Second Degree SOR (SDSOR) Method

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Abstract: **In this paper, we discuss and study the second degree successive over relaxation (SDSOR) method for the solution of linear systems when the eigenvalues of SOR matrix are real. Few examples are considered to show the transcendency of this developed method.**

Keywords: **First degree iterative method, Second degree iterative method, SOR, Gauss-Seidal, Jacobi**

I. INTRODUCTION

Let us consider the system of linear equations of the form

$$AX = b \qquad \dots (1.1)$$

where A is non-singular with non-vanishing diagonal elements and a positive definite matrix of order $n \times n$, X and b are unknown and known n-dimensional vectors. We split the coefficient matrix A without any loss of generality, as

$$A = I - L - U \qquad \dots (1.2)$$

where I is the unit matrix, L and U are strictly lower and upper triangular parts of A.

The linear stationary first degree Successive over relaxation (SOR) for the solution of (1.1) is given by

$$X^{(k+1)} = L_\omega X^{(k)} + \omega\left(I-\omega L\right)^{-1} b \qquad \left(k=0,1,2...\right) \qquad \dots (1.3)$$

Here,

$$L_\omega = \left(I - \omega L\right)^{-1}\left[\left(1-\omega\right)I + \omega U\right] \qquad \dots (1.4)$$

and L_ω is the iteration matrix of SOR method.

If $\bar{\rho}$ be the spectral radius of L_ω, then

$$\bar{\rho} = \frac{\bar{\mu}^2}{\left(1+\sqrt{1-\bar{\mu}^2}\right)^2} \qquad \dots (1.5)$$

where $\bar{\mu}$ is the spectral radius of Jacobi iteration matrix $J = L + U$.

II. SECOND DEGREE SOR (SDSOR) METHOD

The linear stationary second degree method [2] is given by

$$X^{(k+1)} = X^{(k)} + \alpha\left(X^{(k)} - X^{(k-1)}\right) + \beta\left(X^{(k+1)} - X^{(k)}\right) \dots (2.1)$$

Where α and β are computational parameters.

Substitute (1.3) in (2.1)

$$X^{(k+1)} = X^{(k)} + \alpha\left(X^{(k)} - X^{(k-1)}\right) + \beta\left(L_\omega X^{(k)} + \omega\left(I-\omega L\right)^{-1} b - X^{(k)}\right)$$

$$\Rightarrow X^{(k+1)} = \left[\left(\alpha - \beta + 1\right)I + \beta L_\omega\right]X^{(k)} - \alpha I X^{(k-1)} + \omega\beta\left(I - \omega L\right)^{-1} b$$

(or)

$$X^{(k+1)} = PX^{(k)} + QX^{(k-1)} + R \dots (2.2)$$

where

$$P = \left(\alpha - \beta + 1\right)I + \beta L_\omega \qquad \dots (2.3)$$

$$Q = -\alpha I \qquad \dots (2.4)$$

$$R = \omega\beta\left(I - \omega L\right)^{-1} b \qquad \dots (2.5)$$

By using the thoery given in young [2] and discussed in [6], we have

$$\begin{pmatrix} X^{(k)} \\ X^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0 & I \\ Q & P \end{pmatrix}\begin{pmatrix} X^{(k-1)} \\ X^{(k)} \end{pmatrix} + \begin{pmatrix} 0 \\ R \end{pmatrix} \dots (2.6)$$

Necessary and sufficient condition for the convergence of method (2.2) for any $X^{(0)}$ and $X^{(1)}$ is that spectral radius of $\begin{pmatrix} 0 & I \\ Q & P \end{pmatrix}$ must be less than unity in magnitude.

Let $$T = \begin{pmatrix} 0 & I \\ Q & P \end{pmatrix} \qquad \dots (2.7)$$

Then, Spectral radius of iteration matrix T is less than one if and only if all roots of λ of

$$\left|\lambda^2 I - \lambda P - Q\right| = 0 \qquad \dots (2.8)$$

are less then unity in modulus.

