Determining Steady-State Solution, Using First Order Linear Differential Equation

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***Abstract*: - This study examines how to determine the steady-state solution of a linear differential equation as a panacea to sustaining Nigerian ailing economy. It further explains the methods of solving a linear differential equation using the separation of variables and integrating factor which produce the same general solution with the applications as in relation to the field of science and engineering. We have used the analytical method to predict the population size over a long period of time. Through the theory which states that over a long period of time, as the independent variables t→∞, the population size will saturate and approach the steady-state value. The results obtained in these are fully presented and discussed.**

I. INTRODUCTION

Any equation that contains the differential coefficient of unknown variables is called differential equation. While the order of a differential equation is the highest-ordered derivative appearing in the equation. Linear differential equation is an ordinary differential equation of order n that has the form:

$$b_0(x) \quad \frac{d^n y}{dx^n} + b_1(x)\frac{d^{n-1}}{dx^{n-1}} + \quad ---+b_{n-1}(x)$$
$$\frac{dy}{dx}(x)y = R(x)$$

Where R(x) and the coefficient bj (x) (j = 0, 1, 2, 3,………n) depend solely on the variable x. Differential equation has many interesting examples in the field of science and Engineering. These application areas include Newton's Law of cooling. Which states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the ambient temperature (that is temperature of its surrounding).

Another Application of Linear differential equation is in chemical reactions. A process that involves rearrangement of the molecular or ionic structure of a substance as distinct from a change in physical form or nuclear reaction. In order word, chemical reaction is a process that leads to the transformation of one set of chemical substances to another.

This study will help us to determine the steady- state solution of first order differential equation, using separation of variables and integrating factor as applicable to law of cooling and mixture of two salt solutions of different concentrations.

Statement of Problem

Despite the wide applications of a differential equation, it is not common to find research thesis that capture the full concepts of steady- state solution of a differential equation.

This is because the researchers do not use the techniques of separation of variables and do not also make use of integrating factor. By using the techniques of separation of variables and integrating factor, it is possible to study the convergent of the general solution as the independent variable t approaches infinity.

The key question is over a very long period of time, will the general solution approach the steady- state solution? This is generally not taken seriously in post graduates mathematics study. This idea is the key problem that must be satisfied before the population size can be estimated.

Aim and Objectives of the Study

The aim of this thesis is to determine the steady-state solution of a linear differential equation and predict the population size over time to achieve this aim, the following objectives are stated:

- To determine the steady state solution of a single model equation
- To solve a linear differential equation using the techniques of separation of variables and integrating factor.
- To predict a population size using a differential equation over a long period of time.

Scope of the Study

This research has only discussed the analytical method of predicting the population size of any linear differential equation, the methods of solving a differential equation and the solutions of a deterministic scenario. The stochastic dimension of this analysis is beyond the scope of this thesis.

II. MATHEMATICAL MODELLING

We looked at the mathematical analysis, prediction and the population size over time, using the analytical method to determine the convergence of the steady state solution. We also looked at the solution of the linear differential equation under several initial conditions.

Prediction of Population Size: This will be the mathematical analysis of the problem, involving how the problems have been solved under several initial condition and prediction of the population size of a linear differential equation over a long period of time. If we consider a giving differential equation say:

$$\frac{dp}{dt} + 0.05p = 90 \qquad \text{………………………} (1)$$

The differential equation has the form:

P(t) = 0.05 and q(t) = 90

and is linear

Here, $\int p(t)dt = \int 0.05dt = 0.05t$

$I.F(t) = e^{\int p(t)dt} = e^{\int 005t}$ …………………(2)

Multiplying equation (1) by the integrating factor of equation (2), we have:

$$e^{0.05}\frac{dp}{dt} + 0.05\,e^{0.05t} = 90\,e^{0.05t}$$

$$\frac{d}{dt}(Pe^{0.05t}) = 90\,e^{0.05t} \qquad \text{………………………}(3)$$

Integrating both sides of equation (3) with respect to t, we have:

$$\frac{d}{dt}(Pe^{0.05t})\,dt = \int 90\,e^{0.05t}\,dt$$

$$Pe^{0.05} = 1800\,e^{0.05} + K$$

$$P = Ke^{0.05t} + 1800$$

$$P(t) = Ke^{0.05t} + 1800$$

Therefore, the methods of integrating factor and separation of variable product have the same general solution.

