The p-k Extended Mittag-Leffler Function and Marichev-Saigo-Maeda Fractional Operators

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***Abstract*- In present paper we find some important compositions of p-k extended Mittag-Leffler function with their special cases by using Marichev-Saigo-Maeda differential and integral operators. These results are further expressed with relation of p-k extended Mittag-Leffler function with Fox-H function and Wright hypergeometric function.In last we obtained some special cases of these functions.**

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***Keywords*- The p-k extended Mittag-Leffler function, Fox H function, Marichev-Saigo-Maeda operators, Pochhammer Smbol, Wright hypergeometric function, Two parameter Gamma function.**

I. INTRODUCTION AND DEFINITIONS

The two parameter Pochhammer symbol is recently given by [2](equation 2.1)

1.1 Definition: Two Parameter Pochhammer Symbol

Let $x \in C; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p-k Pochhammer Symbol (i.e Two Parameter Pochhammer Symbol), ${}_p(x)_{n,k}$ is given by

$$ {}_p(x)_{n,k} = \left(\frac{xp}{k}\right)\left(\frac{xp}{k}+p\right)\left(\frac{xp}{k}+2p\right)\ldots\ldots\ldots\left(\frac{xp}{k}+(n-1)p\right). \tag{1.1}$$

1.2 Definition:Two Parameter Gamma Function

Forx $\in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n \in N$, the p-k Gamma Function (i.e Two Parameter Gamma Function), ${}_p\Gamma_k(x)$is given by [2] (equation 2.6,2.7 and 2.14)

$$ {}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}} \tag{1.2}$$

Or

$$ {}_p\Gamma_k(x) = \frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{x}{k}-1}}{{}_p(x)_{n,k}} \tag{1.3}$$

The integral representation of p-k Gamma Function is given by,

$$ {}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}}\, t^{x-1}dt. \tag{1.4}$$

1.3 Definition: p-k Extended Mittag-Leffler Function

Let $k, p \in R^+ - \{0\}; \lambda, \mu, \eta \in C/kZ^-; Re(\mu) > 0, Re(\eta) > 0, Re(\lambda) > 0, and\ q \in (0,1) \cup N.$

The p-k Mittag-Leffler function denoted by ${}_pE_{k,\mu,\eta}^{\lambda,q}(z)$ and defined as

$$ {}_pE_{k,\mu,\eta}^{\lambda,q}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\lambda)_{nq,k}}{{}_p\Gamma_k(n\mu+\eta)}\frac{z^n}{n!} \tag{1.5}$$

Where ${}_p(\lambda)_{nq,k}$ is the two parameter Pochhammer symbol given by equation (1.1) and ${}_p\Gamma_k(x)$ is two parameter Gamma function given by equation (1.2 and 1.3).

1.4 Definition: Marichev-Saigo-Maeda fractional Operators

Saigo Operators

The fractional integral and differential operators with Gauss Hyper geometric functions as kernels, were introduced by Saigo[11], which are interesting generalizations of the classical Riemann Liouville and Erdelyi-Kober fractional operators(see [8])(also see [4] and [5]).For $\alpha, \beta, \gamma \epsilon C$ and $x \epsilon R^+$ with $Re(\alpha) > 0$, the left-hand and the right-hand sided generalized fractional integral operators associated with Gauss hypergeometric function are defined by [11]

$$ \left(I_{0+}^{\alpha,\beta,\gamma}f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-txftdt\right) \tag{1.6}$$

And

$$ \left(I_{-}^{\alpha,\beta,\gamma}f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \frac{(t-x)^{\alpha-1}}{t^{\alpha+\beta}}\ {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-xtftdt\right) \tag{1.7}$$

respectively. Here, ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function [8] defined for $z \epsilon C, |z| < 1$ and $\alpha, \beta \epsilon C, \gamma \epsilon C \backslash Z_0^-$ by

$$ {}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\frac{z^n}{n!},$$

where$(z)_n = (z)_{n,1}$. The corresponding fractional differential operators are

$$ \left(D_{0+}^{\alpha,\beta,\gamma}f\right)(x) = \left(\frac{d}{dx}\right)^l \left(I_{0+}^{-\alpha+l,-\beta-l,\alpha+\gamma-l}f\right)(x) \tag{1.8}$$

And

$$ \left(D_{-}^{\alpha,\beta,\gamma}f\right)(x) = \left(-\frac{d}{dx}\right)^l \left(I_{-}^{-\alpha+l,-\beta-l,\alpha+\gamma}f\right)(x), \tag{1.9}$$

where$l=[Re(\alpha)]+1$ and $[Re(\alpha)]$ denotes the integer part of $Re(\alpha)$.

