Application of GELS Approximation to Bratu-Gelfand Equation

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Abstract:- **In this paper, the flexibility of the Generalized Euclidean Least Square (GELS) Approximation scheme is explored to obtain a more accurate approximation to the nonlinear part of Bratu-Gelfand equation. The resulting equation is solved using differential transform method and perturbation method. The problem is also solved using the conventional differential transform method and the perturbation method with the Maclaurin's approximation of the nonlinear part. The results obtained are better when GELS approximation is employed before applying any of the two methods.**

Keywords: **Generalized Euclidean Least Square (GELS) Approximation, Differential transform, inverse differential transform, homogeneous ODE, nonhomogeneous ODE.**

I. INTRODUCTION

Nonlinear elliptic eigenvalue problems of the form [2]

$$\Delta u(x) + \lambda f\big(u(x)\big) = 0,\ x \in \Omega \qquad (1)$$

$$u(x) = 0,\ x \in \partial\Omega \qquad (2)$$

is common in science and engineering. In the study of radiative heat transfer, combustion problem in a numerical slab, nanotechnology, fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe, equations of the form (1) and (2) are pivotal. In the simulation of a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction [6, 16, 2, 15]. More so, because of the simplicity of this model, it is widely used as a bench-marking tool for numerical methods. One special case of the equation 1 above is the boundary value problem

$$u'' + \lambda e^u = 0, 0 < x < 1, \qquad u(0) = u(1) = 0$$

called the Bratu-Gelfand equation whose solution is

$$u = -2\ln\left(\frac{\cosh\left(\frac{\theta}{4}(2x-1)\right)}{\cosh\left(\frac{\theta}{4}\right)}\right), \text{where} \theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right).$$

There are zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ or $\lambda < \lambda_c$ respectively and λ_c is estimated from

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)$$

as $\lambda_c = 3.513830719$. Extensive researches have been carried out to find a usable closed form solution to Bratu-Gelfand problem and some of the proposed method of solution include Variational iteration method (VIM) [14], Adomian Decomposition Method (ADM) and Restarted Adomian Decomposition Method (RADM) [15, 3], the use of operational matrix of Bernstein Polynomials (BPs) [10], spline-based approaches [2, 1], sinc-Collocation method [13]. The dynamical behavior of Bratu problem was studied in [5] and the precise existence and multiplicity results for radial solutions of the Liouville–Bratu–Gelfand problem was determined in [9] . Even solution to the fractional type Bratu problem has been sought using Adomian decomposition method [7] and homotopy perturbation method [8]. Most of these methods require the approximation of the nonlinear term by some simpler functions. These approximations are often adopted from the Maclaurin's series of the nonlinear term. This paper adopts the Generalized Euclidean Least Square (GELS) Approximation (derived in [12] as ELS scheme) to approximate the nonlinear part and the the resulting equation is solved using the differential transform method and perturbation method. The results are compared with the results obtained when the Maclaurin's series is adopted.

II. METHODOLOGY

The exponential function in the boundary value Bratu problem

$$u'' + \lambda e^u = 0, 0 < x < 1,\ u(0) = u(1) = 0 \qquad (3)$$

can be approximated by the Maclaurin series as

$$e^u = \sum_{m=0}^{\infty} \frac{u^m}{m!},$$

but it shall be approximated using the Generalized Euclidean Least Square (GELS) Approximation developed in [12] (where it was denoted as ELS scheme). In what follows, the GELS scheme, the perturbation method and differential transform method shall be detailed.

