *et al.* in [3] developed Beta-G,Nadarajah and Kotz in [4] developed Beta-Gumbel, Nadarajah and Gupta in [5], the Beta Fr´echet distribution was established. Famoye et al. in [6] presented the Beta-Weibull, Nadarajah and Kotz in [7]; Beta exponential by Akinsete*et al.*, in [8]; beta-Pareto, Barreto-Souza *et al.*, in[9]; beta generalized exponential.

Motivated by the recent developments in developing more flexible distributions and the need for continuous extension and generalization to more complex situations, in this study we'll transform the Exponential-Gamma developed by Ogunwale*et al*., in [10] using the technique that generates T-X family of distributions.

II. EXPONENTIAL-GAMMA DISTRIBUTION (EGD)

Definition 1: Let X be a continuous independent random variables such that; $x \sim EGD(x;\alpha,\lambda)$ as defined by Ogunwale *et al.*, in [10] the probability density function is given as;

$$f(x)=\frac{x^{\alpha-1}\lambda^{\alpha+1}e^{-2\lambda x}}{\Gamma(\alpha)}, x,\alpha,\lambda>0 \quad (1)$$

***Definition 2*:** The cumulative distribution function of exponential-gamma distribution is given as;

$$F(x)=\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)} \quad (2)$$

***Definition 3*:** The survival function of exponential-gamma distribution is given as:

$$S(x)=1-\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)} \quad (3)$$

***Definition 4*:** The hazard function of exponential-gamma distribution is given as;

$$h(x)=\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}2^{\alpha}}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)} \quad (4)$$

***Lemma 1*:** Given that a random variable X such that$x \sim EGD(x;\alpha,\lambda)$, the cumulative hazard function is defined as;

$$H(x)=\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}$$

Proof

$$H(x)=W(F(x))=-\log(1-F(x))\equiv\int_0^x h(x)dx$$

$$H(x)=\int_0^x\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}2^{\alpha}}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}dx$$

$$=\frac{\lambda^{\alpha+1}2^{\alpha}}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}\int_0^x x^{\alpha-1}e^{-2\lambda x}dx \quad (5)$$

$$H(x)=\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)} \quad (6)$$

III. TRANSFORMATION OF THE EXPONENTIAL-GAMMA DISTRIBUTION (EGD)

Recently, Alzaatreh*et al.* in [2] extended the beta-generated family of distributions by using any non-negative continuous random variable *T* as the generator, in place of the beta random variable. The new class of distributions is defined as

$$G(x) = \int_0^{-\log(F(x))} r(t)dt \qquad (7)$$

where$r(t)$ is the PDF of a non-negative continuous random variable *T*. The corresponding PDF to the CDF in (7) is given by

$$g(x) = \frac{f(x)}{1-F(x)} r\left(-\log(1-F(x))\right) \qquad (8)$$

In this new class, the distribution of the random variable *T* is the generator. The new family of distributions generated from (8) is called “*T-X* distribution”, the upper limit for generating the *T-X* distribution is $-\log(1-F(x))$. It is clear that one can define a different upper limit for generating different types of *T-X* distributions. In this article, we define the upper limit to be $-\log(1-F(x))$, which leads to a new family of exponential-gamma *T-X* distributions.

IV. THE EXPONENTIAL-GAMMA-X FAMILY

***Theorem 1*:** Let $r(t)$ be the PDF of a non-negative continuous random variable *T* defined on $[0,\infty)$, and let $F(x)$ denote the CDF of a random variable *X* such that $x \sim EGD(x;\ \alpha,\ \lambda)$, we define the CDF for the exponential-gamma-*X* class of distributions for a random variable *X* as

$$G(x) = \frac{\lambda}{2^{\alpha}\Gamma(\alpha)} \gamma\left(\alpha, -\log(F(x))\right), x, \alpha, \lambda > 0$$

Proof:

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt$$

where $r(t) = \dfrac{\lambda^{\alpha+1} t^{\alpha-1} e^{-2\lambda t}}{\Gamma(\alpha)}$ is the pdf of Exponential-Gamma distribution then,

$$G(x) = \int_0^{-\log(1-F(x))} \frac{\lambda^{\alpha+1} t^{\alpha-1} e^{-2\lambda t}}{\Gamma(\alpha)} dt$$

$$= \frac{\lambda^{\alpha+1}}{\Gamma(\alpha)} \int_0^{-\log(1-F(x))} t^{\alpha-1} e^{-2\lambda t} dt \qquad (9)$$

are is hazard function, pdf and cumulative hazard function of Exponential-Gamma distribution respectively,

Table 1: Some families of generalized distribution with different *T* distribution