MSM Operators

Marichev [9] introduced the generalization of the Saigo operators as Mellin type convolution operators with Appell function as the kernel, which were later extended and studied by Saigo and Maeda [10]. For $\alpha,\alpha',\beta,\beta',\gamma\in C$ and $x\in R^+$ with $Re(\gamma)>0$, the left and right hand sided Marichev–Saigo-Maeda (MSM) fractional integral operators associated with third Appell function are defined by

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x\frac{(x-t)^{\gamma-1}}{t^{\alpha'}}\mathrm{F}_3\left(\alpha,\alpha',\beta,\beta',\gamma;1-tx,1-xtft\,dt\right.\qquad(1.10)$$

and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty\frac{(t-x)^{\gamma-1}}{t^{\alpha}}\mathrm{F}_3\left(\alpha,\alpha',\beta,\beta',\gamma;1-xt,1-txft\,dt\right.\qquad(1.11)$$

respectively. The corresponding fractional differential operators are

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(\frac{d}{dx}\right)^m\left(I_{0+}^{-\alpha',-\alpha,-\beta'+m,-\beta,-\gamma+m}f\right)(x)\qquad(1.12)$$

and

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(-\frac{d}{dx}\right)^m\left(I_{-}^{-\alpha',-\alpha,-\beta'+m,-\gamma+m}f\right)(x)\quad,\qquad(1.13)$$

where $m=[Re(\gamma)]+1$ .The relation between the MSM fractional operators are given as;

$$\left(I_{0+}^{\alpha,0,\beta,\beta',\gamma}f\right)(x)=\left(I_{0+}^{\gamma,\alpha-\gamma,-\beta}f\right)(x),\qquad(1.14)$$

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{+}^{0,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\qquad(1.15)$$

and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(I_{-}^{\gamma,\alpha-\gamma,-\beta}f\right)(x),\qquad(1.16)$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma}f\right)(x)=\left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma}f\right)(x)\qquad(1.17)$$

1.5 Definition: Generalized Wright Hypergeometric Function:

Let $\alpha_i,\beta_j\,\epsilon R\backslash\{0\}$ and $a_i,b_j\,\epsilon\,C,i=1,2,3\ldots..p\,;j=1,2,3\ldots..q$ then the generalized Wright function is defined by

$${}_{\mathrm{p}}\Psi_q(z)=\ {}_{\mathrm{p}}\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots\ldots..(\alpha_p,A_p)\\(\beta_1,B_1),\ldots\ldots\ldots(\beta_q,B_q)\end{matrix}\middle|z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(\alpha_i+nA_i)}{\prod_{j=1}^{q}\Gamma(\beta_j+nB_j)}\frac{z^n}{n!},\quad z\epsilon C,\qquad(1.18)$$

Where $\Gamma(z)$ is the Gamma function. This function was introduced by Wright [12] and the conditions for its existence along with its representation in terms of Mellin-Bernes integral were established by Kilbas*et al.*[7].

$${}_{\mathrm{p}}\Psi_q\left[\begin{matrix}(\alpha_1,A_1),\ldots\ldots..(\alpha_p,A_p)\\(\beta_1,B_1),\ldots\ldots\ldots(\beta_q,B_q)\end{matrix}\middle|z\right]=H_{p,q+1}^{1,p}\left[-z\middle|\begin{matrix}(1-\alpha_1,A_1),\ldots\ldots..(1-\alpha_p,A_p)\\(0,1),(1-\beta_1,B_1),\ldots\ldots\ldots(1-\beta_q,B_q)\end{matrix}\right]\qquad(1.19)$$

Where $H_{p,q}^{m,n}[.]$denotes the Fox H-fuction ([1] and [3]).