2.1 GELS Scheme

Given a function $f(x)$, the GELS approximation of order s is defined as

$$P_s(x) = \sum_{m=0}^{s} a_m x^m$$

and the coefficients $a_m, (m = 1(1)n)$ can be obtained using

$$A = \alpha^{-1}\beta$$

where

$$\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{s+1} \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_{s+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_s & \alpha_{s+1} & \alpha_{s+2} & \cdots & \alpha_{2s} \end{pmatrix},$$

$$A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_s \end{pmatrix}, \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_s \end{pmatrix},$$

$$\alpha_r = \sum_{i=0}^{n} x_i^r, \text{and} \beta_r = \sum_{i=0}^{n} x_i^r y_i.$$

The choice of the nodes x_i's can be made according to individual choice and can be adjusted until a desired accuracy is obtained. For instance, in the present work, an approximation to the exponential function is needed. Choose the nodes x_i within the interval $[0,1]$ (since the solution to Bratu equation is found in the interval $[0,1]$) so that they are evenly spaced (mesh size = h, say) and with this choice, y_i's are obtained as $y_i = e^{x_i}$. By choosing $h \leq 0.001$, it is easy to compute α_r and β_r $(r = 0,1,2)$ and a second order approximation for the exponential function is obtained as

$$e^x = 1.013 + 0.851x + 0.839x^2$$

in contrast to the second order Maclaurin series

$$e^x = 1 + x + 0.5x^2.$$

2.2 Differential Transform Method

The Differential Transform of a function $u(x)$ about the point $x = 0$ is defined [11] as

$$DT\{u(x)\} = U[k] = \frac{1}{k!}\frac{d^k}{dx^k}U(x)\bigg|_{x=0} \tag{4}$$

and the inverse transform as

$$DT^{-1}\{U[k]\} = u(x) = \sum_{k=0}^{\infty} U[k]x^k \tag{5}$$

and the Mth order approximation of $u(x)$ is defined as

$$u(x) = \sum_{k=0}^{M} U[k]x^k. \tag{6}$$

Some important differential transforms are given below;

1. $DT\{\alpha f(x) \pm \beta g(x)\} = \alpha F[k] + \beta G[k]$, and
$$DT\{f(x)g(x)\} = \sum_{r=0}^{k} F[r]G[k-r].$$
2. $D\{cx^n\} = c\delta_{n,k}$.
3. If $y(x) = \frac{d^n}{dx^n}f(x)$, then $Y[k] = \left(\prod_{r=1}^{n}(k+r)\right)F[k+n]$.
4. Initial conditions: If $\frac{d^n}{dx^n}y(x)\Big|_{x=0} = \alpha$ then $Y[n] = \frac{\alpha}{n!}$.
5. Boundary conditions: If $y(b) = \beta$ then $\beta = \sum_{k=0}^{\infty} Y[k]b^k$.

For more on differential transform, see [11] and according to the theorems proved in [11], we know that the differential transform method is applicable to Bratu equation and that the method will converge to the exact solution.

2.3 Perturbation Method

To solve the Bratu equation 3, assume a solution of the

$$u(x) = \sum_{m=0}^{\infty} \lambda^m u_m(x)$$

and obtain the functions $u_m(x)$ by substituting into the equation 3.

III. SOLUTION OF BRATU EQUATION

We approximate the nonlinearity of the Bratu equation using the Maclaurin series and the ELS scheme and the resulting problem is solved using the perturbation. The results obtained in each case are compared with the other.

3.1 Differential Transform Method of Solution of Bratu Equation

Start by assuming a second order approximation to the exponential function e^u as

$$e^u = a_0 + a_1 u + a_2 u^2,$$

Bratu equation 3 then becomes

$$u'' + \lambda(a_0 + a_1 u + a_2 u^2) = 0. \tag{7}$$

By taking the differential transform of equation 7,

$$(k+2)(k+1)U[k+2] + \lambda\left(a_0\delta_{0,k} + a_1 U[k] + a_2 \sum_{m=0}^{k} U[m]U[k-m]\right) = 0$$

which gives

$$U[2] = -\lambda\left(\frac{a_0 + a_1 U[0] + a_2 U[0]^2}{2}\right), \quad k = 0.$$

$$U[k+2] = -\lambda\left(\frac{a_1 U[k] + a_2 \sum_{m=0}^{k} U[m]U[k-m]}{(k+2)(k+1)}\right), \quad k > 0.$$

