Generalization of Pure-Supplemented Modules

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Abstract: **Let R be a ring and M be an R-module. We generalized the concepts pure-lifting and pure-supplemented module and introduce weak distribution with fully invariant. We prove every pure g-lifting is pure g-supplemented module. Let M be a weak distribution pure g-supplemented module, then M/A is pure g-supplemented module for every submodule A of M. Let $M = M_1 \oplus M_2$ be a weakly distributive R-module. Then each M_i, $i \in \{1, 2\}$ is closed weak g-supplemented if and only if M is closed weak g-supplemented.**

Key Words: **g-small, g-supplemented, pure-lifting, pure-supplemented, pure g-supplemented, closed weak g-supplemented, Distributive, weak Distributive modules**.

I. INTRODUCTION

Throughout this paper R is an associative ring with unity and all modules are unitary R-modules. [12] Sahira M. Yasen and W. Khalid Hasan introduce the concepts pure-module and pure-supplemented module with some conditions. Let M be an R-module, a sub module L of module M is denoted by $L \leq M$. submodule L of M is called essential (large) in M, abbreviated $K \leq_e M$, if for every submodule N of M, $L \cap N$ implies $N = 0$. A submodule N of a module M is called small in M, denoted by $N \ll M$, if for every sub module L of M, the equality $N + L = M$ implies $L = M$. [2] A submodule K of m is called generalized small (g-small) submodule of M denoted by $N \ll_g M$, if for every essential submodules T of M with the property $M = K + T$ implies that $T = M$. Supplemented modules and two other generalizations amply supplemented and weakly supplemented modules were studied by Helmut Zoschinger and he posed their whole structure over discrete valuation rings. ¨ After Zoschinger, some variations of supplemented modules were studied. Let M be an R-module and U, V are submodules of M. If $M = U + V$ and V is minimal with respect to property, or equivalently, $M = U + V$ and $U \cap V << V$, then V is called a supplement of U in M. M is called supplemented if every submodule of M has supplement in M. If $M = U + V$ and $U \cap V << M$, then V is called a weak supplement of U in M.[10] M is called weak supplemented if every submodule of M has weak supplement in M. [12] Let M be an R-module. P is called a g-pure sub module of M if $KM \cap P = KP$ for every ideal in R. An R-module M is called lifting if for every submodule N of M there is a decomposition $M = M_1 + M_2$ such that $M_1 \leq N$ and $N \cap M_2 \ll M$. An R-module M is called pure-lifting module if for every submodule A of M there exists a pure submodule P of M, $P \leq A$ such that $M = P + X$ with $A \cap X \ll X$. Let M be a module. M is called Pure g-lifting module for every submodule A of M there exists a g- pure submodule P of M, $P \leq A$ such that $M = P + X$ with $A \cap X << gM$. Every g-lifting module is pure g-lifting module. An R-module M is called pure-supplemented module if for given any submodule A of M there exists a pure submodule P of M such that $M = A + X$ iff $M = P + X$. [2] B. Kosen, C. Nebiyen and a. Pakin, introduce the concept g-supplemented module. Let M be an R-module and U, V are submodules of M. If $M = U + V$ and $M = U + T$ with T is essential in V implies $T = V$, or equivalently, $M = U + V$ and $U \cap V <<_g M$, then V is called g-supplement of U in M.

In this paper we generalized the concepts pure-lifting and pure-supplemented module. The concepts small, e-small, c-small and g-small play a key role in the study of supplemented, weak supplemented pure- supplemented and pure-lifting modules.

Proposition: 1) Every hollow module is pure g-supplemented module.

2) Every lifting module is pure g-supplemented module.

3) Every pure g-supplemented module is weakly g-supplemented module.

Proof: [12].

Theorem: The following are equivalent for an R-module:

1) M is pure-g-lifting module.
2) Every essential submodule N of M can be written as $N = A + K$, where A is g-pure in M and $K <<_g M$.
3) For every essential submodule N of M there exists a pure g-submodule A of N such that $M = A + K$ and $\frac{N}{A} <<_g \frac{M}{A}$.