II. REQUIRED RESULTS

The following MSM integral operators are required here [10, p. 394] to obtain the MSM fractional integration of p-k extended Mittag-Leffler function ${}_pE_{k,\mu,\eta}^{\lambda,q}(z)$

Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\,\epsilon\,C$ such that $Re(\alpha)>0$.

(a) If $Re(\rho)>max\{0,Re(\alpha'-\beta'),Re(\alpha+\alpha'+\beta-\gamma,$ then

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(t)=\frac{\Gamma(\rho)\Gamma(-\alpha'+\beta'+\rho)}{\Gamma(\beta'+\rho)\Gamma(-\alpha-\alpha'+\gamma+\rho)}\times\frac{\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho)}{\Gamma(-\alpha'-\beta+\gamma+\rho)}t^{-\alpha-\alpha'+\gamma+\rho-1}\qquad(2.1)$$

(b) If $Re(\rho)>max\{Re(\beta),Re(-\alpha-\alpha'+\gamma,Re-\alpha-\beta'+\gamma,$ then

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho}\right)(t)=\frac{\Gamma(-\beta+\rho)\Gamma(\alpha+\alpha'-\gamma+\rho)}{\Gamma(\rho)\Gamma(\alpha-\beta+\rho)}\times\frac{\Gamma(\alpha+\beta'-\gamma+\rho)}{\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho)}t^{-\alpha-\alpha'+\gamma-\rho}\qquad(2.2)$$

To obtain the MSM fractional differentiation of the generalized special function ${}_pE_{k,\mu,\eta}^{\lambda,q}(z)$

, following results will be used [6].

Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\epsilon C$.

(a) If $Re(\rho)>max\{0,Re(-\alpha+\beta),Re(-\alpha-\alpha'-\beta'+\gamma,$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\right)(t)=\frac{\Gamma(\rho)\Gamma(-\beta+\alpha+\rho)}{\Gamma(-\beta+\rho)\Gamma(\alpha+\alpha'-\gamma+\rho)}\frac{\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho)}{\Gamma(\alpha+\beta'-\gamma+\rho)}t^{\alpha+\alpha'-\gamma+\rho-1}\qquad(2.3)$$

(b) $Re(\rho)>max\{Re(-\beta'),Re(\alpha'+\beta-\gamma),Re(\alpha+\alpha'-\gamma+Re\gamma+1,$ *then*

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho}\right)(t)=\frac{\Gamma(\beta'+\rho)\Gamma(-\alpha-\alpha'+\gamma+\rho)}{\Gamma(\rho)\Gamma(-\alpha'+\beta'+\rho)}\frac{\Gamma(-\alpha'-\beta+\gamma+\rho)}{\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho)}t^{\alpha+\alpha'-\gamma-\rho}\qquad(2.4)$$

The relation between classical Pochhammer symbol and two parameter Pochhammer symbol are given as[2]

Proposition 1. Let $x\in C/kZ^-;k,p,r,s\in R^+-\{0\}$ and $Re(x)>0,n\in N$, then the following identity holds,

$${}_r\Gamma_s(x)=\frac{k}{s}\left(\frac{r}{p}\right)^{\frac{x}{s}}{}_p\Gamma_k\left(\frac{kx}{s}\right)\qquad(2.5)$$

And particular case,

$${}_r\Gamma_k(x)=\left(\frac{r}{p}\right)^{\frac{x}{k}}{}_p\Gamma_k(x)$$

Also ${}_p\Gamma_k(x)$ is further related to k- Gamma function and classical Gamma function as ${}_p\Gamma_k(x) = \left(\frac{p}{k}\right)^{\frac{x}{k}} \Gamma_k(x) = \frac{p^{\frac{x}{k}}}{k} \Gamma\left(\frac{x}{k}\right)$ (2.6)

Proposition 2. Let $x \in C/kZ^-; k, p \in R^+ - \{0\}$ and $Re(x) > 0, n, q \in N$, then the following identity holds,

$$ {}_p(x)_{nq,k} = \left(\frac{p}{k}\right)^{nq} (x)_{nq,k} \tag{2.7}$$

And particular case,

$$ {}_p(x)_{nq,k} = (p)^{nq} \left(\frac{x}{k}\right)_{nq} \tag{2.8}$$

III. MAIN RESULTS

In this section we present the formula for the MSM fractional integration and differentiation operators associated with the p-k extended Mittag-Leffler function ${}_pE_{k,\mu,\eta}^{\lambda,q}(z)$.

