Evaluation of Special Integrals with Differential Transform Method

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***Abstract*: - Differential transform method (DTM) is established to be useful in evaluating some certain class of special integrals (Fresnel sine integral, Fresnel cosine integrals and error function). DTM is also employed in solving variable coefficient Airy equation. The Error function and Fresnel sine and cosine integrals were converted to recursive relations using differential transform method and then solved. The results converge to the exact solution. The DTM solution of Airy equation also converges to the solution obtained from Power series method.**

***Keywords*:-Differential Transform, Inverse Differential Transform, Airy Equation, Error Function, Fresnel sine integrals, Fresnel cosine integral**

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I. INTRODUCTION

Methods for solving nonlinear ordinary and partial differential equations include Perturbation Methods, Adomian Decomposition Method (ADM), Variational Iterative Method (VIM), Differential Transform Method (DTM), Homotopy Perturbation Method and Homotopy Analysis Method (HAM) ([18, 5, 6, 7, 8, 3, 16, 11]). The Perturbation methods, ADM, DTM and VIM are very effective in handling weakly nonlinear problems while HAM and HPM can handle weakly as well as strongly nonlinear problems. [16] gave a complete proof of the convergence of DTM method for homogeneous and non-homogeneous ordinary differential equations. DTM has been applied to solve linear and nonlinear systems of ordinary differential equations [4, 10, 2, 1, 3, 7, 15, 12, 13] and it is applied to solve some problems in biology [17, 14].

In this paper, differential transform method is used to solve the Airy equation and a closed form series solution is obtained. The error function and Fresnel integrals are evaluated using differential transform method. The result reveals that the complexity involved in evaluating some special integrals can be reduced by using the differential transform method. The results here verifies the results in [16] that DTM solution converges for any homogeneous ordinary differential equation.

1.1 Differential Transform

The Differential Transform of a function $y(x)$ about the point $x = 0$ is defined as ([18, 5, 9])

$$DT\{y(x)\} = Y[k] = \frac{1}{k!}\frac{d^k}{dx^k}y(x)\bigg|_{x=0} \tag{1.1}$$

and the inverse transform defined as

$$DT^{-1}\{Y[k]\} = y(x) = \sum_{k=0}^{\infty} Y[k]\,x^k \tag{1.2}$$

We define the Mth order approximation of $y(x)$ as

$$y(x) = \sum_{k=0}^{M} Y[k]\,x^k \tag{1.3}$$

In what follows, we shall take

$DT\{y(x)\} = Y[k]$, $DT\{f(x)\} = F[k]$, $DT\{g(x)\} = G[k]$.

Theorem 1.1.

$$If\ y(x) = cx^n,\ then\ Y[k] = c\delta_{n,k} \tag{1.4}$$

Theorem 1.2.

$$If\ y(x) = \frac{d^n}{dx^n}f(x),\ then\ Y[k] = \left(\prod_{r=1}^{n}(k+r)\right)F[k+n].$$

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Theorem 1.3.

$$If\ y(x) = f(x)\,g(x)\ then\ Y[k] = \sum_{r=0}^{k} F[r]\,G[k-r]$$

Theorem 1.4.

$$If\ y(x) = \int_0^x f(t)dt\ then\ Y[k] = \begin{cases} 0 & k=0 \\ \frac{F[k-1]}{k} & k>0 \end{cases}.$$

For a proof of the theorems above, see [16].

II. DTM SOLUTION OF THE AIRY EQUATION

The Airy equation is the second order ODE

$$\frac{d^2u}{dt^2} - tu = 0 \tag{2.1}$$

with the initial condition

$$u(0) = u'(0) = \frac{\sqrt{3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}. \tag{2.2}$$

Taking the differential transform, we have

$$\begin{aligned} D\{u''\} - D\{tu\} &= 0 \\ (k+1)(k+2)U[k+2] - U[k-1] &= 0 \\ U[k+2] &= \frac{1}{(k+1)(k+2)}U[k-1] \\ U[k] &= \frac{(k-2)!}{k!}U[k-3] \end{aligned} \tag{2.3}$$

and the differential transform of the initial conditions give

$$U[0] = U[1] = \frac{\sqrt{3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}. \tag{2.4}$$

It is easy to see that the recursive equation cannot be solved completely with the given two conditions, rather one more condition is needed. A new condition can be found by setting $x = 0$ in the original problem so that

$$U''(0) = 0 \Rightarrow U[2] = 0. \tag{2.5}$$

With these three conditions, the problem can then be solved to obtain

$U[2]=U[5] = U[8] = \cdots = U[3n+2] = \cdots = 0,$

$$U_{3n} = \prod_{r=1}^{n}\frac{(3r-2)!}{(3r)!}U_0, \quad U_{3n+1} = \prod_{r=1}^{n}\frac{(3r-1)!}{(3n+1)!}U_1$$

so that the solution is given as

$$\begin{aligned} u(t) &= U[0]\left\{1+\sum_{n=1}^{\infty}\left(t^{3n}\prod_{r=1}^{n}\frac{(3r-2)!}{(3r)!}\right)\right\} + U[1]\left\{1+\sum_{n=1}^{\infty}\left(t^{3n}\prod_{i=1}^{n}\frac{(3r-1)!}{(3r+1)!}\right)\right\}t \\ &= \frac{\sqrt{3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)}\left\{1+\sum_{n=1}^{\infty}\left(t^{3n}\prod_{r=1}^{n}\frac{(3r-2)!}{(3r)!}\right) + \left\{1+\sum_{n=1}^{\infty}\left(t^{3n}\prod_{i=1}^{n}\frac{(3r-1)!}{(3r+1)!}\right)\right\}t\right\} \end{aligned}$$

This series converges for all t and it produces the same result as that obtained using the power series method.