Taking the differential transform of the first boundary condition $u(0) = 0$, then $U[0] = 0$. Using the transform 5 under subsection 2.2 above and setting $U[1] = \xi$, the differential transform of the second boundary condition $u(1) = 0$ becomes

$$0=\sum_{k=0}^{\infty} U[k]. \quad (8)$$

From equation 8, the value of ξ will be estimated by considering M th partial sum. We shall take $\lambda = 3.513830719$. By adopting the Maclaurin's expansion, then $a_0 = 1$, $a_1 = 1$, $a_2 = 0.5$ and thus

$$U[2] = -\lambda\left(\frac{1+U[0]+0.5U[0]^2}{2}\right), \quad k=0.$$

$$U[k+2] = -\lambda\left(\frac{U[k]+0.5\sum_{m=0}^{k} U[m]U[k-m]}{(k+2)(k+1)}\right), \quad k > 0.$$

with the following conditions

$$U[0]=0, \qquad U[1]=\xi, 0=\sum_{k=0}^{\infty} U[k].$$

This implies that

$$U[2]=\frac{-\lambda}{2}, \qquad U[3]=-\frac{\lambda\xi}{6}, \qquad U[4]=\frac{\lambda^2}{24}-\frac{\lambda\xi^2}{24}, \quad U[5]=\frac{\lambda^2\xi}{30},$$

$$U[6]=-\frac{\lambda^3}{180}+\frac{\lambda^2\xi^2}{144}, U[7]=-\frac{19\lambda^3\xi}{5040}+\frac{\lambda^2\xi^3}{1008}, \quad U[8]=\frac{19\lambda^4\xi}{40320}-\frac{3\lambda^3\xi^2}{2240},$$

$$U[9]=\frac{83\lambda^4\xi}{181440}-\frac{5\lambda^3\xi^3}{24192}.$$

Hence,

$$u(x)=\xi x-\frac{\lambda}{2}x^2-\frac{\lambda\xi}{6}x^3+\frac{\lambda^2}{24}x^4-\frac{\lambda\xi^2}{24}x^4+\frac{\lambda^2\xi}{30}x^5-\frac{\lambda^3}{180}x^6+\frac{\lambda^2\xi^2}{144}x^6-\frac{19\lambda^3\xi}{5040}x^7+\frac{\lambda^2\xi^3}{1008}x^7.$$

By taking $\lambda = 3.513830719$ and using the condition

$$0=\sum_{k=0}^{7} U[k]$$

we obtain ξ as $\xi = 2.526$. With this value, the coefficients become

$$U[0]=0, \quad U[1]=2.526; \; U[2]=-1.75692; \; U[3]=-1.47932, U[4]=-0.419735, \; U[5]=1.03962, \quad U[6]=0.306071, \; U[7]=-0.215717$$

and the result is

$$u(x)=2.526x-1.75692x^2-1.47932x^3-0.419735x^4+1.03962x^5+0.306071x^6-0.215717x^7.$$

By using GELS approximation, we take $a_0 = 1.013$, $a_1 = 0.851$, $a_2 = 0.839$ and thus

$$U[2]=-\lambda\left(\frac{1.013+0.851U[0]+0.839U[0]^2}{2}\right), \quad k=0.$$

$$U[k+2] = -\lambda\left(\frac{0.851U[k]+0.839\sum_{m=0}^{k} U[m]U[k-m]}{(k+2)(k+1)}\right), \quad k>0.$$

with the following conditions

$$U[0]=0, \qquad U[1]=\xi, 0=\sum_{k=0}^{\infty} U[k].$$

This implies that

$$U[2]=-0.5065\lambda, \qquad U[3]=-\frac{851\lambda\xi}{6000}, \quad U[4]=0.0359193\lambda^2-\frac{839\lambda\xi^2}{12000},$$

$$U[5]=0.0485304\lambda^2\xi, \quad U[6]=-0.00819354\lambda^3+\frac{713989\lambda^2\xi^2}{72000000},$$

$$U[7]=-0.0052885\lambda^3\xi+\frac{703921\lambda^2\xi^3}{252000000},$$

Substituting $U[k]$'s and rearranging, we have

$$u(t)=\xi x-\frac{1013\lambda}{2000}x^2-\frac{851\lambda\xi}{6000}x^3+\left(0.0359193\lambda^2-\frac{839\lambda\xi^2}{12000}\right)x^4+0.0485304\lambda^2\xi x^5+\left(-0.00819354\lambda^3\frac{+713989\lambda^2\xi^2}{72000000}\right)x^6+\left(-0.0052885\lambda^3\xi+\frac{703921\lambda^2\xi^3}{252000000}\right)x^7.$$

