Numerical Studies for Solving Third Order Ordinary Differential Equation

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Abstract: **In this paper, the concept of Variational Iteration method(VIM) in solving third-order ordinary differential equations is considered. Some test problems were considered and a comparative study were carried with existing methodsbut a case study was the Modified Adomian Decomposition method (MADM). The third example is represented with on a graph and Error analysis was also plotted.From the chart, the convergence of the VIM was so rapid and involved only a few terms of the series and by far better than the MADM, Seven-Step Block method, Differential Transform method and Picard Iteration method.**

Keywords: **Ordinary differential equations, Variational Iteration method, Modified Adomian Decomposition method, Initial value problems(IVP), Correction functional, Lagrange multiplier.**

I. INTRODUCTION

Most phenomena in sciences, economics, management, engineering etc. can be modeled by differential and integral theories. Interestingly, solutions to most of the differential equations arising from such models do not have analytic solutions necessitating the development of numerical techniques.However, most of these methods require tedious analysis, large computational work, and lack of existing rule in the selection of initial value, scientifically unrealistic assumptions, perturbation, linearization and discretization of the variable which givesrise to rounding off errors causing loss of Physical nature of the problem [2].Researchers were aiming to establish reliable methods capable for solving a large class of linear and nonlinear differential and integral equations without the tangible restrictive assumptions or discretization of the variables [3]. The Variation iteration method proposed in this work is easy to apply, it requires no linearization or discretization. Recently, there has been great development of new powerful methods capable of handling linear and nonlinear equations that over come most of the classical methods [3].The Adomiande composition method, the variation aliteration method, and the homotopy perturbation method are examples of the newly developed methods. The VIM provides efficient algorithm for analytic approximate solutions and numeric simulations for real-world applications in sciences[4–8].Unlike the Adomiande composition method, where computational algorithms are normally used to deal with the nonlinear terms[3]. VIM method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists. The VIM approaches linear and nonlinear problems directly in a like manner [3]. In this paper, we aim to apply the VIM to third order ODEs. Secondly, reliability purposes, a comparative study will be done with exiting methods and conclusions will be drawn from the result obtained.

II. FUNDAMENTALS OF THE VARIATIONAL ITERATION METHOD (VIM)

Consider the equation

$$Lu + Nu = g(x), \qquad (1)$$

Where $L$and N are linear and nonlinear operators respectively. And $g(x)$ is the source inhomogeneous term. The variational iteration method admits the use of a correction functional for equation (1)

In the form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(t)\,(Lu_n(t) + N\tilde{u}_n(t) - g(t))\,dt, \qquad (2)$$

Where λ is a general Lagrange's multiplier, which can be identified optimally through the variational theory, and $\tilde{u}_n$ as a restricted variation which means $\delta\tilde{u}_n = 0$. The Lagrange multiplierλ is crucial and critical in the method, and it can be a constant or a function. Having λ determined, an iteration formula should be used for the determination of the successive approximations $u_{n+1}(x),\ n \geq 0$ of the solution $u(x)$. The zeroth approximationu_0 can be any selective function. However, using the initial values $u(0), u'(0),$ and $u''(0)$ are preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x). \qquad (3)$$

The details of the derivation of each distinct Lagrange multipliers λ in [1] is skipped while the summary of the obtained results is seen in what follows.

The third-order ODE is of the form:

$$u'''(x) + au'' + bu' + cu = g(x),$$
$$u(0) = \alpha, \quad u'(0) = \beta, \quad u''(0) = \gamma, \qquad (4)$$

We found that

$$\lambda = -\frac{1}{2!}(t-x)^2 \qquad (5)$$

and the correction functional takes the form

$$u_{n+1}(x) = u_n(x) - \frac{1}{2!}\int_0^x (t-x)^2 \left(u'''_n(t) + au''_n(t) + bu'_n(t) + cu_n(t) - g(t)\right)dt, \quad (6)$$

Although the zeroth approximation $u_0(x)$ is any selective function, but it is preferable to select it in the form

$$u_0(x) = u(0) + u'(0)x + u''(0)\frac{1}{2!}x^2 \qquad (7)$$

In what follows we present the following illustrative examples. We will examine some ODE's with initial value problems and do a comparative study with the existing methods like Modified Adomian decomposition method (MADM).