III. EVALUATION OF THE ERROR FUNCTION USING DTM

The error function is defined as

$$y(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{3.1}$$

and taking the differential transform gives

$$Y[0] = 0;\ Y[k] = \frac{2}{\sqrt{\pi}}\frac{E[k-1]}{k},\ k = 1, 2, \cdots$$

where

$$E[k] = D\left\{e^{-t^2}\right\} = \frac{1}{k!}\frac{d^k}{dx^k}\left(e^{-t^2}\right)\Big|_{t=0} = \begin{cases} 1 & k=0 \\ 0 & k=2m-1, \quad m=1,2,\cdots \\ \frac{(-1)^m}{(2m)!}\prod_{n=1}^{m}(4n-2) & k=2m \end{cases}.$$

so that and

$$E[k-1] = \begin{cases} \frac{2}{\sqrt{\pi}} & k=1 \\ 0 & k=2m \\ \frac{(-1)^m}{(2m)!}\prod_{n=1}^{m}(4n-2) & k=2m+1 \end{cases}, \quad m=1,2,\cdots.$$

$$Y[0]=0,\ Y[k] = \begin{cases} \frac{2}{\sqrt{\pi}} & k=1 \\ 0 & k=2m \\ \frac{2(-1)^m}{(2m+1)!\sqrt{\pi}}\prod_{n=1}^{m}(4n-2) & k=2m+1 \end{cases}, \quad m=1,2,\cdots$$

The solution becomes

$$y(x) = \sum_{k=0}^{\infty}Y[k]x^k = \frac{2}{\sqrt{\pi}}x\left(1+\sum_{m=1}^{\infty}\left(\frac{(-1)^m}{(2m+1)!}\prod_{n=1}^{m}(4n-2)\right)x^{2m}\right.$$

and by substituting

$$\prod_{n=1}^{m}(4n-2) = \frac{2(2m-1)!}{(m-1)!}$$

we have

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\sum_{m=0}^{\infty}\left((-1)^m\frac{x^{2m+1}}{m!(2m+1)}\right)$$

which converges to the exact solution.

IV. EVALUATION OF FRESNEL INTEGRALS USING DTM

Fresnel cosine integral

Fresnel cosine integral is defined as

$$c(x) = \int_0^x \cos\left(t^2\right)dt$$

and taking the differential transform gives

$$Fc[0] = 0;\ Fc[k] = \frac{C[k-1]}{k},\ k = 1, 2, \cdots$$

where

$$C[k] = D\left\{\cos\left(t^2\right)\right\} = \frac{1}{k!}\frac{d^k}{dt^k}\left(\cos\left(t^2\right)\right)\Big|_{t=0} = \begin{cases} 0 & k \neq 4m \\ (-1)^m\frac{1}{(2m)!} & k=4m \end{cases}, \quad m=1,2,\cdots$$

and

$$Fc\left[k\right] = \begin{cases} 0 & k \neq 4m+1 \\ (-1)^m \frac{1}{(2m)!(4m+1)} & k = 4m+1 \end{cases}, \quad m = 0, 1, 2, \cdots$$

so that the solution becomes

$$\int_0^x \cos\left(t^2\right) dt = \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m+1}}{(2m)!\,(4m+1)}$$

Fresnel sine integral

Fresnel sine integral is defined as

$$s\left(x\right) = \int_0^{\infty} \sin\left(x^2\right) dx.$$

and taking the differential transform gives

$$Fs\left[0\right] = 0;\;\; Fs\left[k\right] = \frac{S\left[k-1\right]}{k},\;\; k = 1, 2, \cdots$$

where

$$S\left[k\right] = D\left\{\sin\left(x^2\right)\right\} = \frac{1}{k!}\frac{d^k}{dx^k}\left(\sin\left(x^2\right)\right)\bigg|_{t=0} = \begin{cases} 0 & k \neq 4m+2 \\ (-1)^m \frac{1}{(2m+1)!} & k = 4m+2 \end{cases}, \quad m = 0, 1, 2, \cdots$$

and

$$Fs\left[k\right] = \begin{cases} 0 & k \neq 4m+3 \\ (-1)^m \frac{1}{(2m+1)!(4m+3)} & k = 4m+3 \end{cases}, \quad m = 0, 1, 2, \cdots$$

so that the solution becomes

$$\int_0^x \sin\left(t^2\right) dt = \sum_{m=0}^{\infty} (-1)^m \frac{x^{4m+3}}{(2m+1)!\,(4m+3)}.$$

V. CONCLUSION

Efficiency of differential transform method is demonstrated in this paper by solving Airy equation and the results of [16] is established. Special integrals, Fresnel integrals and error function, are then evaluated using the differential transform method. Differential transform method reduces the problems to recursive relations which are easily solved. The result from the DTM gives the Airy polynomials exactly and without much rigor. Error function and Fresnel integrals are also solved using DTM and the solution coincides with the exact solution. Many steps are required to evaluate the special integrals but all these rigorous steps are avoided by simply employing differential transform method. It is therefore established that the DTM solution converges to the exact solution of the Airy equation, error function and Fresnel integrals.

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