A Block Scheme with Bernstein Single Step Method for Direct Solution Second Order Differential Equations

Ojo Ezekiel Olukunle[1], Okoro Felix M.[2]
[1, 2]Department of Mathematics, Ambrose Alli University Ekpoma, Nigeria

Abstract: **- A continuous one-step hybrid block method with two-off grid points using Bernstein polynomial as basis function for solving directly the general second order initial value problems of ordinary differential equations is derived. The scheme is based on collocation and interpolation techniques at desired off grid points and implemented as block mode so as to obtain approximate solution at both step and off step points. The method was applied on linear and non-linear ODE and found to be consistent, zero stable and convergent. Numerical results of the proposed scheme show efficiency over some existing schemes.**

Keywords: **Bernstein polynomial; Collocation; Interpolation; Block method; Zero Stability; Consistency; Region of Absolute stability.**

I. INTRODUCTION

The desire of many scholars to obtain more accurate approximate solution to mathematical models, arising from engineering, medicine, science and science social, in the form of ordinary differential equations (ODEs) which cannot be solve analytically have led them to proposed several different numerical methods.

Researchers over the years have considered different approaches of generating numerical solution to second order initial value problem of ODE the form

$$y'' = f(x, y(x), y'(x)), \quad y(0) = \eta_0, \quad y'(a) = \eta_1 \quad (1.1)$$

The development of numerical methods for IVP of the form (1.1) has given rise to two major discrete variable methods; One step (or single step) method and multistep method especially the linear multistep method [2]

These methods employ polynomials as a trial or basis function. Polynomials which have played a central role in approximation theory as well as in numerical analysis for decades, have a great variety of functions, differentiable and integrable [12].In solving IVP of (1.1), many scholars have worked by using single step and multistep methods with different polynomials.

For example, [11], [17], [18] and [19] used power series polynomial as the basis function to proposed linear multistep methods for initial value problems of the form (1.1) in the predictor corrector mode and Taylor series algorithm to supply starting values.

[7] employed Chebyshev polynomial as basis function to proposed three- step implicit numerical method capable of solving (1.1). [4] used the Newton's polynomials to generate predictors-corrector method and Taylor series algorithm to supply starting values.

Also [3] and [17] developed predictor-corrector methods for the solution of (1.1) using Chebyshev polynomials as basis. A single step method for solving (1.1) based on power series as basis function was proposed by [8], [10], [21] and [2]. While [16] employed shifted Legendre polynomial as basis function to proposed a continuous block linear multistep methods for initial value problems of the form (1.1).

Resently, [1] presented a method for solving (1.1 & 1.2) using Lucas Polynomial as basis function.Several numerical methods based on the use of polynomial functions (Power series, Legendre, Chebyshev, Lucas e.t.c) have been used as basis function to develop numerical methods for direct solution of higher order IVP using interpolation and collocation procedure.

In this paper, we propose one step hybrid block method using Bernstein polynomial as basis function in collocation interpolation approach.

The Bernstein polynomials of degree m are defined on the interval [0, 1], as [12]

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, i = 0, 1, \dots, m \quad (1.2)$$

In general, we approximate any function y(x) over [0, 1] as Bernstein basis function

$$y(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = C^T \emptyset(x) \quad (1.3)$$

where $C^T = [c_0, c_1 \dots, c_m]$,

are the coefficients to be determined and

$\emptyset(x) = [B_{0,m}(x),\ B_{1,m}(x), \dots,\ B_{m,m}(x)]^T$,

is the Bernstein polynomial of degree m.

This paper is organization as follows: Section 2 is methodology. Section 3, analysis of the basic properties of the method is presented. Numerical implementation of the scheme is in Section 4. Section 5 is discussion of result, and in Section 6 the conclusion