Theorem 3.1 Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \epsilon C$ such that $Re(\alpha) > 0$, $k, p \in R^+ - \{0\}; \lambda, \mu, \eta \in C/kZ^-; Re(\mu) > 0, Re(\eta) > 0, Re(\lambda) > 0, and\ q \in (0,1) \cup N$. If $Re(\rho) > max\{0, Re(\alpha' - \beta'), Re(\alpha + \alpha' + \beta - \gamma)\}$, then for $t > 0$

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) =$$

$$\frac{k\, t^{-\alpha-\alpha'+\gamma+\rho-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)} \; {}_4\Psi_4\left[\begin{matrix} \left(\frac{\lambda}{k}, q\right), (\rho,1), (-\alpha'+\beta'+\rho,1), (-\alpha-\alpha'-\beta+\gamma+\rho,1) \\ \left(\frac{\eta}{k}, \frac{\mu}{k}\right), (\beta'+\rho,1), (-\alpha-\alpha'+\rho+\gamma,1), (-\alpha'-\beta+\gamma+\rho,1) \end{matrix}; t\, p^{q-\frac{\mu}{k}}\right] \tag{3.1}$$

Proof: On using (1.5) and taking left hand side MSM fractional integral operator inside the summation

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \sum_{n=0}^{\infty} \frac{{}_p(\lambda)_{nq,k}}{{}_p\Gamma_k(n\mu+\eta)} \frac{1}{n!} \left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{n+\rho-1}\right)$$

Making use of (2.1) ,(2.6) and (2.8)

$$= t^{-\alpha-\alpha'+\gamma+\rho-1} \sum_{n=0}^{\infty} \frac{(p)^{nq}\left(\frac{\lambda}{k}\right)_{nq}}{\left(\frac{p}{k}\right)^{\frac{n\mu+\eta}{k}} \Gamma_k(n\mu+\eta)} \frac{\Gamma(\rho+n)\Gamma(-\alpha'+\beta'+\rho+n)}{\Gamma(\beta'+\rho+n)\Gamma(-\alpha-\alpha'+\gamma+\rho+n)} \times \frac{\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho+n)}{\Gamma(-\alpha'-\beta+\gamma+\rho+n)} \frac{t^n}{n!}$$

$$= \frac{k\, t^{-\alpha-\alpha'+\gamma+\rho-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\lambda}{k}+n\right)}{\Gamma_k\left(\frac{n\mu+\eta}{k}\right)} \frac{\Gamma(\rho+n)\Gamma(-\alpha'+\beta'+\rho+n)}{\Gamma(\beta'+\rho+n)\Gamma(-\alpha-\alpha'+\gamma+\rho+n)} \times \frac{\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho+n)}{\Gamma(-\alpha'-\beta+\gamma+\rho+n)} \frac{t^n}{n!}$$

The required result of (3.1) can be obtained by using (1.18)

The right hand side expression of (3.1) is further reduced in Fox-H function as follows

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{k\, t^{-\alpha-\alpha'+\gamma+\rho-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}$$

$$\times H_{4,5}^{1,4}\left[-t\, p^{q-\frac{\mu}{k}}; \begin{matrix} \left(1-\frac{\lambda}{k}, q\right), (1-\rho,1), (1+\alpha'-\beta'-\rho,1), (1+\alpha+\alpha'+\beta-\gamma-\rho,1) \\ (0,1), \left(1-\frac{\eta}{k}, \frac{\mu}{k}\right), (1-\beta'-\rho,1), (1+\alpha+\alpha'-\rho-\gamma,1), (1+\alpha'+\beta-\gamma-\rho,1) \end{matrix}\right]$$

Corollary 3.1.1 The result in (3.1) is reduced to another Fox H function by setting $p = k, q = 1, k = 1, \lambda = 1, \eta = 1\ and\ \rho = 1$

