Differential Transform Method (DTM) for the Solution of Linear Ordinary Differential Equations of First Order, Second Order and Third Order

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***Abstract:* - In this study, the analysis of Differential Transform Method (DTM) for the solution Ordinary Differential Equation was explored. The solution to the Differential Equations addressed in this study were presented via Differential Transform Method for problems such as 1st, 2nd and 3rd order linear differential equations respectively. Similarly, comparisons between the exact solution to the problems and their DTM results were made in tabular form. The outcome of the comparison showed that our numerical results compete favourably with the exact solutions of the problems considered.**

***Keywords:* Transforms, Differential Transform Method (DTM), Numerical solution, first, second and third Order Differential Equations, Tabular comparison, exact solution.**

I. INTRODUCTION

Under this subheading, we reviewed a few number of researchers that have utilized DTM in solving ordinary differential equations. A numerical expert like [1] used Differential transform method to solve cholera model, and he thereafter recommends researchers to use the method for solution of similar other problems because of its easy-to-use mode. Equally, [4] used Differential transform method to solve fourth order differential equation and thereafter recommends researchers to use the method to solve other problems because of it reduces computational difficulties compared to other numerical methods. To obtain a solution to a proposed Dynamical behaviour of pest management model and impulsive effect and nonlinear incidence rate, the expert [6] and [5] utilized DTM for his analysis. Consequently [3] authored a text recommending DTM as one of the suitable methods of handling ordinary differential equations (ODE). Buttressing his remarks, [7] in his lecture notes recommends the same DTM as a new means of handling (ODE)

In like manner, addressing nonlinear problems [2] used differential transform method to solve Riccati equation and he thereafter he concluded that the method is reliable and introduces a significant improvement in solving differential equations because of its small size of computation in comparison with the computational size required in numerical methods and the rapid convergence.

II. RESEARCH METHODOLOGY

In this section, we utilized the format of table 2.0 to formulate a numerical scheme, perform the necessary iterations for all our research numerical examples presented in section 3.0 below.

Table 2.1: Differential Transforms of some functions

S/No	Original function	Transformed functions
1	$y(x) = g(x) \pm h(x)$	$Y(k) = G(k) \pm H(k)$
2	$y(x) = \alpha g(x)$	$Y(k) = \alpha G(k)$
3	$y(x) = \frac{dg(x)}{dx}$	$Y(k) = (k+1)Y(k+1)$
4	$y(x) = \frac{d^2 g(x)}{dx^2}$	$Y(k) = (k+1)(k+2)Y(k+2)$
5	$y(x) = 3$	$Y(k) = 3\delta(k)$
6	$y(x) = g(x)h(x)$	$Y(k) = \sum_{m=0}^{k} H(m)G(k-m)$

III. RESULTS AND DISCUSSIONS

The DTM computation results are as presents in the table and numerical examples in 3.1, 3.2 and 3.3 below.

Example 3.1

Solve $y' + 4y = 0, y(0) = 1$

Solution

Taking the differential transform of the given linear Differential Equation above gives,

$$(k+1)Y(k+1) + 4Y(k) = 0$$

$$Y(k+1) = \frac{1}{k+1}\left[-4Y(k)\right]$$

When k=0

$$Y(1) = -4Y(0)$$

$$Y(1) = -4$$

When k=1

$$Y(2) = \frac{-4Y(1)}{2} = 8$$

When k=2

$$Y(3) = \frac{-4Y(2)}{3}$$

$$Y(3) = \frac{-32}{3}$$

When k=3

$$Y(4) = \frac{-4Y(3)}{4}$$

$$Y(4) = \frac{128}{12}$$

When k=4

$$Y(5) = \frac{-4Y(4)}{5}$$

$$Y(5) = -\frac{128}{15}$$

When k=5

$$Y(6) = \frac{-4Y(5)}{6}$$

$$Y(6) = \frac{512}{90}$$

When k=6

$$Y(7) = \frac{-4Y(6)}{6}$$

$$Y(7) = -\frac{2048}{630}$$

When k=7

$$Y(8) = \frac{-4Y(7)}{7}$$

$$Y(8) = \frac{8192}{5040}$$

When k=8

$$Y(9) = \frac{-4Y(8)}{9}$$

$$Y(9) = -\frac{8192}{11340}$$

$$\therefore y(t) = \sum_{k=0}^{\infty} Y(K)t^K = Y(0)t^0 + Y(1)t^1 + Y(2)t^2 + \cdots$$