II. METHODOLOGY

We define a basis function in the form of Bernstein polynomialas,

$$y(t) = \sum_{k=0}^{c+i-1} a_k B_{k,n}(t) \qquad (2.1)$$

where c and i are number of distinct collocation and interpolation points respectively, $a_k$is the coefficients to be determined and $B_{k,n}(t)$ is the Bernstein Polynomial derived from the recursiverelation

$$B_{k,m}(t) = (1-t)B_{k,m-1}(t) + tB_{i-1,k-1}(t) \quad (2.2)$$

Differentiating (2.1) twice and substituting into (1.1) gives:

$$f(x, y(x), y'(x)) = \sum_{k=0}^{c+i-1} a_k B''_{k,n}(t) \ (2.3)$$

We consider a grid point of step length one and off step point at $x = x_{n+\frac{2}{7}}, x_{n+\frac{5}{7}}$. Collocating (2.3) at points$x = x_n$, $x_{n+\frac{2}{7}}$, $x_{n+\frac{5}{7}}$ and x_{n+1}, and interpolating (2.1) at $x = x_{n+\frac{2}{7}}$ and $x_{n+\frac{5}{7}}$, give a system of five equations which are solved using Gaussian elimination method to obtained the parameters$a'_j s, j = 0,1, \dots ,5$.The parameters $a'_j s$ obtained are then substituted back into (2.1) to give the continuous hybrid one step method of the form;

$$y(x) = \alpha_0 y_n + \alpha_{\frac{2}{7}} y_{n+\frac{2}{7}} + \alpha_{\frac{5}{7}} y_{n+\frac{5}{7}} + h^2 \left[\beta_0 f_n + \beta_{\frac{2}{7}} f_{\frac{2}{7}} + \beta_{\frac{5}{7}} f_{\frac{5}{7}} + \beta_1 f_{n+1}\right] \qquad (2.4)$$

where α_0 and β_0 are continuous coefficients. The continuous method (2.4) is used to generate the main method. That is, we evaluate at $x = x_n$ and x_{n+1} we obtain the methods as follows

$$y_{n+1} = -\frac{2}{3} y_{n+\frac{2}{7}} + \frac{5}{3} y_{n+\frac{5}{7}} + \frac{1}{8820} h^2 \begin{bmatrix} -24f_n + 116f_{n+\frac{2}{7}} \\ + \ 719f_{n+\frac{5}{7}} \ + 39f_{n+1} \end{bmatrix} \qquad (2.5)$$

$$y_n = \frac{5}{3} y_{n+\frac{2}{7}} - \frac{2}{3} y_{n+\frac{5}{7}} + \frac{1}{8820} h^2 \left[39f_n + 719f_{n+\frac{2}{7}} - 24f_{n+1}\right] \qquad (2.6)$$

In order to incorporate the second initial condition at (1.2) in the derived schemes, we differentiate (2.4) and evaluate at points $x = x_n$, $x_{n+\frac{2}{7}}$, $x_{n+\frac{5}{7}}$ and x_{n+1}, we obtained the following discrete derivative schemes:

$$hy' + \frac{7}{3} y_{n+\frac{2}{7}} - \frac{7}{3} y_{n+\frac{5}{7}} = \frac{1}{4200} h^2 \begin{bmatrix} -401f_n - 1521f_{n+\frac{2}{7}} - 194f_{n+\frac{5}{7}} \\ +16f_{n+1} \end{bmatrix} \qquad (2.7)$$

$$hy'_{n+\frac{2}{7}} + \frac{7}{3} y_{n+\frac{2}{7}} - \frac{7}{3} y_{n+\frac{5}{7}} = \frac{1}{29400} h^2 \left[513f_n - 4767f_{n+\frac{2}{7}} - 2478f_{n+\frac{5}{7}} + 432f_{n+1}\right] \qquad (2.8)$$

$$hy'_{n+\frac{5}{7}} + \frac{7}{3} y_{n+\frac{2}{7}} - \frac{7}{3} y_{n+\frac{5}{7}} = \frac{1}{29400} h^2 \left[-432 + 2478f_{n+\frac{2}{7}} + 4767f_{n+\frac{5}{7}} - 513f_{n+1}\right] \qquad (2.9)$$

$$hy'_{n+1} + \frac{7}{3} y_{n+\frac{2}{7}} - \frac{7}{3} y_{n+\frac{5}{7}} = \frac{1}{4200} h^2 \begin{bmatrix} -16f_n + 194f_{n+\frac{2}{7}} \\ +1521f_{n+\frac{5}{7}} + 401f_{n+1} \end{bmatrix} \qquad (3.0)$$