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\right)\,{}_1E_{1,\mu,1}^{1,1}(t)=t^{-\alpha-\alpha'+\gamma}H_{3,4}^{1,3}\left[-t;\begin{matrix}(0,1),(\alpha'-\beta',1),(\alpha+\alpha'+\beta'-\gamma,1)\\(0,\mu),(-\beta',1),(\alpha+\alpha'-\gamma,1),(\alpha'+\beta-\gamma,1)\end{matrix}\right]$$

on setting $\mu=1$ in (3.1.1) following result can be obtained

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\right)\,{}_1E_{1,1,1}^{1,1}(t)=t^{-\alpha-\alpha'+\gamma}\mathfrak{C}\ \ {}_3F_3\left[\begin{matrix}1,1-\alpha'+\beta',1-\alpha-\alpha'-\beta'+\gamma\\1+\beta',1-\alpha-\alpha'+\gamma,1-\alpha'-\beta+\gamma'\end{matrix};t\right]$$

Here $\mathfrak{C}=\frac{\Gamma(1-\alpha'+\beta')\Gamma(1-\alpha-\alpha'-\beta'+\gamma)}{\Gamma(1+\beta')\Gamma(1-\alpha-\alpha'+\gamma)\Gamma(1-\alpha'-\beta+\gamma)}$

In view of (1.14), the corresponding result of (3.1) for the operator (1.6) is as follows

Corollary 3.1.2 Let $\alpha,\beta,\gamma,\rho\ \epsilon C$ such that $Re(\alpha)>0,\ k,p\in R^{+}-\{0\};\ \lambda,\mu,\eta\in C/kZ^{-};Re(\mu)>0,Re(\eta)>0,Re(\lambda)>0, and\ q\in(0,1)\cup N$, then for $t>0$,

$$\left(I_{0+}^{\alpha,\beta,\gamma}t^{\rho-1}\right)\,{}_pE_{k,\mu,\eta}^{\lambda,q}(t)=\frac{kt^{-\beta+\rho-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}\ {}_3\Psi_3\left[\begin{matrix}\left(\frac{\lambda}{k},q\right),(\rho,1),(-\beta+\gamma+\rho,1)\\\left(\frac{\eta}{k},\frac{\mu}{k}\right),(-\beta+\rho,1)(\gamma+\alpha+\rho,1)\end{matrix};tp^{q-\frac{\mu}{k}}\right]$$

Theorem 3.2 Let $\alpha,\alpha',\beta,\beta',\gamma,\rho\ \epsilon C$such that $Re(\alpha)>0,\ k,p\in R^{+}-\{0\};\lambda,\mu,\eta\in C/kZ^{-};Re(\mu)>0,Re(\eta)>0,Re(\lambda)>0, and\ q\in(0,1)\cup N$. If $Re(\rho)>max\{0,Re(\alpha'-\beta'),Re(\alpha+\alpha'+\beta-\gamma)\}$, then for $t>0$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho}\right)\,{}_pE_{k,\mu,\eta}^{\lambda,q}(t)=\frac{k\,t^{-\alpha-\alpha'+\gamma-\rho}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}$$

$$\times{}_4\Psi_4\left[\begin{matrix}\left(\frac{\lambda}{k},q\right)\ \ ,(-\beta+\rho,-1),(\alpha+\alpha'-\gamma+\rho,-1),(\alpha+\beta'+\rho-\gamma,-1)\\\left(\frac{\eta}{k},\frac{\mu}{k}\right),(\rho,-1),(\alpha+\rho-\beta,-1),(\alpha+\alpha'+\beta'+\rho-\gamma,-1)\end{matrix};t\,p^{q-\frac{\mu}{k}}\right]\qquad(3.2)$$

Proof: On using (1.5) and taking right hand side MSM fractional integral operator inside the summation

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho}\right)\,{}_pE_{k,\mu,\eta}^{\lambda,q}(t)=\sum_{n=0}^{\infty}\frac{{}_p(\lambda)_{nq,k}}{{}_p\Gamma_k(n\mu+\eta)}\frac{1}{n!}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{n-\rho}\right)$$