Table 3.1: Tabular values for example 3.1

t	DTM	EXACT	ERROR
0.1	0.67032	0.67032	2.78794E-11
0.2	0.449329	0.449329	2.75726E-08
0.3	0.301193	0.301194	1.53717E-06
0.4	0.20187	0.201897	2.64151E-05
0.5	0.135097	0.135335	0.000238281
0.6	0.089288	0.090718	0.001430259
0.7	0.054327	0.06081	0.006482639
0.8	0.016836	0.040762	0.02392665
0.9	-0.04818	0.027324	0.075499426

Remark: The DTM results agree with Exact solution in the table above example.

Example 3.2

$$y' - 3y = 0, y(0) = 2$$

Solution

Taking transform of the linear equation

$$(k+1)Y(k+1) - 3Y(k) = 0$$

$$Y(k+1) = \frac{1}{k+1}\left[3Y(k)\right]$$

When k=0

$$Y(1) = 3Y(0)$$

$$Y(1) = 6$$

When k=1

$$Y(2) = \frac{3Y(1)}{2}$$

$$Y(2) = 9$$

When k=2

$$Y(3) = \frac{3Y(2)}{3}$$

$Y(3) = 9$

When k=3

$$Y(4) = \frac{3Y(3)}{4}$$

$$Y(4) = \frac{27}{4}$$

When k=4

$$Y(5) = \frac{3Y(4)}{5}$$

$$Y(5) = \frac{81}{20}$$

When k=5

$$Y(6) = \frac{3Y(5)}{6}$$

$$Y(6) = \frac{243}{120}$$

When k=6

$$Y(7) = \frac{3Y(6)}{7}$$

$$Y(7) = \frac{243}{280}$$

When k=7

$$Y(8) = \frac{3Y(7)}{8}$$

$$Y(8) = \frac{729}{2240}$$

When k=8

$$Y(9) = \frac{3Y(8)}{8}$$

$$Y(9) = \frac{2187}{19683}$$

$$\therefore y(t) = \sum_{k=0}^{\infty} Y(K)t^K = Y(0)t^0 + Y(1)t^1 + Y(2)t^2 + \cdots$$

Table 3.2: Tabular results for Example 3.2

t	DTM	EXACT	ERROR
0.1	2.699718	2.699718	7.1676E-13
0.2	3.644238	3.644238	2.1778E-09
0.3	4.919206	4.919206	1.5742E-07
0.4	6.640231	6.640234	3.1366E-06
0.5	8.963347	8.963378	3.1593E-05
0.6	12.09909	12.09929	0.00020809
0.7	16.33131	16.33234	0.00102526
0.8	22.04226	22.04635	0.0040904
0.9	29.74556	29.75946	0.01390181
1	40.12941	40.17107	0.04165916

Remark: DTM agrees with the Exact solution in the table above example.