III. NUMERICAL SOLUTION

The following differential problems are considered for comparing purposes.

1. Consider the following third-orderordinary differential equation
   $u''' = -u(t)$Kuboye(2015)$u(0) = 1, \quad u'(0) = -1, u''(0) = 1$

   Exact solution: $u(t) = e^{-t}$

Problem (1) stated above was solved by OkaiJ.O(2018) with MADM(n=2). Our new method VIM(n=1) was also applied to solve the same problem.

Applying the VIM and (5)

The correctional functional becomes:

$$u_{n+1}(t) = u_n - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_n(t) + u_n(t)))d\beta, n \geq 0$$

We obtain the following iterations

$$u_0(t) = u(0) + tu'(0) + \frac{1}{2!}t^2u''(0) = 1 - t + \frac{t^2}{2!}$$

$$u_1(t) = u_0(t) - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_0(\beta) + u_0(\beta)))d\beta$$
$$= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!}$$

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$$u_n(t) = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \cdots, \ (8)$$

Applying (3) to (8), the solution becomes

$u(t) = e^{-t}$

Which coincides with the exact solution.

2. Consider the following third-orderordinary differential equation
   $u''' = e^t$OkaiJ.O(2018)$u(0) = 3, u'(0) = 1, u''(0) = 5$
   Exact solution: $u(t) = 2 + 2t^2 + e^t$

Applying the VIM and (5)

The correctional functional becomes:

$$u_{n+1}(t) = u_n - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_n(\beta) - e^\beta))d\beta, \quad n \geq 0$$

We obtain the following iterations

$$u_0(t) = u(0) + tu'(0) + \frac{1}{2!}t^2u''(0) = 3 + t + \frac{5t^2}{2!}$$

$$u_1(t) = u_0(t) - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_0(\beta) - e^\beta))d\beta$$
$$= 2 + 2t^2 + e^t \qquad (9)$$

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$u_2(t) = u_3(t) = u_4(t) = 0$

Applying (3) to (9), the solution becomes

$u(t) = 2 + 2t^2 + e^t$

Which coincides with the exact solution.

3. Consider the following third-order ordinary differential equation
   $u'''(t) + 4u'(t) = t$, Okai O.J(2018)
   The initial conditions are:
   $u(0) = 0, \ u'(0) = 0, u''(0) = 1$ exact-solution: $u(t) = \left(\frac{3}{16}\right)(1 - cos2t) + \frac{t^2}{8}$

Problem (3) stated above was solved by OkaiJ.O(2018) with MADM(n=2). Our new method VIM(n=2) was also applied to solve the same problem. The table of values is presented below.

Applying the VIM and (5)

The correctional functional becomes:

$$u_{n+1}(t) = u_n - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_n(\beta) + 4u'_n(\beta) - \beta))d\beta, \quad n \geq 0$$

We obtain the following iterations

$$u_0(t) = u(0) + tu'(0) + \frac{1}{2!}t^2u''(0) = \frac{t^2}{2!}$$

$$u_1(t) = u_0(t) - \frac{1}{2!}\int_0^t ((\beta - t)^2 (u'''_0(\beta) - e^\beta))d\beta$$

$$= \frac{t^2}{2!} - \frac{t^4}{8} \quad (10)$$

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$$u_n(t) = \frac{t^2}{2!} - \frac{t^4}{8}$$