Proof: $1 \Rightarrow 2$. Let M is pure-g-lifting module i.e. for every submodule N of M there exists a g- pure submodule P of M, $P \leq N$ such that $M = P + X$ with $N \cap X << X$. Hence $N \cap X <<_g M$. We have $N = N \cap M = N \cap (P + X) = N \cap T + N \cap X = P + (N \cap X)$. If $A = P$, then $K = N \cap X$ with ($K <<_g M$).

$2 \Rightarrow 3$. Let N be an essential submodule of M, since $N = A + K$, where A is g-pure in M and $K << gM$. We have $M = N + L$, therefore $\frac{M}{A} = \frac{A+K}{A} + \frac{L}{A}$, this implies $A + K + L = M$.

Since $K << gM$, therefore $A + K = M$ and $\frac{N}{A} <<_g \frac{M}{A}$.

$3 \Rightarrow 1$. Let N be an essential submodule of M, there exists a pure g-submodule A of N such that M = A + K and $\frac{N}{A} \ll_g \frac{M}{A}$ to prove that N ∩ K<< K. Suppose that N ∩ K + T = K, where T ≤ K. Then M = A + K = A + N ∩ K + T implies

$$\frac{M}{A} = \frac{A+(N+K)+T}{A} = \frac{(N+K)+A}{A} + \frac{A+T}{A} = \frac{N}{A} + \frac{A+T}{A}$$

, since $\frac{N}{A} \ll \frac{M}{A}$ therefore $\frac{M}{A} = \frac{A+T}{A}$. Hence A + T = M and T = K. Thus N ∩ K << K. //

Proposition: Every Pure g-lifting is pure g-supplemented module.

Proof: Let M be pure g-lifting module and A be a a submodule of M. i.e. for every submodule A of M there exists a g- pure submodule P of M, P ≤ A such that M = P + X with

A ∩ X<< gM. Suppose that M = A + X then M = P + T, where P ≤ A and P << gM and

A ∩T<< gM. We have A = A ∩ M = A ∩ (P + T) = A ∩ P + A ∩ T = P + (A ∩ T), then

Let M = A + X = P + (A ∩ T) + X. Since (A ∩ T) << gM. (A ∩ T),then M = P + X thus

M = P + X, since P ≤ A such that M = P + X with A ∩ X<< gM. //

Proposition: Let M be an R-module is pure g-supplemented module if and only if for every submodule A of M there exist a g-pure submodule P of M such that $\frac{A+P}{P} \ll_g \frac{M}{P}$ and $\frac{A+P}{A} \ll_g \frac{M}{A}$.

Proof: ⇒ Let M be an R-module is pure g-supplemented module i.e. every submodule A of M there exists a g- pure submodule P of M, P ≤ A such that M = A + X iff M = P + X with

A ∩ X<< gX. Let K ≤ M and suppose that $\frac{A+P}{P} + \frac{K}{P} = \frac{M}{P}$ then $\frac{A+P+K}{P} = \frac{M}{P}$ implies $\frac{A+K}{P} = \frac{M}{P}$.

Then A + K = M. Since M is pure g-supplemented, therefore A + K = M = P + X, P ≤ A, then

A + X ≤ K, implies M ≤ K. This shows K= M i.e. $\frac{K}{P} = \frac{M}{P}$

therefore $\frac{A+P}{P} \cap \frac{K}{P} \ll_g \frac{K}{P}$ then $\frac{A+P}{P} \ll_g \frac{M}{P} \Rightarrow \frac{A+P}{A} \ll_g \frac{M}{A}$ with $\frac{A+P}{P} \cap \frac{K}{P} \ll_g \frac{M}{P}$.