The block methods are derived by combining equation (2.5) to (3.0) and solved simultaneously to obtain the following explicit result:

$$y_{n+\frac{2}{7}} = y_n + \frac{2}{7} hy'_n + \frac{1}{11025} h^2 \left[252f_n + 242f_{n+\frac{2}{7}} - 62f_{n+\frac{5}{7}} + 18f_{n+1}\right]$$

$$y_{n+\frac{5}{7}} = y_n + \frac{5}{7} hy'_n + \frac{1}{3528} h^2 \left[225f_n + 625f_{n+\frac{2}{7}} + 50f_{n+\frac{5}{7}}\right]$$

$$y_{n+1} = y_n + hy'_n + \frac{1}{1800} h^2 \left[159f_n + 539f_{n+\frac{2}{7}} + 196f_{n+\frac{5}{7}} + 6fn+1 \right. \qquad (3.1)$$

$$y'_{n+\frac{2}{7}} = y'_n + \frac{1}{735} h \left[83f_n + 147f_{n+\frac{2}{7}} - 28f_{n+\frac{5}{7}} + 8f_{n+1}\right]$$

$$y'_{n+\frac{5}{7}} = y'_n + \frac{1}{1176} h \left[95f_n + 525f_{n+\frac{2}{7}} - 245f_{n+\frac{5}{7}} - 25f_{n+1}\right]$$

$$y'_{n+1} = y'_n + \frac{1}{120} h \left[11f_n + 49f_{n+\frac{2}{7}} + 49f_{n+\frac{5}{7}} + 11f_{n+1}\right]$$

III. ANALYSIS OF THE BASIC PROPERTIES OF THE METHOD

In this section, we analyze the derived scheme by determining the order and error constant, consistency, zero stability and region of absolute stability of the scheme.

3.1 Order and Error constant

Definition 3.1 The one-step implicit hybrid block linear method and the associated linear difference operator are said to have order p if $C_0 = C_1 = C_2 = C_3 = \cdots C_p = C_{p+1}$ and $C_{p+2} \neq 0$see [20] for details. According to Fatunla [15] and [20],, we expand (3.1) using Taylor's series and combining the coefficient of the like terms in h^n, the following result are obtained.

Hence The block method has order p=(4, 4, 4, 4, 4, 4)^T with error constant (−1.2372e−5,−1.8446e−5,−1.4172e−5,−7.9993e−5, 0.5.1648e−5,−2.8345e−5). The region of absolute stability of the method is between (-9.9328, 0.00)

3.2 Consistency of the Scheme

Definition 3.2 A numerical method is said to be consistent, if it has order greater than one (p≥ 1) see [14]and [20] for details.

Hence our methods are consistent since the order is greater than one

3.3 Zero Stability

Definition 3.3*A* block method is said to be zero stable if the roots$Z_r; r1,..,n$ of the first characteristic polynomial ρ(z), defined by $p(z) = \det|ZQ - T|$satisfies |z| ≤ 1and every root with$|Z_r| = 1$has multiplicity not exceeding two in the limit as h → 0 (see Fatunla [14] for details)

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Substituting, we have

$$\rho(z) = z^5(z-1) = 0$$

gives z = 0 or z = 1. Hence the block is zero stable since |z|=1

3.4 Convergence of the method

Definition 3.4 The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. [13]

Hence our methods are convergent since is consistent and zero stable

IV. NUMERICAL EXAMPLES

We solve the following example to illustrate our method (2.9).

Problem I:

$$y'' = y', \ y(0) = 0, \ y'(0) = -1, \ h = 0.1$$

Exact solution $\qquad y(x) = 1 - \exp(x)$

Source: Kayode et al ([18])

Problem II:

$$y'' = 2y - y', \ y(0) = 0, \qquad y'(0) = 1$$

Exact solution: $\quad y(x) = \frac{e^x - e^{-2x}}{3}, \ 0 \leq x \leq 1$

Source: Adeyefa et al ([7])

Problem II :

$$y'' - x(y')^2 = 0, y(0) = 1, y'(0) = -\frac{1}{2}, \qquad h = 0.01$$

Exact solution $\quad y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Source: Anake et al ([9])