Substituting P and Q in (2.8)

$$\left|\lambda^2 I-\lambda\left[(\alpha-\beta+1)I+\beta L_\omega\right]+\alpha I\right|=0$$

$$\Rightarrow\left|L_\omega+\left(\frac{\alpha-\beta+1}{\beta}\right)I-\left(\frac{\lambda^2+\alpha}{\lambda\beta}\right)I\right|=0 \quad \ldots (2.9)$$

The eigenvalues λ of T are related to the eigenvalues ρ of L_ω by

$$\rho+\frac{(\alpha-\beta+1)}{\beta}=\frac{\lambda^2+\alpha}{\lambda\beta} \quad \ldots (2.10)$$

Let $\lambda=Ve^{i\theta}$ … (2.11)

Putting (2.11) in (2.10), we obtain

$$\rho+\frac{(\alpha-\beta+1)}{\beta}=\frac{\left(Ve^{i\theta}\right)^2+\alpha}{\left(Ve^{i\theta}\right)\beta}$$

$$\Rightarrow\rho+\frac{(\alpha-\beta+1)}{\beta}=\frac{V^2e^{2i\theta}+\alpha}{\left(Ve^{i\theta}\right)\beta}$$

$$\Rightarrow\rho+\frac{(\alpha-\beta+1)}{\beta}=\frac{Ve^{i\theta}}{\beta}+\frac{\alpha e^{-i\theta}}{V\beta}$$

$$\Rightarrow\rho+\frac{(\alpha-\beta+1)}{\beta}=\frac{V(\cos(\theta)+i\sin(\theta))}{\beta}+\frac{\alpha(\cos(\theta)-i\sin(\theta))}{V\beta}$$

$$\Rightarrow\rho=-\frac{(\alpha-\beta+1)}{\beta}+\left(\frac{V}{\beta}+\frac{\alpha}{V\beta}\right)\cos(\theta)+i\left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)\sin(\theta) \quad \ldots (2.12)$$

On comparison

$$\operatorname{Re}\rho=-\frac{(\alpha-\beta+1)}{\beta}+\left(\frac{V}{\beta}+\frac{\alpha}{V\beta}\right)\cos(\theta)$$

$$\Rightarrow\cos(\theta)=\frac{\operatorname{Re}\rho+\frac{(\alpha-\beta+1)}{\beta}}{\left(\frac{V}{\beta}+\frac{\alpha}{V\beta}\right)} \quad \ldots (2.13)$$

and

$$\operatorname{Im}\rho=\left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)\sin(\theta)$$

$$\Rightarrow\sin(\theta)=\frac{\operatorname{Im}\rho}{\left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)} \quad \ldots (2.14)$$

Squaring and adding of (2.13) and (2.14), we get

$$\left[\frac{\operatorname{Re}\rho+\frac{(\alpha-\beta+1)}{\beta}}{\left(\frac{V}{\beta}+\frac{\alpha}{V\beta}\right)}\right]^2+\left[\frac{\operatorname{Im}\rho}{\left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)}\right]^2=1$$

$$\Rightarrow\frac{\left[\operatorname{Re}\rho+\frac{(\alpha-\beta+1)}{\beta}\right]^2}{\left(\frac{V}{\beta}+\frac{\alpha}{V\beta}\right)^2}+\frac{\left[\operatorname{Im}\rho\right]^2}{\left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)^2}=1 \quad \ldots (2.15)$$

is an ellipse with

$$\text{Centre}=\left(-\frac{(\alpha-\beta+1)}{\beta},0\right) \quad \ldots (2.16)$$

$$\text{Foci}=(\xi_1,0)=\left(-\frac{(\alpha-\beta+1)}{\beta}-\frac{2\sqrt{\alpha}}{\beta},0\right) \quad \ldots (2.17)$$

$$\text{Foci}=(\xi_2,0)=\left(-\frac{(\alpha-\beta+1)}{\beta}+\frac{2\sqrt{\alpha}}{\beta},0\right) \quad \ldots (2.18)$$

Theorem 1:

If the eigenvalues of ρ of L_ω are real and lie in the interval $\zeta_1\le\rho\le\zeta_2<1$, then the optimal choices of α and β satisfy the following conditions.