⇐ Let A be a submodule of M, then there exist a g-pure submodule P of M such that $\frac{A+P}{P} \ll_g \frac{M}{P}$ and $\frac{A+P}{A} \ll_g \frac{M}{A}$. If M = A + X (X ≤M) then $\frac{M}{P} = \frac{A+P}{P} + \frac{X+P}{P}$. But $\frac{A+P}{P} \ll_g \frac{M}{P}$. Then $\frac{M}{P} = \frac{X+P}{P}$, thus M = X + P ⇒ M = X + A and X + A $\ll_g$M. //

Proposition: Let M be pure g-Supplemented module and A be a sub module of M. If for every g-pure submodule of M $\frac{A+P}{A} \ll_g \frac{M}{A}$ then $\frac{M}{A}$ is pure g-supplemented module.

Proof: Let N ≤ M and let M = N + X for X is a submodule of M. Then $\frac{N}{A} \le \frac{M}{A}$ and $\frac{M}{A} = \frac{N}{A} + \frac{X}{A}$, for A ≤ X. Since M = N + X, therefore M = P + X, where P is g-pure in M. Then $\frac{M}{A} = \frac{P+X}{A}$

$\frac{M}{A} = \frac{P+A}{A} + \frac{X}{A}$. Since $\frac{P+A}{A} \cap \frac{X}{A} \ll_g \frac{M}{A}$ therefore $\frac{M}{A} = \frac{X}{A}$ implies $\frac{M}{A}$ is pure g-supplemented. //

Recall that a sub module N of M is called fully invariant, if f(N) ≤ N, for each f $\in End_R(M)$. The set $f^{-1}(N)$ = {m ∈ M : f (m) ∈ N}. Note that $f^{-1}(N)$ is a submodule of M and that f ($f^{-1}(N)$) ≤ N. Note further that f ($f^{-1}(N)$) = N in case f is an epimorphism. Moreover, for any submodules L ≤ N of M, we have $f^{-1}(L) \le f^{-1}(N)$.A module M is called duo module if every essential submodule is fully invariant . [9] Let M be an R- module and U≤ M. A submodule U is said to be a distributive submodule of M if U = U ∩ X + U ∩ Y for all X, Y ∈ M. A module M is called distributive if and only if for every submodules K, L, N of M such that N + (K ∩ L) = (N + K) ∩ (N + L) or N ∩ (K + L) = (N ∩ K) + (N ∩ L). Weakly distribution module are proper generalization of distributive modules. A submodule U is said to be a weak distributive submodule of M if U = U ∩ X + U ∩ Y for all X, Y ∈ M such that X + Y = M. A module M is said to be weakly distributive if for every submodule of M is a weak distributive submodule of M. A ring R is weakly distributive if R is a weakly distributive left R-module.

Proposition: Let M be a weak distribution pure g-supplemented module, then M/A is pure g-supplemented module for every submodule A of M.

Proof: Let X be direct summand of M, then M = X ⊕ Y for some Y submodule of M.

Since M = X + Y, therefore $\frac{M}{A}=\frac{X}{A}+\frac{Y}{A}$ and $\frac{U}{A}\leq\frac{M}{A}$.

Since M is a weak distributive pure g-supplemented module. $U=(U\cap X)+(U+Y)$ i.e.

$$\frac{U}{A}=\frac{(U\cap X)}{A}+\frac{(U\cap Y)}{A}=\left(\frac{U}{A}\cap\frac{X}{A}\right)+\left(\frac{U}{A}\cap\frac{Y}{A}\right)=\frac{U}{A}\cap\left(\frac{X}{A}+\frac{Y}{A}\right)=\frac{U}{A}\cap\frac{M}{A}$$

with $\frac{X}{A}\cap\frac{Y}{A}=\{0\}$

$\Rightarrow\ \frac{M}{A}=\frac{X\cap A}{A}\oplus\frac{Y\cap A}{A}$. Hence $\frac{X\cap A}{A}$ is a direct summand of $\frac{M}{A}\Rightarrow\frac{M}{A}=\frac{X\cap A}{A}+\frac{Y\cap A}{A}$.