By taking $\lambda = 3.513830719$ and using the condition

$$0=\sum_{k=0}^{7} U[k]$$

we obtain ξ as $\xi = 2.20472$. With this values, the coefficients become

$$U[0]=0, \; U[1]=2.20472; \; U[2]=-1.77976; \; U[3]=-1.09878, \; U[4]=-0.750676, \; U[5]=1.32108, \; U[6]=0.23967, \; U[7]=-0.136248$$

and the result is

$$u(x)=2.20472x-1.77976x^2-1.09878x^3-0.750676x^4+1.32108x^5+0.23967x^6-0.136248x^7$$

3.2 Perturbation Method of Solution of Bratu Equation

By substituting the series

$$u(x) = \sum_{m=0}^{\infty} \lambda^m u_m(x)$$

and the boundary conditions

$u_0(0) = 0,\ u(1) = 0, u_1(0) = u_2(0) = \cdots = 0,\ u_1(1) = u_2(1) = \cdots = 0,$

into equation7, it gives

$$\sum_{m=0}^{\infty} \lambda^m u_m'' + \lambda\left(a_0 + a_1 \sum_{m=0}^{\infty} \lambda^m u_m + a_2 \sum_{m=0}^{\infty} \lambda^m \left(\sum_{i=0}^{m} u_i u_{m-i}\right)\right) = 0$$

and by rearranging,

$$\sum_{m=1}^{\infty} \lambda^m \left(u_m'' + a_1 u_{m-1} + a_2 \sum_{i=0}^{m-1} u_i u_{m-1-i}\right) = -u_0'' - \lambda a_0.$$

Next is to comparing coefficients of both sides. The terms void of λ gives

Constant term: $u_0'' = 0, u_0(0) = 0, u_0(1) = 0,$

Coefficients of λ: $u_1'' + a_1 u_0 + a_2 u_0^2 = -a_0, u_1(0) = 0, u_1(1) = 0,$ (10)

Coefficient of λ^m:

$u_m'' + a_1 u_{m-1} + a_2 \sum_{i=0}^{m-1} u_i u_{m-1-i} = 0, u_m(0) = 0, u_m(1) = 0,\ m \geq 2.$

and solving equation 9 gives $u_0 = 0$. On substituting u_0 into equation 10 and solving,

$$u_1 = \frac{a_0}{2}(x - x^2).$$

For $m \geq 2$,

$$u_m'' + a_1 u_{m-1} + a_2 \sum_{i=0}^{m-1} u_i u_{m-1-i} = 0, u_m(0) = 0, u_m(1) = 0, \quad m \geq 2.$$

Hence, for $m = 2$,

$$u_2'' + a_1 u_1 + 2a_2 u_0 u_1 = 0, u_2(0) = 0, u_2(1) = 0. \tag{12}$$

By substituting u_0 and u_1 into equation 12, we get

$$u_2'' + \frac{a_1 a_0}{2}(1 - x^2) = 0, u_2(0) = 0, u_2(1) = 0$$
$$\Rightarrow u_2 = \frac{a_0 a_1}{24}(x^4 - 2x^3 + x)$$

For $m = 3$,

$$u_3'' + a_1 u_2 + 2a_2(2u_0 u_2 + u_1^2) = 0, u_3(0) = 0, u_3(1) = 0. \tag{13}$$

By substituting u_0, u_1 and u_2 into equation 13, we get

$$u_2'' + \frac{a_1 a_0}{2}(1 - x^2) = 0, u_2(0) = 0, u_2(1) = 0$$
$$\Rightarrow u_2 = \frac{a_0 a_1}{24}(x^4 - 2x^3 + x).$$