Making use of (2.2),(2.6) and (2.8)

$$=t^{-\alpha-\alpha'+\gamma-\rho}\sum_{n=0}^{\infty}\frac{(p)^{nq}\left(\frac{\lambda}{k}\right)_{nq}}{\left(\frac{p}{k}\right)^{\frac{n\mu+\eta}{k}}\Gamma_k(n\mu+\eta)}\frac{\Gamma(-\beta+\rho-n)\Gamma(\alpha+\alpha'-\gamma+\rho-n)\Gamma(\alpha+\beta'-\gamma+\rho-n)}{\Gamma(\rho-n)\Gamma(\alpha-\beta+\rho-n)\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho-n)}\frac{t^n}{n!}$$

$$=\frac{kt^{-\alpha-\alpha'+\gamma-\rho}}{\Gamma\left(\frac{\lambda}{k}\right)p^{\frac{\eta}{k}}}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\lambda}{k}+nq\right)\Gamma(-\beta+\rho-n)\Gamma(\alpha+\alpha'-\gamma+\rho-n)\Gamma(\alpha+\beta'-\gamma+\rho-n)}{\Gamma\left(\frac{n\mu}{k}+\frac{\eta}{k}\right)\Gamma(\rho-n)\Gamma(\alpha-\beta+\rho-n)\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho-n)}\frac{\left(tp^{q-\frac{\mu}{k}}\right)^n}{n!}$$

The required result of (3.2) can be obtained by using (1.18)

In view of (1.16),the corresponding result of (3.2) for the operator (1.7) is as follows

Corollary 3.2.1 Let $\alpha,\beta,\gamma,\rho\ \epsilon C$ such that $Re(\alpha)>0,\ k,p\in R^{+}-\{0\};\ \lambda,\mu,\eta\in C/kZ^{-};Re(\mu)>0,Re(\eta)>0,Re(\lambda)>0, and\ q\in(0,1)\cup N$, then for $t>0$,

$$\left(I_{-}^{\alpha,\beta,\gamma}t^{-\rho}\right)\,{}_pE_{k,\mu,\eta}^{\lambda,q}(t)=\frac{kt^{-\beta-\rho}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}\ {}_3\Psi_3\left[\begin{matrix}\left(\frac{\lambda}{k},q\right),(\gamma+\rho,-1),(\beta+\rho,-1)\\\left(\frac{\eta}{k},\frac{\mu}{k}\right),(\rho,-1)(\beta+\gamma+\alpha+\rho,-1)\end{matrix};tp^{q-\frac{\mu}{k}}\right]$$

Theorem 3.3 Let $\alpha,\alpha',\beta,\beta',\gamma,\rho \in C$ such that $Re(\alpha)>0,\ k,p \in R^{+}-\{0\}; \lambda,\mu,\eta \in C/kZ^{-}; Re(\mu)>0, Re(\eta)>0, Re(\lambda)>0, and\ q \in (0,1)\cup N$. If $Re(\rho)>max\{0, Re(-\alpha+\beta), Re(-\alpha-\alpha'-\beta'+\gamma)\}$, then for $t>0$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{k\, t^{\alpha+\alpha'+\rho-\gamma-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}$$

$${}_4\Psi_4\left[\begin{matrix}\left(\frac{\lambda}{k},q\right), (\rho,1),(\alpha-\beta+\rho,1),(\alpha+\alpha'+\beta'+\rho-\gamma,1)\\ \left(\frac{\eta}{k},\frac{\mu}{k}\right),(-\beta+\rho,1),(\alpha+\alpha'+\rho-\gamma,1),(\alpha+\beta'-\gamma+\rho,1)\end{matrix}; t\,p^{q-\frac{\mu}{k}}\right]$$

Proof: On using (1.5) and taking left hand side MSM fractional differential operator (2.3) inside the summation

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \sum_{n=0}^{\infty}\frac{{}_p(\lambda)_{nq,k}}{{}_p\Gamma_k(n\mu+\eta)}\frac{1}{n!}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{n+\rho-1}\right)$$

On using (2.3),(2.6) and (2.8)

$$= \sum_{n=0}^{\infty}\frac{(p)^{nq}\left(\frac{\lambda}{k}\right)_{nq}}{\frac{p^{\frac{n\mu+\eta}{k}}}{k}\Gamma\left(\frac{n\mu+\eta}{k}\right)}\frac{\Gamma(\rho+n)\Gamma(-\beta+\alpha+n+\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho+n)}{\Gamma(-\beta+\rho+n)\Gamma(\alpha+\alpha'-\gamma+\rho+n)\Gamma(\alpha+\beta'-\gamma+\rho+n)} t^{\alpha+\alpha'-\gamma+\rho+n-1}$$

