Method of Taylor's Series for the Primitive of Linear Second Kind Non-Homogeneous Volterra Integral Equations

Sudhanshu Aggarwal[1*], Kavita Bhatnagar[2], Arti Dua[3]

[1]*Assistant Professor, Department of Mathematics, National P.G. College, Barhalganj, Gorakhpur-273402, U.P., India*
[2]*Assistant Professor, Department of Mathematics, Noida Institute of Engineering & Technology, Greater Noida-201306, U.P., India*
[3]*Assistant Professor, Department of Applied Science & Humanities, I.T.S. Engineering College, Greater Noida-201308, U.P., India*

Abstract: **Integral equations are playing an increasingly important role in obtaining the solution of many scientific and engineering problems such as determination of potentials, seismic travel time, optical fibers and system identification. In this paper, authors have solved linear second kind non-homogeneous Volterra integral equations (V.I.E.) using Taylor series method. Authors have been considered three numerical examples for explaining the complete methodology. Results of numerical examples show that Taylor series method is very useful and effective numerical method for handling the problem of obtaining the primitives of linear second kind non-homogeneous V.I.E.**

Keywords: **Taylor series method; Volterra integral equation; Power series.**

AMS Subject Classification 2010: **45D05, 45A05, 35C10**

I. INTRODUCTION

With the remarkable advancement in engineering, science, and technology, today more than ever before, the study of integral equations has become essential. For, to have an exhaustive understanding of subjects like waves and electromagnetic, chemistry, fluid dynamics, physics, statistics, mechanics, heat transfer, chemical science, mathematical biology, aerodynamics, electricity the knowledge of determining the solution to integral equations is absolutely necessary. These integral equations may be linear or nonlinear. Finding and interpreting the solutions of these integral equations is therefore a central part of applied mathematics and a thorough understanding of integral equations is essential for any scholars. Aggarwal with others [1-5] used different integral transformations for obtaining the solutions of V.I.E. of second kind. The primitives of first kind V.I.E. were obtained by Aggarwal et al. [6-11] by applying Laplace; Kamal; Mahgoub; Aboodh; Elzaki; Shehu integral transformations on them. Aggarwal and others scholars [12-18] determined the exact solution of famous problem of mechanics (Abel's problem) by applying Laplace; Kamal; Mohand; Aboodh; Sumudu; Shehu; Sadik integral transformations on it. This problem was a special case of V.I.E.

The goal of this paper is to determine the solutions of linear second kind non-homogeneous V.I.E. by applying Taylor series method on them.

II. POWER SERIES (TAYLOR SERIES) OF FREQUENTLY USED FUNCTIONS IN ENGINEERING AND MATHEMATICAL SCIENCE

$$e^{\tau} = \left[1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \frac{\tau^5}{5!} + \cdots \ldots \ldots \ldots \ldots\right]$$

$$e^{-\tau} = \left[1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} - \frac{\tau^5}{5!} + \cdots \ldots \ldots \ldots\right]$$

$$e^{a\tau} = \left[1 + a\tau + \frac{(a\tau)^2}{2!} + \frac{(a\tau)^3}{3!} + \frac{(a\tau)^4}{4!} + \cdots\right]$$

$$e^{-a\tau} = \left[1 - a\tau + \frac{(a\tau)^2}{2!} - \frac{(a\tau)^3}{3!} + \frac{(a\tau)^4}{4!} - \cdots \ldots .\right]$$

$$sin\tau = \left[\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \frac{\tau^7}{7!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . .\right]$$

$$cos\tau = \left[1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .\right]$$

$$tan\tau = \left[\tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\right]$$

$$sinh\tau = \left[\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5!} + \frac{\tau^7}{7!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots\right]$$

$$cosh\tau = \left[1 + \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + \frac{\tau^6}{6!} + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .\right]$$

$$sin^{-1}\tau = \left[\tau + \frac{1}{2}\left(\frac{\tau^3}{3}\right) + \frac{1.3}{2.4}\left(\frac{\tau^5}{5}\right) + \cdots \ldots ., \tau^2 < 1\right]$$

$$tan^{-1}\tau = \left[\tau - \frac{\tau^3}{3} + \frac{\tau^5}{5} - \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . .\right]$$

$$log(1+\tau) = \left[\tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} - \frac{\tau^4}{4} + \cdots, -1 < \tau \leq 1\right]$$