(i) $V^2=\alpha$

(ii) $\frac{\zeta_1+\zeta_2}{2}=-\frac{\alpha-\beta+1}{\beta}$

(iii) $\frac{\zeta_2-\zeta_1}{2}=\frac{2V}{\beta}$

(iv) $2V=\frac{(\zeta_2-\zeta_1)}{2-(\zeta_1+\zeta_2)}\left(1+V^2\right)$

Proof:

(i) If ρ is real then $\operatorname{Im}\rho=0$

$$\Rightarrow \left(\frac{V}{\beta}-\frac{\alpha}{V\beta}\right)\sin(\theta)=0$$

$$\Rightarrow V^2=\alpha$$

(ii) From (2.17) and (2.18)

$$\xi_1=-\frac{(\alpha-\beta+1)}{\beta}-\frac{2\sqrt{\alpha}}{\beta}$$

$$\xi_2=-\frac{(\alpha-\beta+1)}{\beta}+\frac{2\sqrt{\alpha}}{\beta}$$

$$\xi_1+\xi_2=-\frac{(\alpha-\beta+1)}{\beta}-\frac{2\sqrt{\alpha}}{\beta}-\frac{(\alpha-\beta+1)}{\beta}+\frac{2\sqrt{\alpha}}{\beta}$$

$$=-\frac{2(\alpha-\beta+1)}{\beta}$$

Therefore, $\dfrac{\xi_1+\xi_2}{2}=-\dfrac{(\alpha-\beta+1)}{\beta}$

(iii)

$$\xi_2-\xi_1=-\frac{(\alpha-\beta+1)}{\beta}+\frac{2\sqrt{\alpha}}{\beta}-\left(-\frac{(\alpha-\beta+1)}{\beta}-\frac{2\sqrt{\alpha}}{\beta}\right)$$

$$=\frac{4\sqrt{\alpha}}{\beta}$$

$$=\frac{4V}{\beta}$$

Therefore, $\dfrac{\xi_2-\xi_1}{2}=\dfrac{2V}{\beta}$

(iv)

$$\frac{\zeta_1+\zeta_2}{2}=-\frac{\alpha-\beta+1}{\beta}$$

$$\Rightarrow -\frac{\zeta_1+\zeta_2}{2}=\frac{\alpha-\beta+1}{\beta}$$

$$\Rightarrow 1-\frac{\zeta_1+\zeta_2}{2}=1+\frac{\alpha-\beta+1}{\beta}$$

$$\Rightarrow \frac{2-(\zeta_1+\zeta_2)}{2}=\frac{\beta+\alpha-\beta+1}{\beta}$$

$$\Rightarrow \frac{2-(\zeta_1+\zeta_2)}{2}=\frac{1+\alpha}{\beta}$$

Divide (iii) of Theorem 1 by the above inequality, we have

$$\frac{\frac{\xi_2-\xi_1}{2}}{\frac{2-(\zeta_1+\zeta_2)}{2}}=\frac{\frac{2V}{\beta}}{\frac{1+\alpha}{\beta}}$$

$$\Rightarrow \frac{\xi_2-\xi_1}{2-(\zeta_1+\zeta_2)}=\frac{2V}{1+\alpha}$$

$$\Rightarrow 2V=\left(\frac{\xi_2-\xi_1}{2-(\zeta_1+\zeta_2)}\right)(1+V^2)$$

Theorem 2: If the eigenvalues ρ of L_ω lie in the interval

$\xi_1\le\rho\le\xi_2<1$ and $-\xi_1=\xi_2=\overline{\rho}$

then the optimum values of the parameters in (2.2) are

$$\alpha=\frac{\overline{\rho}^2}{\left(1+\sqrt{1-\overline{\rho}^2}\right)^2} \quad \text{and} \quad \beta=\frac{2}{\left[1+\sqrt{1-\overline{\rho}^2}\right]}$$

Proof:

