Comparison of Saturation Term and Disease Induced in Sir Epidemic Model

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Abstract: **-The paper considered a SIR epidemic model having saturation terms and disease induced death which were neglected in the previous literature. We determine the basic reproduction number of the modified models by Linearization method. We also investigate the effect of disease induced death by providing Numerical Simulation using Runge-kutta of order 4 method. The results show that saturation term has an appreciable effect than disease induced death on the epidemic model.**

Keywords: **Basic Reproduction Number, Endemic Equilibrium, Disease induced death, Local Stability.**

I. INTRODUCTION

In [1], Stability Analysis of an SIR epidemic model with Non-Linear Incidence Rate and Treatment was considered.[2], [3] and [4] also studied Mathematical modeling and control of infectious disease. Also, [5] and [6] investigated the numerical simulation of epidemic model.[7],[8] and[9] analyzed the Disease free Equilibrium (DFE)and concluded that it is Locally and Globally asymptotically stable.

In this paper, we considered an SIR epidemic model with saturation terms and disease induced death, where $\frac{KSI}{1+\alpha S^p+\beta I^q}$ is the nonlinear incidence rate with transmission rate K, and $p = q > 1$, p and q being positive constant, α *and* β are the parameters which measure the effects of Sociological, Physiological or other mechanisms.

II. THE BASIC MATHEMATICAL MODEL

In this paper, model of [1] was adopted and modified by removing treatment term τ and incorporating disease induced death m .

The Existing model of [1]

$$\frac{dS}{dt} = b - dS - \frac{KSI}{1+\alpha S^p+\beta I^q} + \gamma R$$
$$\frac{dI}{dt} = \frac{KSI}{1+\alpha S^p+\beta I^q} - (d+\mu)I - \tau I$$
$$\frac{dR}{dt} = \mu I - (d+\gamma)R + \tau I \qquad (1)$$

2.1 Proposed Model

$$\frac{dS}{dt} = b - dS - \frac{KSI}{1+\alpha S^p+\beta I^q} + \gamma R$$
$$\frac{dI}{dt} = \frac{KSI}{1+\alpha S^p+\beta I^q} - (d+\mu+m)I$$
$$\frac{dR}{dt} = \mu I - (d+\gamma)R \qquad (2)$$

2.2 Disease Free Equilibrium (DFE)

At disease free equilibrium, $I = 0, R = 0$

$$b - dS - \frac{KSI}{1+\alpha S^p+\beta I^q} + \gamma R = 0$$
$$b - dS = 0$$
$$S = \frac{b}{d}$$
$$(S, I, R) = (\frac{b}{d}, 0, 0) \qquad (3)$$

2.3 The Endemic Equilibrium

At endemic equilibrium, $I \neq 0, p = q = 2$

Therefore, from equation (2), we have;

$$\mu I - (d+\gamma)R = 0$$
$$\mu I = (d+\gamma)R$$
$$R^* = \frac{\mu}{d+\gamma}$$
$$\left(\frac{KS^* \pm \sqrt{(KS^*)^2 - 4\beta(b - dS^* - \gamma R^*)^2(1+\alpha S^{*2})}}{2\beta(b - dS^* - \gamma R^*)}\right) \qquad (4)$$

Also to get S^* from equation (2), we say,

$$KS = (1+\alpha S^2+\beta I^2)(d+\mu+m)$$
$$KS - (\alpha S^2)(d+\mu+m) = (1+\beta I^2)(d+\mu+m)$$
$$S^* = \frac{K \pm \sqrt{K^2 - 4\alpha(d+\mu+m)^2(1+\beta I^{*2})}}{2\alpha(d+\mu+m)} \qquad (5)$$

Also to get I^* from equation (2), we get;

$$b - dS - \frac{KSI}{1+\alpha S^p + \beta I^q} + \gamma R = 0$$

$$b - dS - \gamma R = \frac{KSI}{1+\alpha S^2 + \beta I^2}$$

$$(b - dS - \gamma R)(1+\alpha S^2 + \beta I^2) = KSI$$

$$\beta I^2 (b - dS - \gamma R) + (b - dS - \gamma R)(1+\alpha S^2) = KSI$$

$$(b - dS - \gamma R)(1+\alpha S^2) = KSI - \beta I^2 (b - dS - \gamma R)$$

$$I^* = \frac{KS^* \pm \sqrt{(KS^*)^2 - 4\beta(b - dS^* - \gamma R^*)^2(1+\alpha S^{*2})}}{2\beta(b - dS^* - \gamma R^*)} \quad (6)$$