Given the following initial conditions P(0) = 15, P(0) = 12, P(0) =

8, P(0) = 6, P(0) = 18, P(0) = 2.5, = P(0) = 15.5, P(0) = 22 and P(0) = 30.

We proposed to find the particular solution for each of these initial conditions.

Considering the initial condition of P(0) = 15

$P(0) = 15 = 1800 + K,$ Since $e^0 = 1$

$1800 = k = 15$

$K = 15 - 1800$

Hence, $K = -1{,}785$

$$P(t) = 1800 - 1785e^{-0.05t}$$

We consider another initial condition $P(0) = 12,$

$P(0) = 12 = 1800 + K$ Since $e^0 = 1$

$1800 = k = 12$

$K = 12 - 1800$

$K = -1{,}788$

Hence, $P(t) = 1800 - 1{,}788\,e^{-0.05t}$

$K = -1{,}788$

$P(0) = 8 = 1800 + K$ Since $e^0 = 1$

$1800 = k = 8$

$K = 8 - 1800$

$K = -1{,}792$

Hence

$$P(t) = 1800 - 1{,}792e^{-0.05t}$$

We shall predict P(t) as the parameter t varies at 50, 150, 250, 300, 350, 400, 450, & 500 for each of the particular solution obtained from the given initial conditions:

If we consider the particular solution in equation (3.3)

P(t) $= 1800 - 1785e^{-0.05t}$.

P(50) $= 1800 - 1785e^{-0.05(50)}$

P(50) $= 1800 - 1785e^{-0.05(50)}$

$= 1800 - 146.5217225$

$= 1{,}653.478277$

P(100) $= 1800 - 1785e^{-0.05(100)}$

$= 1800 - 12.02733539$

$= 1{,}787.972765$

P(t) $= 1800 - 1785e^{-0.05t}$

III. RESULTS

Varying t and keeping k constant, we tabulate below:

Example	t(days)	Constant	$Ke^{-0.05(t)}$ K= 1770	P(t)
1	50	1800	146.5217225	1,653.478277
2	100	1800	12.027223539	1,787.972765
3	150	1800	0.9872556	1799.012744
4	200	1800	0.081038874	1799.918961
5	250	1800	0.006652075912	1799.993348
6	300	1800	0.0005460356421	1799.999454
7	350	1800	0.00004482133493	1799.999955
8	400	1800	0.000003679159216	1799.999996
9	450	1800	0.0000003020037792	1800.000000
10	500	1800	0.0000024789978	1800.000000

From the analysis above, this model equation can be use to determine the population behaviour of a particular area. Hence, we predict that the population will grow exponentially and converge to the steady-state after 200 days and fully saturate after 450 days as P(t) →x.

From the general solution y(t) = 288 - 278 e keeping C constant (i.e. C = 278) and varying t = 10, 15,…, 275. We tabulate as below:

Example	T(days)	Constant	$Ke^{-0.0083}$ C =27 K= 1770	P(t)
1	0	278	278	10
2	10	278	121.2217016	166.7782984
3	25	278	80.04757246	207.9524275
4	20	278	52.85863647	235.141363
5	25	278	43.9046684	23.0953132
6	30	278	2.04897071	264.950293
7	35	278	15.22016379	272.7798362
8	40	278	01.0504873	27.9495128
9	45	278	0.636741557	281.36584
10	50	278	4.382507783	281.6174922
11	55	278	2.893946419	285.1060536
12	60	278	1.910989391	286.0890106
13	65	278	1.261903271	286.7380967
14	70	278	1.261903271	286.7380967
15	75	278	0.55025202	287.449748
16	80	278	0.363353573	287.6366464
17	85	278	0.239937	287.760063
18	90	278	0.15844066	287.8415594
19	95	278	0.104624358	287.895754
20	100	278	0.069087677	287.9399123
21	105	278	0.045621376	287.953786