From (iv) of theorem 1, we have

$$2V=\left(\frac{\xi_2-\xi_1}{2-(\zeta_1+\zeta_2)}\right)(1+V^2)$$

If $\overline{\rho}$ be the spectral radius of L_ω, then

$$\Rightarrow 2V=\overline{\rho}(1+V^2)$$

$$\Rightarrow \overline{\rho}V^2-2V+\overline{\rho}=0$$

$$\Rightarrow V=\frac{2\pm\sqrt{4-4\overline{\rho}^2}}{2\overline{\rho}}$$

$$V=\frac{1\pm\sqrt{1-\overline{\rho}^2}}{\overline{\rho}}$$

Let $1+V^2=\dfrac{2}{1+\sqrt{1-\overline{\rho}^2}}$

Since $V^2 = \alpha$,

$$\alpha = \frac{2}{1+\sqrt{1-\bar{\rho}^2}} - 1$$

$$\Rightarrow \alpha = \frac{1-\sqrt{1-\bar{\rho}^2}}{1+\sqrt{1-\bar{\rho}^2}}$$

$$= \frac{\bar{\rho}^2}{\left(1+\sqrt{1-\bar{\rho}^2}\right)^2}$$

From(iii) of Theorem 1,

$$\frac{\xi_2 - \xi_1}{2} = \frac{2V}{\beta}$$

$$\Rightarrow \beta = \frac{2(2V)}{\zeta_2 - \zeta_1} = \left(\frac{2}{\zeta_2 - \zeta_1}\right)\left(\frac{\xi_2 - \xi_1}{2-(\zeta_1+\zeta_2)}\right)\left(\frac{2}{1+\sqrt{1-\bar{\rho}^2}}\right)$$

$$= \frac{4}{\left(2-(\zeta_1+\zeta_2)\right)\left(1+\sqrt{1-\bar{\rho}^2}\right)}$$

Since $-\xi_1 = \xi_2$,

$$\beta = \frac{2}{\left(1+\sqrt{1-\bar{\rho}^2}\right)}$$

Therefore, the spectral radius of T is $\alpha^{\frac{1}{2}} = \dfrac{\bar{\rho}}{\left(1+\sqrt{1-\bar{\rho}^2}\right)}$

III. NUMERICAL EXAMPLES

In this section, we consider two linear positive definite systems of the form $AX = b$ and obtain their solutions by using first and second degree Jacobi, Gauss-Seidal and SOR methods up to an error not more than 0.5×10^{-10}

taking null vector as a initial guess to get the exact solutions (1,1) and (1,1,1) of respective systems. We tabulate the obtained results here under.

Example 1:

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

S.N0	Method	Spectral radius	Number of iterations taken to achieve the solution
1	Jacobi	0.5	35
	Second degree Jacobi	0.2679	22
2	Gauss-Seidal	0.25	19
	Second degree Gauss-Seidal	0.1270	15
3	SOR	0.0718	12
	Second degree SOR	0.0359	11

Example 2:

Let $A = \begin{bmatrix} 20 & 1 & 0 \\ 6 & 10 & -1 \\ 0 & -1 & 18 \end{bmatrix}$ and $b = \begin{bmatrix} 21 \\ 15 \\ 17 \end{bmatrix}$

S.N0	Method	Spectral radius	Number of iterations taken to achieve the solution
1	Jacobi	0.1886	16
	Second degree Jacobi	0.0.0951	13
2	Gauss-Seidal	0.0356	9
	Second degree Gauss-Seidal	0.0178	8
3	SOR	0.0091	8
	Second degree SOR	0.0045	7

IV. CONCLUSION

As seen in the tabulated results that second degree SOR method has less spectral radius compared to the other methods and hence SDSOR method has greater rate of convergence.

REFERENCES

[1]. David M.Young, Second Degree Iterative Methods for the solution of Large Linear Systems, Center for Numerical Analysis University of Texas, Austin, October 9,1970.

[2]. D.M. Young, Linear Solution of Large Linear systems, Academic press, New York and London 1971.

[3]. David M. Young and David R. Kincard, Linear Stationary Second Degree Methods for the solution of Large Linear Systems, (July 9, 1990)

[4]. David R. Kincard and David M. Young, Stationary Second Iterative Methods and Recurrences, Elsevier Science Publisher B.V.North-Holland,1992.

[5]. T.A.Manteuffel, Optimal Parameters for Linear Second Degree Stationary Iterative Methods, Vol.4, Aug.1982.

[6]. S.R. Koneru & V.B.Kumar Vatti (1987) Extrapolated accelerated gauss-seidal methods, International Journal of Computer Mathematics, 21:3-4,311-318,DOI:10.1080/00207168708803573.