We continue in this manner and obtain other u_m's. Using the coefficients from Maclaurin series, we have the solution as

$$u(x) = 2.70511x - 1.75692x^2 - 1.489x^3 - 0.202139x^4 + 0.934822x^5 + 0.0236383x^6 - 0.287353x^7 + 0.0718384x^8,$$

while using the coefficients from GELS approximation, we have the solution

(9)

$$u(x) = 2.69568x - 1.77976x^2 - 1.25168x^3 - 0.722518x^4 + 1.33218x^5 + 0.0323507x^6 - 0.408353x^7 + 0.102088x^8.$$

IV. DISCUSSION OF RESULTS

Solution of the Bratu equation (11)

$$u'' + \lambda e^u = 0, 0 < x < 1, \qquad u(0) = u(1) = 0$$

has been obtained using the differential transform method and perturbation method. Using DTM with Maclaurin approximation, the result is

$$u(x) = 2.526x - 1.75692x^2 - 1.47932x^3 - 0.419735x^4 + 1.03962x^5 + 0.306071x^6 - 0.215717x^7$$

while DTM with GELS approximation gives

$$u(x) = 2.20472x - 1.77976x^2 - 1.09878x^3 - 0.750676x^4 + 1.32108x^5 + 0.23967x^6 - 0.136248x^7.$$

x	Exact	Mac	GELS	Absolute Error (Mac)	Absolute Error (GELS)
0.1	0.201598449	0.23352	0.201514	0.031921551	0.0000844489
0.2	0.360247143	0.422767	0.360198	0.062519857	0.0000491426
0.3	0.468653927	0.559039	0.468844	0.090385073	0.000190073
0.4	0.521145762	0.635418	0.521872	0.114272238	0.000726238
0.5	0.51475311	0.648208	0.517118	0.13345489	0.00236489
0.6	0.449845762	0.598262	0.457588	0.148416238	0.007742238

Table 1: Comparison of the results from Differential Transform Method.

By solving the Bratu equation using perturbation method with Maclaurin approximation gives

$$u(x) = 2.70511x - 1.75692x^2 - 1.489x^3 - 0.202139x^4 + 0.934822x^5 + 0.0236383x^6 - 0.287353x^7 + 0.0718384x^8,$$

while perturbation method with GELSA scheme gives

$$u(x) = 2.69568x - 1.77976x^2 - 1.25168x^3 - 0.722518x^4 + 1.33218x^5 + 0.0323507x^6 - 0.408353x^7 + 0.102088x^8.$$

x	Exact	Mac	GELS	Absolute Error (Mac)	Absolute Error (GELS)
0.1	0.251308186	0.251442	0.25046	0.000133814	0.000848186
0.2	0.457410925	0.458807	0.457199	0.001396075	0.000211925
0.3	0.608103751	0.613796	0.612049	0.005692249	0.003945249
0.4	0.694518207	0.709665	0.708011	0.015146793	0.013492793
0.5	0.710953024	0.741904	0.740228	0.030950976	0.029274976
0.6	0.656268973	0.708504	0.706362	0.052235027	0.050093027

Table 2: Comparison of the results from Perturbation Method.

The results are shown in tables (1) and (2). It is clear from the tables that the DTM and Perturbation methods produce better accuracy when the GELS approximation is adopted than when the Maclaurin's expansion is used.

V. CONCLUSION

The Gelfand-Bratu equation is solved using the differential transform and perturbation methods. In each case, the nonlinear term e^u in the problem is approximated using Maclaurin's series and the Generalized Euclidean Least Square (GELS) Approximation. It was revealed that the two methods give better results when the GELS approximation is adopted. This paper has demonstrated that if accuracy is of prime importance, then GELS approximation should be adopted to approximate rather than Maclaurin series. Nevertheless, because of the rigor involved in obtaining the approximation, Maclaurin series can be adopted when elegance is of priority above accuracy.

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