22	110	278	0.030125632	287.9698744
23	115	278	0.01989368	287.9801068
24	120	278	0.01313626	287.9868637
25	125	278	0.008674402025	287.9913256
26	230	278	0.005728057068	287.9942719
27	135	278	0.003782466812	287.9962175
28	140	278	0.002497715197	287.9975023
29	145	278	0.00164934154	287.9983507
30	150	278	0.001089126729	287.9989109
31	155	278	0.000719194382	287.9992808
32	160	278	0.0004749130201	287.999251
33	165	278	0.000313604194	287.9996864
34	170	278	0.0002070854835	287.9997929
35	175	278	0.0001367468863	287.9998633
36	180	278	0.00009029947728	287.999097
37	185	278	0.00005962838217	287.9999404
38	190	278	0.00003937502262	287.9999606
39	195	278	0.00002600071349	287.999974
40	200	278	0.00001716945051	287.9999828
41	205	278	0.0000113767977	287.9999887
42	210	278	0.000007486726643	287.9999925
43	215	278	0.000004943787173	287.9999951
44	220	278	0.000003264581809	287.9999967
45	225	278	0.000002155734868	287.9999978
46	230	278	0.000001423518568	287.9999986
47	235	278	0.0000009400066509	287.9999991
48	240	278	0.0000006207242557	287.9999994
49	245	278	0.0000004098892293	287.9999996
50	250	278	0.0000002706663687	287.9999997
51	255	278	0.0000001821602785	287.9999998
52	260	278	0.0000001180238769	287.9999999
53	265	278	0.00000007793592	287.9999999
54	270	278	0.00000005146422727	287.0000000
55	275	278	0.0000000339839023	287.9999967

Summary, we have used the analytical method to confirm that the population begins to saturate or tends to the steady-state solution after 75 days and fully saturate after 275days. This conclusion is consistent with a major theory in differential equation which states that over a very long period of time (i.e. $t \to \infty$), P(t) will approach the steady-state solution.

IV. APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

In this section we looked at a few standard applications of linear differential equation in the field of science. For example, Newtons' law of cooling, chemical reactions, the mixture of two salt solutions of different concentration, falling bodies and air resistance.

Newton's Law of Cooling: According to Newton's empirical law of cooling, the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium, the so-called ambient temperature (Zill 1997).

If we let T(t) represent the temperature of a body at any time t, T_m represent the constant temperature of the surrounding medium and represent the rate at which a body coos, then Newton's law of cooling translates into the mathematical statement.

$$\frac{dT}{dt} \alpha \ (T - T_m) \ \frac{T}{dt} = K(T - T_m)$$

Where K is a constant of proportionality. Since we have assumed the body is cooling, we must have $T > T_m$, and so it stands to reason that $K < 0$.

Given the model equation of Newton's law of cooling as:

$$\frac{dT}{dt} = 0 \ K(T - T_m) = 0$$

At a steady-state $\frac{dT}{dt} = 0$

$$K(T - T_m) = 0$$

Let $T = T_e$ such that

$$K(T_e - T_m) = 0$$

$$If \ K \neq 0, T_{e,} - T_m = 0$$

$$T_e = T_m$$

On the basis of this assumption, we are able to find the value of the new dependent variable T_e when $\frac{dT}{dt} = 0$

Now, the method of integrating factor shall be used in solving equation 4.1

$$\frac{dT}{dt} = K(T - T_m)$$

$$\frac{dT}{dt} = KT - KT_m$$

$$\frac{dT}{dt} = KT = -KT_m$$

By the method of integrating factor.