Table 4.1: Comparison of the error for problem 1

x	Error in [17]	Error in [2]	Error in[18]	Error in our method
0.1	0.82×10^{-06}	2.220×10^{-08}	3.460×10^{-09}	1.6454×10^{-11}
0.2	0.31×10^{-05}	1.250×10^{-07}	5.676×10^{-09}	6.772×10^{-11}
o.3	0.65×10^{-05}	3.250×10^{-07}	7.641×10^{-09}	1.609×10^{-10}
0.4	0.66×10^{-05}	6.424×10^{-07}	1.050×10^{-08}	3.044×10^{-10}
0.5	0.11×10^{-05}	1.099×10^{-06}	1.450×10^{-08}	5.075×10^{-10}
0.6	1.80×10^{-04}	1.721×10^{-06}	1.878×10^{-08}	7.814×10^{-10}
0.7	0.26×10^{-04}	2.538×10^{-06}	2.280×10^{-08}	1.139×10^{-09}
0.8	0.37×10^{-04}	3.583×10^{-06}	2.826×10^{-08}	1.594×10^{-09}
0.9	0.51×10^{-04}	4.896×10^{-06}	3.555×10^{-08}	2.163×10^{-09}
1.0	0.67×10^{-04}	6.522×10^{-06}		2.866×10^{-09}

Table 4.2: Comparison of the error for problem 1I.

x	Exact Solution	Computed Solution	Error in the Proposed Method	Error in [7]
0.1	0.0954800549992223	0.0954800547459173	2.533054×10^{-10}	9.766814×10^{-06}
0.2	0.183694237374844	0.183694236418694	9.56150×10^{-10}	1.831503×10^{-05}
0.3	0.267015723827325	0.267015721836013	1.991312×10^{-09}	6.510125×10^{-05}
0.4	0.347498577841350	0.347498574566664	3.274686×10^{-09}	1.024784×10^{-04}

Table 4.3:Comparison of the error for problem III

x	Error in [9]	Error in [5]	Error in [6]	Error in our scheme
0.1	6.2172×10^{-15}	7.5028×10^{-13}	4.8627×10^{-14}	1.5102×10^{-15}
0.2	2.4425×10^{-14}	9.7410×10^{-12}	2.1604×10^{-13}	1.8121×10^{-15}
0.3	5.6843×10^{-14}	3.7638×10^{-11}	5.2557×10^{-13}	2.2673×10^{-15}
0.4	1.0347×10^{-13}	9.7765×10^{-11}	1.0254×10^{-12}	2.7847×10^{-14}
0.5	1.6742×10^{-13}	2.0825×10^{-10}	1.8032×10^{-12}	4.5220×10^{-14}
0.6	2.5091×10^{-13}	3.9604×10^{-10}	3.0078×10^{125}	1.0259×10^{-14}
0.7	3.6016×10^{-13}	7.0460×10^{-10}	4.8991×10^{-12}	1.9090×10^{-14}
0.8	5.0493×10^{-13}	1.2095×10^{-09}	7.9460×10^{-12}	1.1456×10^{-13}
0.9	6.9522×10^{-13}	2.0511×10^{-09}	1.3702×10^{-11}	2.0926×10^{-13}
1.0	9.4836×10^{-13}	3.5066×10^{-09}	2.1885×10^{-11}	2.6478×10^{-13}

V. DISCUSSION OF RESULT

A new one-step hybrid block Bernstein method with two off-step points of order 4 is proposed for the direct solution of general second order ordinary differential equations. The main method and the additional methods were obtained from the same continuous method derived via interpolation and collocation procedures and then applied in block form as simultaneous numerical integrators over non-overlapping interval .The properties of the method are also discussed.

In Tables 4.1, 4.2 and 4.3, we compared the accuracy of proposed method with some existing methods, the proposed method display better accuracy.

VI. CONCLUSION

A block scheme with bernstein single step method generated in this paper is accurate, efficient and can compete favorably with existing schemes.

REFERENCES

[1]. Adeniran, A.O.& Longe,I.O (2019). Solving Directly Second Order Initial Value Problems with Lucas Polynomial Journal of Advance in Mathematics Computer Science 32(4):1-7.

[2]. Adeniran, A. O., Odejide S. A. & Ogundare, B. S. (2015) 'One-Step Hybrid Numerical Scheme for Direct solution of General Second Order Ordinary Differential Equations., International Journal of Applied Mathematics Volume 28 No. 3 pp.197-212

[3]. Adeniyi, R. & Adeefa, E. (2013). On chebyshev collocation approach for a continuous formulation of implicit hybrid methods for vips in second order odes. Journal of Mathematics, 12, 9 -12.