Example 3:3

$$y'' - 7y' + 12y = 0 \quad y(0) = 0, y'(0) = 1$$

Taking the Differential Transform of both sides yields

$$(k+1)(k+2)Y(k+2) - 7(k+1)Y(k+1) + 12Y(k) = 0$$

$$(k+1)(k+2)Y(k+2) = 7(k+1)Y(k+1) - 12Y(k)$$

$$Y(k+2) = \frac{7(k+1)Y(k+1) - 12Y(0)}{(k+1)(k+2)}$$

When k=0

$$Y(2) = \frac{7(1)Y(1) - 12Y(0)}{(1)(2)}$$

$$Y(2) = \frac{7}{2}$$

$$Y(2) = 3.5$$

When k=1

$$Y(1+2) = \frac{7(1+1)Y(1+1) - 12Y(1)}{(1+1)(1+2)}$$

$$Y(3) = \frac{7(2)Y(2) - 12(1)}{(2)(3)}$$

$$Y(3) = \frac{37}{6}$$

$$Y(3) = 6.166667$$

When k=2

$$Y(4) = \frac{7(3)Y(3) - 12Y(2)}{(3)(4)}$$

$$Y(4) = \frac{7(3)(6.166667) - 12(3.5)}{12}$$

$$Y(4) = \frac{87.500007}{12}$$

$$Y(4) = 7.291667$$

When k=3

$$Y(5) = \frac{7(4)Y(4) - 12Y(3)}{(4)(5)}$$

$$Y(5) = \frac{7(4)(7.291667) - 12(6.166667)}{20}$$

$$Y(5) = \frac{130.166679}{20}$$

$$Y(5) = 6.508334$$

When k=4

$$Y(6) = \frac{7(5)Y(5) - 12Y(4)}{(5)(6)}$$

$$Y(6) = \frac{7(5)(6.508334) - 12(7.291667)}{30}$$

$$Y(6) = \frac{140.291686}{30}$$

$$Y(6) = 4.676389$$

When k=5

$$Y(7) = \frac{7(6)Y(6) - 12Y(5)}{(6)(7)}$$

$$Y(7) = \frac{7(6)(4.676389) - 12(6.508334)}{42}$$

$$Y(7) = \frac{118.30833}{42}$$

$$Y(7) = 2.816865$$

When k=6

$$Y(8) = \frac{7(7)Y(7) - 12Y(6)}{(7)(8)}$$

$$Y(8) = \frac{81.909717}{56}$$

$$Y(8) = 1.462674$$

When k=7

$$Y(9) = \frac{7(8)Y(8) - 12Y(7)}{(8)(9)}$$

$$Y(9) = \frac{48.107364}{72}$$

$$\therefore\ y(t) = \sum_{k=0}^{\infty} Y(K)t^K = Y(0)t^0 + Y(1)t^1 + Y(2)t^2 + \cdots$$

Table 3.3: Tabular results of example 3.3

t	DTM	EXACT	ERROR
0.1	0.141966	0.141966	6.00345E-07
0.2	0.403441	0.403422	1.91733E-05
0.3	0.860658	0.860514	0.000144005
0.4	1.633496	1.632916	0.000580974
0.5	2.908917	2.907367	0.001549874
0.6	4.97609	4.973529	0.002561448
0.7	8.278272	8.278477	0.00020467
0.8	13.48801	13.50935	0.021340474
0.9	21.61411	21.7185	0.104390275
1.0	34.15075	34.51261	0.36185911