I.F = $e^{\int kdt} = e^{-kt}$

$$e^{-kt} \frac{dT}{dt} - ke^{-kt} T = -kT_m e^{-kt}$$

Multiply ugh by the integrating factor

Taking the compact form on the LHS of the expression.

Integrating both sides

$$\frac{d}{dt}(e^{-kt} T) = kT_m e^{-kt}$$

$$e^{-kt} T = -kT_m \int e^{-kt} + C$$

$$e^{-kt} T = \frac{-k}{-k} T_m e^{-kt} + C$$

$$T(t) = T_m + Ce^{kt}$$

Assuming K < 0, therefore. The method of integrating factor and method of separation of variables produce the same general solution.

Given $T_m = 35^0$ k = 0.06

$T(t) = 35 = Ce^{-0.06t}$

We propose to find the particular solution given the initial condition T(0) = 15,

$$T(0) = 15 = 35 + C, e^0 = 1$$

$$35 + C = 15$$

$$C = 15 - 35$$

$$C = -20$$

Varying t and keeping C constant, we tabulate below:

Example	t(days)	Constant	$Ce^{-0.20t}$ where c = -20 =27 K= 1770	We tabulate below; T(t)
1	0	35	20	15
2	10	35	10.97623272	24.02376728
3	20	35	6.023884238	28.97611576
4	30	35	3.305977764	31.69402224

5	40	35	0.995741367	33.18564093
6	50	35	0.995741367	34.00425863
7	60	35	0.546474448	34.45352555
8	70	35	0.299911536	34.70008846
9	80	35	0.008229747049	34.99177025
10	90	35	0.0903331618	34.90966838
11	100	35	0.049575043	34.95042496
12	110	35	0.02720736	34.97279264
13	120	35	0.014931716	34.98506828
14	130	35	0.00819469958	34.9918053
15	140	35	0.004497456484	34.9918053
16	160	35	0.002468196082	34.9975318
17	170	35	0.00135457473	34.99864543
18	180	35	0.0007434063737	34.99925659
19	190	35	0.0004079900682	34.99989201
20	200	35	0.0002239096969	34.99977609
21	210	35	0.0001228842471	34.99987712
22	200	35	0.00006744030468	34.99993256
23	210	35	0.00003701202395	34.99996299
24	220	35	0.00002031262942	34.99997969
25	230	35	0.00001114780739	34.99998885
26	240	35	0.0000011804641	34.99999388
27	250	35	0.00000335765506	34.99999664
28	260	35	0.000001842720167	34.99999816
29	270	35	0.00000101130627	34.99999899
30	290	35	0.000005550166484	34.99999944
31	300	35	0.0000003045995949	34.99999997
32	310	35	0.000000167167802	34.99999983
33	320	35	0.0000000174363493	34.99999991
34	330	35	0.00000005034997439	34.99999995
35	340	35	0.00000002763265182	34.99999997
36	350	35	0.00000001516512086	34.99999998
37	360	35	0.000000008322794788	34.99999999
38	370	35	0.000000004567646625	34.0000000

Using the analytical method we are able to show that the temperature of the cooling body begins to saturate after 60 minutes and fully saturate after 370 minutes over a long period therefore, this can be used to determine the temperature of cooling (body(T(t) over a long period of time.

The Mixture of Two Salt Solutions of Differing Concentration

The mixing at two salt solutions of differing concentration gives rise to a first-order differential equation for the amount of salt contained in the mixture (Zill, 1997). Let us suppose that a large mixing tank holds 300 gallons of water in which salt has been dissolved. Another brine solution is pumped into the large tank at a rate of 3gal/min. and then when the solution is well stored it is pumped out at the same rate. If the concentration of the solution entering is 21b/gal, determine a model for the amount of salt in the tank at any time.