Name	density function *r(t)*	Density function of the family *g(x)*
Exponential	$\theta e^{-\theta t}$	$\theta f(x)(1-F(x))^{\theta-1}$
Beta-Exponential	$\dfrac{\lambda e^{-\lambda\beta x}(1-e^{\lambda x})^{\alpha-1}}{B(\alpha,\beta)}$	$\dfrac{\lambda f(x)}{B(\alpha,\beta)}\left(1-F(x)\right)^{\lambda\beta-1}\left\{1-\left(1-F(x)\right)^{\lambda}\right\}^{\alpha-1}$
Exponentiated-exponential	$\dfrac{\alpha\lambda(1-e^{-\lambda x})^{\alpha-1}}{e^{\lambda x}}$	$\alpha\lambda f(x)\left\{1-\left(1-F(x)\right)^{\lambda}\right\}^{\alpha-1}\left(1-F(x)\right)^{\lambda-1}$
Gamma	$\dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{-t/\beta}$	$\dfrac{f(x)}{\Gamma(\alpha)\beta^{\alpha}}\left(-\log(F(x))\right)^{\alpha-1}\left(1-F(x)\right)^{\frac{1}{\beta}-1}$
Half-normal	$\dfrac{1}{\sigma}\left(\dfrac{2}{\pi}\right)^{1/2} e^{-t^2/2\sigma^2}$	$\dfrac{1}{\sigma}\left(\dfrac{2}{\pi}\right)^{1/2}\dfrac{f(x)}{1-F(x)}\exp\left(-\left\{\log(1-F(x))\right\}^2/2\sigma^2\right)$
Levy	$\left(\dfrac{c}{2\pi}\right)^{1/2}\dfrac{e^{-c/2t}}{t^{3/2}}$	$\left(\dfrac{c}{2\pi}\right)^{1/2}\dfrac{f(x)}{1-F(x)}\dfrac{\exp(-c/2\log(1-F(x)))}{\left\{-\log(1-F(x))\right\}^{3/2}}$

Log logistics	$\frac{\beta(t/\alpha)^{\beta-1}}{\alpha\left\{1+(t/\alpha)^{\beta}\right\}^{2}}$	$\frac{\beta}{\alpha\beta}\frac{f(x)}{1-F(x)}\left\{-\log(1-F(x))\right\}^{\beta-1}$ $\times\left\{1+(-\log(1-F(x))/\alpha)^{\beta}\right\}^{-2}$
Rayleigh	$\frac{t}{\sigma^{2}}e^{-t^{2}/2\sigma^{2}}$	$\frac{-f(x)\log(1-F(x))}{\sigma^{2}(1-F(x))}\exp(-\left\{\log(1-F(x))\right\}^{2}/2\sigma^{2})$
Weibull	$\frac{c}{\gamma}\left(\frac{t}{\gamma}\right)^{c-1}e^{-(t/\gamma)^{c}}$	$\frac{c}{\gamma}\frac{f(x)}{1-F(x)}\left\{-(1/\gamma)\log(1-F(x))\right\}^{c-1}$ $\times\exp(-\left\{-(1/\gamma)\log(1-F(x))\right\}^{c})$
Exponential-Gamma	$\frac{t^{\alpha-1}\lambda^{\alpha+1}e^{-2\lambda t}}{\Gamma(\alpha)}$	**Exponential-Gamma-X (Proposed)**

Therefore;

$$g(x)=\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}2^{\alpha}}{\left(2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)\right)}$$

$$\frac{\lambda^{\alpha+1}}{\Gamma(\alpha)}\left[\left(\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}\right)^{\alpha-1}\exp\left\{-(2\lambda)\left(\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}\right)\right\}\right]$$

$$g(x)=\frac{\lambda^{\alpha+1}}{\Gamma(\alpha)}\times\frac{\lambda^{\alpha+1}x^{\alpha-1}e^{-2\lambda x}2^{\alpha}}{\left(2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)\right)}$$

$$\left[\left(\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}\right)^{\alpha-1}\exp\left\{-(2\lambda)\left(\frac{\lambda\gamma(\alpha,x)}{2^{\alpha}\Gamma(\alpha)-\lambda\gamma(\alpha,x)}\right)\right\}\right]$$

$$g(x)=\frac{\lambda^{\alpha+1}f(x)}{\Gamma(\alpha)(1-F(x))}\left[-\log(1-F(x)\right]^{\alpha-1} \quad (11)$$

$$\exp\left[-2\lambda\left(-\log(1-F(x))\right], x,\alpha,\lambda>0\right.$$

The CDF and the PDF of exponential-gamma-X distribution given in Equations (10) and (11) can be expressed as $G(x) = R(-\log(1-F(x)) = R(H(x))$ and $g(x) = h(x)\, r(H(x))$, where $h(x)$ and $H(x)$ are the hazard and cumulative hazard functions of the random variable X , hence, the Exponential-Gamma-X distribution can be considered as a family of distributions arising from the hazard functions.

V. CONCLUSION

Statistical distributions are very crucial in describing and predicting real world phenomena. Although many distributions have been developed, there are always rooms for developing distributions which are either more flexible or for fitting specific real world scenarios. In this study we've transformed the Exponential-Gamma developed by Ogunwale*et al.*, in [10] using the technique that generates T-X family of distributions and we hope that this new T-X family of distributions will provide more channels for generalizing more flexible family of probability distributions.

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