$$= \frac{k\, t^{\alpha+\alpha'+\rho-\gamma-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\lambda}{k}+nq\right)\Gamma(\rho+n)\Gamma(-\beta+\alpha+n+\rho)\Gamma(\alpha+\alpha'+\beta'-\gamma+\rho+n)}{\Gamma\left(\frac{n\mu+\eta}{k}\right)\Gamma(-\beta+\rho+n)\Gamma(\alpha+\alpha'-\gamma+\rho+n)\Gamma(\alpha+\beta'-\gamma+\rho+n)}\frac{\left(tp^{q-\frac{\mu}{k}}\right)^n}{n!}$$

The result follows on using (1.18).

The right-hand sided expression of (3.3) is further expressed in terms of Fox's H function as

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{k\, t^{\alpha+\alpha'+\rho-\gamma-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}$$

$$H_{4,5}^{1,4}\left[-tp^{q-\frac{\mu}{k}};\begin{matrix}\left(1-\frac{\lambda}{k},q\right), (1-\rho,1),(1-\alpha+\beta-\rho,1),(1-\alpha-\alpha'-\beta'-\rho+\gamma,1)\\ (0,1),\left(1-\frac{\eta}{k},\frac{\mu}{k}\right),(1-\rho+\beta,1),(1-\alpha-\alpha'+\gamma-\rho,1),(1-\alpha-\beta'+\gamma-\rho,1)\end{matrix}\right]$$

The next theorem yields the right- hand side MSM fractional derivative of ${}_pE_{k,\mu,\eta}^{\lambda,q}$.

Corollary 3.3.1 The result in (3.3) is reduced to another Fox H function by setting $p=k, q=1, k=1, \lambda=1, \eta=1\ and\ \rho=1$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\right) {}_1E_{1,\mu,1}^{1,1}(t) = t^{\alpha+\alpha'-\gamma} H_{3,4}^{1,3}\left[-t;\begin{matrix}(0,1),(\beta-\alpha,1),(-\alpha-\alpha'-\beta'+\gamma,1)\\ (0,\mu),(\beta,1),(-\alpha-\alpha'+\gamma,1),(-\alpha-\beta'+\gamma,1)\end{matrix}\right]$$

On setting $\mu=1$ in (3.3.1) the result is obtained in Gauss Hypergeometric function as follows

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\right) {}_1E_{1,1,1}^{1,1}(t) = t^{\alpha+\alpha'-\gamma}\mathcal{C}\ {}_3F_3\left[\begin{matrix}1, 1-\beta+\alpha, 1+\alpha+\alpha'+\beta'-\gamma\\ 1-\beta, 1+\alpha+\alpha'+\beta'-\gamma, 1+\alpha+\beta'-\gamma\end{matrix}; t\right]$$

Here $\mathcal{C}=\frac{\Gamma(1-\beta+\alpha)\Gamma(1+\alpha+\alpha'+\beta'-\gamma)}{\Gamma(1-\beta)\Gamma(1+\alpha+\alpha'+\beta'-\gamma)\Gamma(1+\alpha+\beta'-\gamma)}$

In view of (1.15), the corresponding result of (3.3) for the operator (1.8) is as follows

Corollary 3.3.2 Let $\alpha,\beta,\gamma,\rho \in C$ such that $Re(\alpha)>0,\ k,p\in R^{+}-\{0\};\ \lambda,\mu,\eta \in C/kZ^{-}; Re(\mu)>0, Re(\eta)>0, Re(\lambda)>0, and\ q\in(0,1)\cup N$, then for $t>0$,

$$\left(D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{k\, t^{\beta+\rho-1}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)} \times \; {}_3\Psi_3\left[\begin{matrix}\left(\frac{\lambda}{k}, q\right), (\rho,1), (\beta+2\gamma,1) \\ \left(\frac{\eta}{k},\frac{\mu}{k}\right), (\beta+\rho,1), (\gamma+\rho,1)\end{matrix}; t\, p^{q-\frac{\mu}{k}}\right]$$