Remark: The DTM results agrees with those of the Exact solution method

Example 3.4

$$y'' - 2y' - 35y = 0, y'(0) = 2, y(0) = 0$$

Solution

Taking Differential Transform Method of both sides gives

$$(k+1)(k+2)Y(k+2) - 2[(k+1)Y(k+1)] - 35Y(k)$$

$$(k+1)(k+2)Y(k+2) = 2[(k+1)Y(k+1)] + 35Y(k)$$

$$Y(k+2) = \frac{2[(k+1)Y(k+1)] + 35Y(k)}{(k+1)(k+2)}$$

When k=0

$$Y(2)=\frac{2[(1)Y(1)]+35Y(0)}{(1)(2)}$$

$$Y(2)=19.5$$

When k=1

$$Y(3)=\frac{2[(2)Y(2)]+35Y(1)}{(2)(3)}$$

$$Y(3)=24.666666667$$

When k=2

$$Y(4)=\frac{2[(3)Y(3)]+35Y(2)}{(3)(4)}$$

$$Y(4)=69.20833335$$

When k=3

$$Y(5)=\frac{[(4)Y(4)]+35Y(3)}{(4)(5)}$$

$$Y(5)=70.85000001$$

When k=4

$$Y(6)=\frac{2[(5)Y(5)]+35Y(4)}{(5)(6)}$$

$$Y(6)=104.3597222$$

When k=5

$$Y(7)=\frac{2[(6)Y(6)]+35Y(5)}{(6)(7)}$$

$$Y(7)=88.85873016$$

When k=6

$$Y(8)=\frac{2[(7)Y(7)]+35Y(6)}{(7)(8)}$$

$$Y(8)=87.74593254$$

When k=7

$$Y(9)=\frac{2[(8)Y(8)]+35Y(7)}{(8)(9)}$$

$$Y(9)=62.69431217$$

$$\therefore\ y(t)=\sum_{k=0}^{\infty}Y(K)t^K=Y(0)t^0+Y(1)t^1+Y(2)t^2+\cdots$$

Table3.4: Results of example 3.4

Y(t)	DTM	EXACT	ERROR
0.1	1.42741019	1.42741019	1.81762E-09
0.2	2.51881181	2.51881641	4.60473E-06
0.3	4.85625566	4.85657002	0.00031435
0.4	9.64288768	9.64910032	0.00621264
0.5	19.2879535	19.35154910	0.06359553
0.6	38.4887879	38.92110439	0.43231650

Remark: The DTM results agrees with those of the Exact solution method to the problem

3.5: Linear Differential Equation of 3rd order

Example 3:5

$$y'''+y''-10y'+8y=0,\ y''(0)=y'(0)=1,\ y(0)=2$$

Taking the Differential Transform Method of the equation

$$(k+1)(k+2)(k+3)Y(k+3)+(k+1)(k+2)Y(k+2)-10[(k+1)Y(k+1)]+8Y(k)=0$$

$$(k+1)(k+2)(k+3)Y(k+3)=-[(k+1)(k+2)Y(k+2)]+10[(k+1)Y(k+1)]-8Y(k)$$

$$Y(k+3)=\frac{-[(k+1)(k+2)Y(k+2)]+10[(k+1)Y(k+1)]-8Y(k)}{(k+1)(k+2)(k+3)}$$

When k=0

$$Y(3)=\frac{-[(1)(2)Y(2)]+10[(1)Y(1)]-8Y(0)}{(1)(2)(3)}$$

$$Y(3)=-1.333333333$$

When k=1

$$Y(4)=\frac{-[(2)(3)Y(3)]+10[(2)Y(2)]-8Y(1)}{(2)(3)(4)}$$

$$Y(4)=0.833333333$$

When k=2

$$Y(5)=\frac{-[(3)(4)Y(4)]+10[(3)Y(3)]-8Y(2)}{(3)(4)(5)}$$

$$Y(5) = -0.96666666$$

When k=3

$$Y(6) = \frac{-[(4)(5)Y(5)] + 10[(4)Y(4)] - 8Y(3)}{(4)(5)(6)}$$

$$Y(6) = 0.52777777$$

When k=4

$$Y(7) = \frac{-[(5)(6)Y(6)] + 10[(5)Y(5)] - 8Y(4)}{(5)(6)(7)}$$

$$Y(7) = -0.59027777$$

When k=5

$$Y(8) = \frac{-[(6)(7)Y(7)] + 10[(6)Y(6)] - 8Y(5)}{(6)(7)(8)}$$

$$Y(8) = 0.19104662$$

When k=6

$$Y(9) = \frac{-[(7)(8)Y(8)] + 10[(7)Y(7)] - 8Y(6)}{(7)(8)(9)}$$

$$Y(9) = -0.111587852$$

$$\therefore y(t) = \sum_{k=0}^{\infty} Y(K)t^K = Y(0)t^0 + Y(1)t^1 + Y(2)t^2 + \cdots$$

Table3.5: Results of example 3.5

T	DTM	EXACT	ERROR
0.1	2.108740804	2.103726935	0.005013869
0.2	2.230383988	2.208771784	0.021612204
0.3	2.358666994	2.302789564	0.055877430
0.4	2.487391952	2.36740504	0.119986912
0.5	2.609371647	2.374996043	0.234375604
0.6	2.715016301	2.283914628	0.431101674
0.7	2.790273434	2.031371465	0.758901969

Remark: The DTM results agrees with those of the Exact solution method

IV. CONCLUSION

This differential Transform Method (DTM) has helped us in successfully solving the lists of Differential Equations addressed by this study. A quick inspection of all the problems considered revealed that our DTM results corresponded with the esact solutions to the problems. Similarly, it could be observed that the DTM was implemented without any further need for discretisation, perturbation and linearization. Thus, for ease of solution to any linear Differential equation/ problems without cumbersome algebraic computations, this study therefore recommends DTM for future and as an alternative to exact method of solution.

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