Applying (3) to (10), the solution becomes

$$u(t) = \frac{t^2}{2!} - \frac{t^4}{8}$$

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Table: Results of theVIM(n=2) are compared with Okai J.O [2]

x	Exact	MADM(n=2)	VIM (n=2)	ERROR ANALYSIS(MADM)	ERROR ANALYSIS (VIM)
0.1	0.00498751665	0.00500832222	0.00498751667	0.00002080557	0.00000000001
0.2	0.01980106362	0.02013262222	0.01980106667	0.00033155860	0.00000000304
0.3	0.04399957220	0.04566690000	0.04399965000	0.00166732780	0.00000007780
0.4	0.07686749200	0.08208782222	0.07686826667	0.00522033022	0.00000077467
0.5	0.11744331765	0.13003472222	0.11744791667	0.01259140457	0.00000459902
0.6	0.16455792104	0.19028160000	0.16457760000	0.02572367896	0.00001967896
0.7	0.21688116071	0.26370112222	0.21694831667	0.04681996152	0.00006715596
0.8	0.27297491043	0.35122062222	0.27316906667	0.07824571179	0.00019415624
0.9	0.33135039275	0.45377010000	0.33184485000	0.12241970725	0.00049445725
1	0.39052753185	0.57222222222	0.39166666667	0.18169469037	0.00113913481

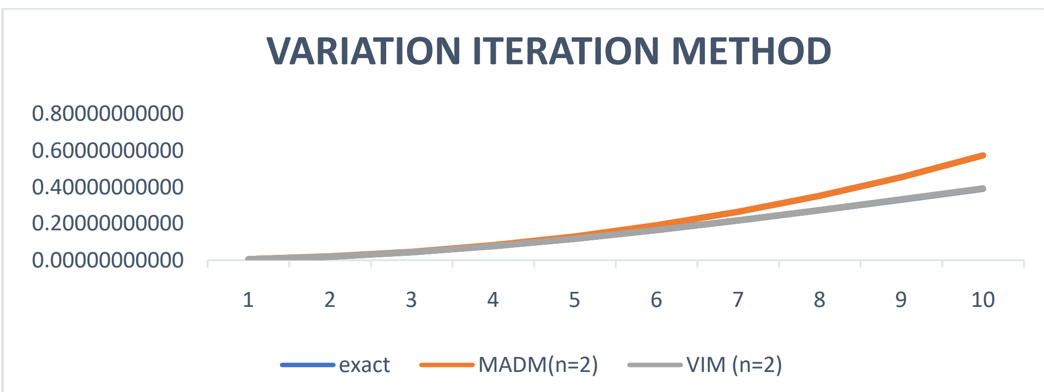

Figure 1:*Comparison between the Exact, VIM and MADM*

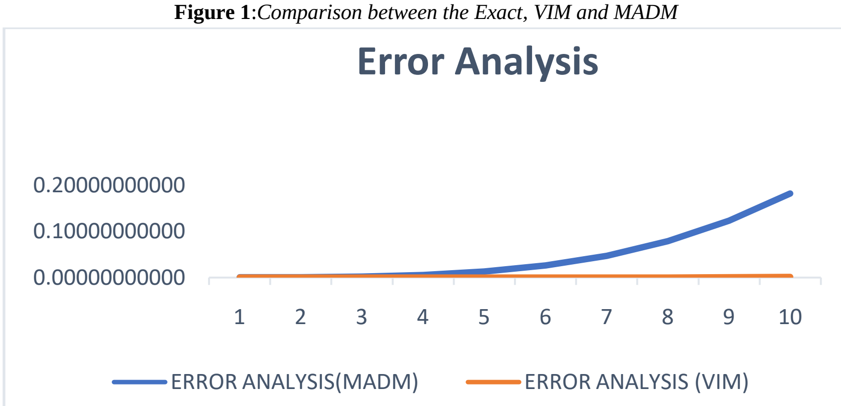

Figure 2: Graph showing the deviation from the Exact

IV. CONCLUSION

The Variational Iteration Method is a promising method for the solution of initial value problems in the treatment of third-order differential equations. In the first two examples considered, the results obtained coincide with the exact solution with one iteration taken and the third example converges faster and in better agreement with the exact solution when compared to the MADM as seen in the graph above. Also, computational difficulties in other traditional methods were reduced.

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