Theorem 3.4 Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in C$such that $Re(\alpha) > 0,\ k, p \in R^+ - \{0\}; \lambda, \mu, \eta \in C/kZ^-; Re(\mu) > 0, Re(\eta) > 0, Re(\lambda) > 0, and\ q \in (0,1) \cup N$.If$Re(\rho) > max\{Re(-\beta'),\ Re(\alpha' + \beta - \gamma), Re(\alpha + \alpha' - \gamma) + [Re(\gamma) + 1]\}$ then for $t > 0$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{k\, t^{\alpha+\alpha'-\rho-\gamma}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)}$$

$$\times_4 \Psi_4\left[\begin{matrix}\left(\frac{\lambda}{k}, q\right), (\beta'+\rho,-1), (-\alpha-\alpha'+\gamma+\rho,-1), (-\alpha'-\beta+\rho+\gamma,-1) \\ \left(\frac{\eta}{k},\frac{\mu}{k}\right), (\rho,-1), (-\alpha'+\beta'+\rho,-1), (-\alpha-\alpha'-\beta+\gamma+\rho,-1)\end{matrix}; t\, p^{q-\frac{\mu}{k}}\right]$$

Proof: On using (1.5) and (2.4) ,we have

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{-\rho}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \sum_{n=0}^{\infty} \frac{{}_p(\lambda)_{nq,k}}{{}_p\Gamma_k(n\mu+\eta)} \frac{1}{n!} \left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{n-\rho}\right)$$

Making use of (2.6) and (2.8) with (2.4)

$$= \sum_{n=0}^{\infty} \frac{(p)^{nq}\left(\frac{\lambda}{k}\right)_{nq}}{\frac{p^{\frac{n\mu+\eta}{k}}}{k}\Gamma\left(\frac{n\mu+\eta}{k}\right)} \frac{\Gamma(\beta'+\rho-n)\Gamma(-\alpha-\alpha'+\gamma+\rho-n)\Gamma(-\alpha'-\beta+\rho+\gamma-n)}{\Gamma(\rho-n)\Gamma(-\alpha'+\beta'+\rho-n)\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho-n)} \frac{t^{\alpha+\alpha'-\gamma-\rho+n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{k\, p^{n\left(q-\frac{\mu}{k}\right)}}{p^{\frac{\eta}{k}}\Gamma\left(\frac{\lambda}{k}\right)} \frac{\Gamma\left(\frac{\lambda}{k}+nq\right)\Gamma(\beta'+\rho-n)\Gamma(-\alpha-\alpha'+\gamma+\rho-n)\Gamma(-\alpha'-\beta+\rho+\gamma-n)}{\Gamma\left(\frac{n\mu+\eta}{k}\right)\Gamma(\rho-n)\Gamma(-\alpha'+\beta'+\rho-n)\Gamma(-\alpha-\alpha'-\beta+\gamma+\rho-n)} \frac{t^{\alpha+\alpha'-\gamma-\rho+n}}{n!}$$

The required result of (3.4) can be easily obtained by using (1.18).

In view of (1.17),the corresponding result of (3.4) for the operator (1.9) is as follows

Corollary 3.4.1 Let $\alpha, \beta, \gamma, \rho \in C$ such that $Re(\alpha) > 0,\ k, p \in R^+ - \{0\};\ \lambda, \mu, \eta \in C/kZ^-; Re(\mu) > 0, Re(\eta) > 0, Re(\lambda) > 0, and\ q \in (0,1) \cup N$, then for $t > 0$,

$$\left(D_{-}^{\alpha,\beta,\gamma} t^{-\rho}\right) {}_pE_{k,\mu,\eta}^{\lambda,q}(t) = \frac{kp^{-\frac{\eta}{k}}t^{\beta-\rho}}{\Gamma\left(\frac{\lambda}{k}\right)} \times_3 \Psi_3\left[\begin{matrix}\left(\frac{\lambda}{k}, q\right), (2\gamma+\rho,-1), (-\beta+\rho,-1) \\ \left(\frac{\eta}{k},\frac{\mu}{k}\right), (\rho,-1), (\gamma-\beta+\rho,-1)\end{matrix}; t\, p^{q-\frac{\mu}{k}}\right]$$

IV. CONCLUSION

The above results may also be reduced in number of results including generalised Mittag-Leffler function, Classical Mittag-Leffler function and many more due to the general properties of hypergeometricfunctions.Our findings compliment and extend many results obtained by several authors in this direction.

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