Falling Bodies and Air Resistance

Under some circumstances a falling body of mass in encounters air resistance proportional to its instantaneous velocity v (Zill, 1997). If we take, in this circumstance, the positive direction to be oriented downward, then the net force acting on the mass is given by mg-kv, where the weight mg of the body is a force acting in the opposite, or upward, direction.

V. CONCLUSION

The key achievements of this project are briefly stated as follows;

A steady-state solution for a linear differential equation has been calculated.

Given an appropriate initial condition, the value of the dependent variable over time can be predicted. The idea of a steady-state solution has been illustrated with three real life examples drawn from the field of science.

The idea of predicting the dependent variable over a long period of time has been successfully predicted in this project. However, the stochastic analysis of this level was not considered in this present study.

A rare discovery has been made in the case of determining steady-state solution from the Newton's law of cooling. The constant of proportionality must always be negative in order to establish the link between the particular solution and the steady-state value. The positive value of the constant of proportionality is a counter intuitive example of this idea because at $t\rightarrow\infty$, the particular solution loses its steady-state value.

REFERENCES

[1] Arthur, M. and MIT (2003). *Ordinary differential Equation.* New York: Dover Publications Incorporated.

[2] Boyd, S. and Vanderberghe, L. (2004). *Introduction to Matrix Methods and Application*, Drilllmlls and Sons Publications, Georgia.

[3] Canor, M. *History of modem Mathematics - Differential...,* Geschichte der Mahematik, vol. III. p. 429; and excellent historical view; reviews by Matthews in Nature, vol. LII. p.513.

[4] Ekaka-a E. N. (2009). Computational and Mathematical Modeling Lolka Voltera System on Data Set

[5] Erwin K. (2006). *Advance Engineering Mathematics,* 9th Edition, John Willey & Sons Publication Inc.

[6] Florida. "McNair Scholars Research Journal. Vol. 1, Article 8. Available at http://www.commons.erau.edu/mcnair/vol.1/1ss1/8

[7] Gabriel N. (2006). Ordinary Differential Equation. Mathematics Department,

[8] Gupta B.D. (2010). Mathematical Physics 4th edition. India: Vikas Publishing House PVT LTD

[9] Hain B. and Felix B. (1977). *Advances in Mathematics* 135. 76-144, Article No. A1971713. Ordinary Differential Equation m the 20th Century, Rutgers University, New Brumswick. New Jersey. 08903 (Received September 11, 1997).

[10] James Stewart (2008). Single Variable Calculus: 6th Edition, Publisher-Thompson.

[11] Lie, S. (1878). *Some development in the theory of Ordinary Differential equations*, Bulletin. New York: Mathematical Society, Vol. II, p. 119, Grunert's archive fiir Mathematik, Vol. LIV.

[12] Michigan State University, East Lansing, MI, 48824.

[13] Nse, C.A. (2012) Fundamentals of Differential Equations. Alphabet Nigeria Publisher © Celestin A. Nse 2012 ISBN 978-046-746-7 First Published 2012.

[14] Seymour L. and Mac L. (199). *Shuaum Outlines of theory and problem of Linear Algebra,* Third Edition. McGraw Hill. Inc. New York.

[15] Stroud K.A. (2007). Engineering Mathematics 6th Edition Published by PLAGRAVE MACMILLAN Hundmills. Basingstoke. Hampshire RTG21 6 X S AND 175 Fifth Avenue, New York. N Y. 10010 Companies and Representation throughout the world.

[16] Stroud, K. A. and Booth, D.J. (2001). *Advance Engineering Mathematics.*

[17] Thomas H. (2000). *The theory of Matrices in the 19th Century,* retrieved 20, September, 2015, from http: oogleScholar.com.

[18] William R.D. and Stanley I.G. (1984). *Elementary Differential Equation.* Crowtel Publications, Chicago.

[19] Zill D.G. (1997). A First Course in Differential Equation with Modeling Application 6th Edition, Brooks/Cole Publishing Company.