2.4 Basic Reproduction Number R_0

By Linearization,

$$\frac{dI}{dt} = \frac{KSI}{1+\alpha S^p + \beta I^q} - (d + \mu + m)I \quad (7)$$

The inverse of v becomes,

$$R_0 = \frac{K\frac{b}{p}}{(1+\alpha(\frac{b}{d})^p)(d+\mu+m)}$$

2.5 Local Stability of Disease Free Equilibrium

$$\frac{dS}{dt} = b - dS - \frac{KSI}{1+\alpha S^p + \beta I^q} + \gamma R$$

$$\frac{dI}{dt} = \frac{KSI}{1+\alpha S^p + \beta I^q} - (d + \mu + m)I$$

$$\frac{dR}{dt} = \mu I - (d + \gamma)R \quad (8)$$

The Jacobian matrix becomes,

$$J(E_0) = \begin{bmatrix} -d & \frac{-KS}{1+\alpha S^p} & \gamma \\ 0 & \frac{KS}{1+\alpha S^p} - (d+\mu+m) & 0 \\ 0 & \mu & -(d+\gamma) \end{bmatrix}$$

The determinant of matrix becomes,

$$|J - \lambda I| = 0$$

$$J(E_0) = \begin{vmatrix} -d-\lambda & \frac{-KS}{1+\alpha S^p} & \gamma \\ 0 & -(d+\mu+m)[R_0 - 1] - \lambda & 0 \\ 0 & \mu & -(d+\gamma) - \lambda \end{vmatrix} = 0$$

$$\lambda_1 = -d$$

$$\lambda_2 = -(d+\mu+m)[R_0 - 1]$$

$$\lambda_3 = -(d+\gamma) \quad (9)$$

All Eigen values solved at the equilibrium points contain negative real part therefore the system is Locally Asymptotically Stable

2.6 Global Stability of Disease Free Equilibrium

We consider the Lyapunov function,

$$V = \frac{KSI}{1+\alpha S^p + \beta I^q} - (d+\mu+m)I$$

$$\frac{dV}{dt} = \left[\frac{KS}{1+\alpha S^p} - (d+\mu+m)\right]I$$

$$\frac{dv}{dt} = (d+\mu+m)[\frac{Kbd^p}{d(d^p+\alpha b^p)(d+\mu+m)} - 1]I$$

$$= (d+\mu+m)[R_0 - 1] \le 0$$

$$if \quad R_0 \le 1$$

$$\frac{dV}{dt} \le 0 \quad (10)$$

Hence, the disease free equilibrium is globally asymptotically stable.

2.7 Local Stability of Endemic Equilibrium

Let

$$S - S^* = w, I - I^* = y, R - R^* = z$$

$$\frac{ds}{dt} = \frac{dw}{dt}, \frac{dE}{dt} = \frac{dx}{dt}, \frac{dI}{dt} = \frac{dy}{dt}$$

$$\frac{dx}{dt} = b - d(x+S^*) - \frac{K(x+S^*)(y+I^*)}{1+\alpha S^p + \beta I^q} + \gamma(z+R^*)$$

$$\frac{dy}{dt} = \frac{K(x+S^*)(y+I^*)}{1+\alpha S^p + \beta I^q} - (d+\mu+m)(y+I^*)$$

$$\frac{dz}{dt} = \mu(y+I^*) - (d+\gamma)(z+R^*) \quad (11)$$

The Jacobian matrix becomes,

$$J(E^*) = \begin{bmatrix} \left(-d - \frac{KI^*}{1+\alpha S^{*p} + \beta I^{*q}}\right) & -\frac{KS^*}{1+\alpha S^{*p} + \beta I^{*q}} & \gamma \\ \frac{KI^*}{1+\alpha S^{*p} + \beta I^{*q}} & \frac{KS^*}{1+\alpha S^{*p} + \beta I^{*q}} - (d+\mu+m) & 0 \\ 0 & \mu & -(d+\gamma) \end{bmatrix}$$

$$|J - \lambda I| = 0$$

The determinant of matrix

$$J(E^*)=\begin{vmatrix} \left(-d-\frac{KI^*}{1+\alpha S^{*p}+\beta I^{*q}}\right)-\lambda & -\frac{KS^*}{1+\alpha S^{*p}+\beta I^{*q}} & \gamma \\ \frac{KI^*}{1+\alpha S^{*p}+\beta I^{*q}} & \frac{KS^*}{1+\alpha S^{*p}+\beta I^{*q}}-(d+\mu+m)-\lambda & 0 \\ 0 & \mu & -(d+\gamma)-\lambda \end{vmatrix}=0$$