[4]. Adesanya, A.O., Anake, T.A. and Oghoyon, G.J. (2009). Continuous Implicit Method for the Solution of General Second Order Ordinary Differential Equations. Journal of Nigerian Association of Mathematical Physics, 15, 71-78.11

[5]. Adesanya, A.O., Odekunle, M.R., Adeyeye, A.O., (2012), Continuous Block Hybrid-Predictor-Corrector Method for the Solution of y″ = f(x, y, y′), Intern. J. of Math. and Soft Computing, 2(2), pp.35-42.

[6]. Adesanya, A.O., Odekunle, M.R., Alkali, A.A., (2012), Order Six Block Predictor-Corrector Method for the Solution of y″ = f(x, y, y′), Cana- dian J. of Sci. and Engr., 3(4), pp.180-185.

[7]. Adeyefa,E.O., Joseph, F.L. & Ogwumu O.D.(2014). Three-Step Implicit Block Method for Second Order Odes, International Journal of Engineering Science Invention, Vol.3 PP.34-38

[8]. Anake, T. (2011). Continuous implicit hybrid one step methods for the solution of initial value problems of general second order ordinary differential equation (Unpublished doctoral dissertation). Covenant University.

[9]. Anake, T.A, Bishop, S.A., & Agboola,O.O. (2015). On a Hybrid Nunerical Algorithm for the Solutions of Higher Order Ordinary Differential Equations. TWMS J. Pure Appl. Math., V.6, N.2, 2015, pp.24-29

[10]. Anake, T.A, Awoyemi, D. O. & Adesanya, A. (2012a). One step implicit hybrid block method for the direct solution of general second order ordinary differential equations. IAENG International Journal of Applied Mathematics, 42(4), 224 -228.

[11]. Awoyemi, D. O. and Kayode S.J. (2005a) 'A maximal order collocation method for direct solution IVPs of general second ordinary differential equations', In Proceedings of the conference organized by the National Mathematical Center, Abuja

[12]. Aysegul A.D, and Nese.I.A (2003), Bernstein Collocation Method for Solving Nonlinear Differential Equations. Numerical Computational Application, 18(3) 293-300.

[13]. Dahlquist, G. (1978) 'On accuracy and unconditional stability of LMM for second order differential equations', BIT, Vol. 18, pp. 133-136, 1978.

[14]. Fatunla, S.O. (1992). Parallel method for second order ordinary differential equation, In: Proceedings of the National Conference of Computational Mathematics, held at University of Benin, Nigeria (1992), 87 99.

[15]. Fatunla, S. O. (1991) 'Block methods for second order IVPs', International Journal of Computer Mathematics, 41 (1991) pp.55-63.

[16]. Kamoh N.M, Abada A.A, & Soomiyol M.C (2018). A block procedure with continuous coefficient for the direct solution of general second order initial value problems of (ODEs) using shifted Legendre polynomial as basis function. International Journal of Multidisciplinary Research andDevelopment Vol. 5 No 236-214.

[17]. Kayode, S. J. and Adeyeye, O. (2013). 'A Two-Step Point Hybrid Method for genereal Second Order Ordinary Differential Equations', African Journal of Mathematics and Computer Science Research, 6(10) pp.191-196 .

[18]. Kayode, S. J. & Adegboro. J. O. (2018). Predictor-Corrector Linear Multistep Method for Direct Solution of Initial Value Problems of Second Order Ordinary Differential Equations, Asian Journal of Physicaland Chemical Sciences 6(1): 1-9.

[19]. Kayoed, S.J. and Obarhua, F. (2015). 3-step y-function hybrid method for direct numerical integration of second order ivps in odes. Theoretical Mathematics Applications, vol 5, no J, 2015, 39 -51, 5(1), 39 -51.

[20]. Lambert, J. D. (1973) 'Computational Methods in Ordinary Differential Equations', New York: John Wiley.

[21]. Sagir, A. (2012). An accurate computation of block hybrid method for solving stiff ordinary differential equations. Journal of Mathematics, 4(18 -21).