Hence $\frac{M}{A}$ is a pure g-supplemented module. //

Proposition: Let A be a sub module of M and eA ≤ A for all $e^2=e\in End_R(M)$ then $\frac{M}{A}$ is pure g-supplemented module. In particular for every fully invariant submodule Y of M, $\frac{M}{Y}$ is pure g- supplemented module.

Proof: Let X be the direct summand of M. Now the projection e : M→X, then $e^2=e\in End_R(M)$ and eA ≤ A, where A is submodule of M. Hence eA = A ∩ X. Then M = X + Y, for some Y∈M,

A = (A ∩ X) + (A ∩ Y). Now $\frac{X+A}{A}=\frac{X\oplus(A\cap Y)}{A}$ and $\frac{Y+A}{A}=\frac{Y\oplus(A\cap X)}{A}$.

M = X ⊕ Y = (X +A) ⊕ (Y +A) = {X ⊕ (A ∩ Y)} + (Y + A).

Then 1. $\frac{M}{A}=\frac{X\oplus(Y+A)}{A}+\frac{Y+A}{A}$

2. {X ⊕ (A ∩ Y)} ∩ (Y + A) = {(X⊕A) ∩ (X⊕Y)}∩ (Y + A)

= {[(X⊕A) ∩Y]∩[(X⊕Y)∩Y}+{[(X⊕A) ∩A]∩[(X⊕Y)∩A}

= (A∩Y)∩(A∩A) = A

Then $\frac{M}{A}=\frac{X\oplus(Y+A)}{A}\oplus\frac{Y+A}{A}$. Therefore $\frac{Y+A}{A}$ is direct summand of $\frac{M}{A}$ with $\frac{Y+A}{A}<<_g\frac{M}{A}$.

Hence $\frac{M}{A}$ is pure g-supplemented. //

Theorem: Let $M=M_1\oplus M_2$ be a weakly distributive R-module. Then each M_i, i∈{1, 2}is closed weak g-supplemented if and only if M is closed weak g-supplemented.

Proof: Let $A\leq^c M$. Since M_i, i∈{1, 2}is closed weak g-supplemented R-modules. Let $M=M_1+M_2$, M_1, M_2 are submodules of M. We have $A\cap M_i\leq^c M_i$. Let $A\cap M_i\leq eB$ in Mi, since M is a weakly distributive R-module.

We have $A=A\cap M=A\cap(M_1\oplus M_2)=(A\cap M_1)\oplus(A\cap M_2)\leq_e B\oplus(A\cap M_2)$ in M. Since $A\leq^c M\ \Rightarrow A=(A\cap M_1)\oplus(A\cap M_2)=B\oplus(A\cap M_2)$, therefore $A\cap M_1=B$, thus $A\cap M_1\leq^c M_1$. Similarly $A\cap M_2\leq^c M_2$. Since M_1, M_2 are closed weak g-supplemented R-modules. Then there are sub modules N_1, N_2 such that $M_1=N_1+(A\cap M_1)$ and $N_1\cap(A\cap M_1)=N_1\cap A<<_g M_1$. Similarly $M_2=N_2+(A\cap M_2)$ and $N_2\cap(A\cap M_2)=N_2\cap A<<_g M_2$. Put $N=N_1\oplus N_2$. So we get

$M=M_1\oplus M_2=\{N_1+(A\cap M_1)\}\oplus\{N_2+(A\cap M_2)\}$

$=(N_1\oplus N_2)+\{(A\cap M_1)\oplus(A\cap M_2)\}$

$=(N_1\oplus N_2)+\{A\cap(M_1\oplus M_2)\}$

$=(N_1\oplus N_2)+\{A\cap M\}=(N_1\oplus N_2)+A$

∴ M = X + A. Since M is weakly distributive module. Now $X\cap A=(N_1\oplus N_2)\cap A$

$=(N_1\cap A)\oplus(N_2\cap A)<<_g(M_1\oplus M_2)=M$. Then X is weak g-supplement of A in M.

hence M is closed weak g- supplemented. //

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