Let,

$$A=\frac{KI^*}{1+\alpha S^{*p}+\beta I^{*q}}, A_1=\frac{KS^*}{1+\alpha S^{*p}+\beta I^{*q}}$$

$$J(E^*)=\begin{vmatrix} (-d-A)-\lambda & -A_1 & \gamma \\ A & A_1-(d+\mu+m)-\lambda & 0 \\ 0 & \mu & -(d+\gamma)-\lambda \end{vmatrix}=0$$

$$-\lambda^3-[(d+A)+(d+\gamma)+(d+\mu+m)+A_1)]\lambda^2-$$
$$[((d+A)+(d+\mu+m)+A_1)+AA_1]\lambda-[(d+A)[(d+\gamma)(1-A_1)$$
$$+(d+\mu+m)]+(d+\gamma(1+AA_1-A_1)+(d+\mu+m)-A\mu\gamma]=0$$

We can write the characteristic equation above as;

$$\lambda^3+B\lambda^2+C\lambda+D=0$$

Where:

$$B=[(d+A)+(d+\gamma)+(d+\mu+m)+A)],$$
$$C=[((d+A)+(d+\mu+m)+A)+AA_1],$$
$$D=[(d+A)[(d+\gamma)(1-A)+(d+\mu+m)]+(d+\gamma(1+AA_1-A)+(d+\mu+m)-A\mu\gamma]$$

Using the Routh-Hurwitz criterion, it can be seen that all the Eigen values of the characteristics equation above have negative real part if and only if :

$$B>0, C>0, D>0, BC-D>0$$

Hence, it is locally asymptotically stable if and only if inequalities above are satisfied

III. RESULTS AND DISCUSSION

m=0.1

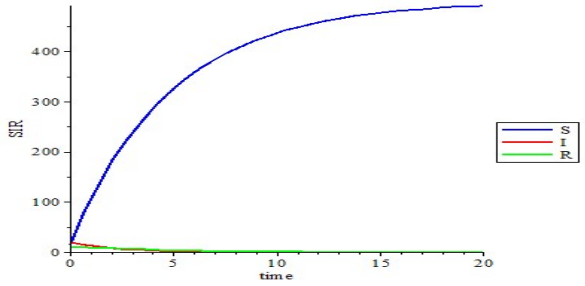

Figure 1: Graph of Susceptible (S), Exposed (E), Infected (I), Recovered (R) against time (t) with

$$\beta=0.1,\ b=100,\ \mu=0.19,\ d=0.2,\ m=0.1,$$
$$\alpha=0.5,\ p=0.5,\ q=0.5,\ \Gamma=0.2, K=0.4$$

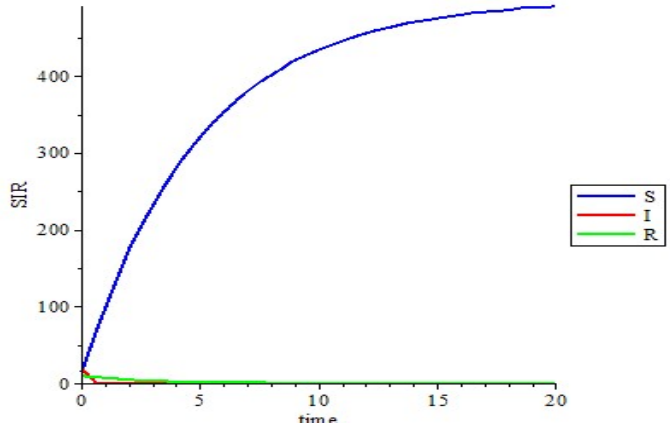

Figure 2: Graph of Susceptible (S), Exposed (E), Infected (I), Recovered (R) against time (t) with

$$\beta=0.1,\ b=100,\ \mu=0.19,\ d=0.2,\ m=50,$$
$$\alpha=0.5,\ p=0.5,\ q=0.5,\ \Gamma=0.2, K=0.4$$

3.1 Discussion

From fig 1-2, it shows a simulation result when disease induced death is considered. The model of [1] has both saturation terms including treatment. The Simulation shows that disease induced death is not a better measure for disease eradication as shown in the simulation result. The presence of saturation terms in the model makes disease induced disease death rate less effective.Therefore, other measures like Treatment and vaccination including saturation terms would have been better. Hence, disease induced death has no significant role to play in disease eradication in the presence of saturation terms.

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