$$log(1-\tau) = \left[-\tau - \frac{\tau^2}{2} - \frac{\tau^3}{3} - \frac{\tau^4}{4} - \cdots, -1 \leq \tau < 1\right]$$

$$\frac{1}{(1-\tau)} = [1 + \tau + \tau^2 + \tau^3 + \cdots \ldots \ldots, |\tau| < 1]$$

$$\frac{1}{(1+\tau)} = [1 - \tau + \tau^2 - \tau^3 + \cdots \ldots \ldots \ldots, |\tau| < 1]$$

$$\frac{1}{(1-\tau)^2} = [1 + 2\tau + 3\tau^2 + 4\tau^3 + \cdots \ldots \ldots, |\tau| < 1]$$

$$\frac{1}{(1-\tau)^3} = [1 + 3\tau + 6\tau^2 + 10\tau^3 + \cdots \ldots, |\tau| < 1]$$

$$(1+\tau)^{\frac{1}{2}} = \left[1 + \frac{\tau}{2} - \frac{\tau^2}{8} + \frac{\tau^3}{16} - \cdots \ldots \ldots \ldots \ldots, |\tau| < 1\right]$$
$$(1+\tau)^{-\frac{1}{2}} = \left[1 - \frac{\tau}{2} + \frac{3\tau^2}{8} - \frac{5\tau^3}{16} + \cdots \ldots \ldots \ldots \ldots, |\tau| < 1\right]$$

III. METHOD OF TAYLOR'S SERIES FOR THE PRIMITIVE OF LINEAR SECOND KIND NON-HOMOGENEOUS V.I.E.

The linear second kind non-homogeneous V.I.E. is given by [19-21]

$$\varphi(\tau) = f(\tau) + \delta \int_0^\tau K(\tau,t)\varphi(t)dt \qquad (1)$$

where

$$\left.\begin{array}{c} \varphi(t) = unknown\ function \\ f(\tau) = known\ function\ (perturbation\ function) \\ \delta = non-zero\ parameter \\ K(\tau,t) = kernel\ of\ integral\ equation \end{array}\right\}$$

Suppose the solution $\varphi(\tau)$ of equation (1) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau) = \sum_{n=0}^{\infty} \beta_n \tau^n \qquad (2)$$

Use equation (2) in equation (1), we have

$$\sum_{n=0}^{\infty} \beta_n \tau^n = T(f(\tau)) + \delta \int_0^\tau K(\tau,t)(\sum_{n=0}^{\infty} \beta_n t^n)dt \qquad (3)$$

where $T(f(\tau))$ is the Taylor series expansion of the function $f(\tau)$.

Equation (3) can be written as

$$\left[\begin{array}{c} \beta_0 + \beta_1\tau + \beta_2\tau^2 \\ +\beta_3\tau^3 + \cdots \ldots \ldots \ldots \\ = T\big(f(\tau)\big) + \delta \int_0^\tau K(\tau,t) \begin{pmatrix} \beta_0 + \beta_1 t \\ +\beta_2 t^2 \\ +\beta_3 t^3 + \cdots \ldots \ldots \ldots \end{pmatrix} dt \end{array}\right] \qquad (4)$$

On simplification, (4) gives a system of algebraic equations in terms of $(\beta_0, \beta_1, \beta_2, \beta_3, \ldots \ldots \ldots \ldots)$. After solving this system, we get a chain of coefficients namely $(\beta_0, \beta_1, \beta_2, \beta_3, \ldots \ldots \ldots \ldots)$. The required solution of equation (1) may be obtained by using these coefficients in equation (2).

Example: 3.1 Consider the following linear second kind non-homogeneous V.I.E.

$$\varphi(\tau) = 1 + \tau + \int_0^\tau (\tau - t)\varphi(t)dt \qquad (5)$$

Suppose the solution $\varphi(\tau)$ of equation (5) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau) = \sum_{n=0}^{\infty} \beta_n \tau^n \qquad (6)$$

Use equation (6) in equation (5), we have

$$\sum_{n=0}^{\infty} \beta_n \tau^n = 1 + \tau + \int_0^\tau (\tau - t)(\sum_{n=0}^{\infty} \beta_n t^n)dt \qquad (7)$$

Equation (7) can be written as

$$\left[\begin{array}{c} \beta_0 + \beta_1\tau + \beta_2\tau^2 + \beta_3\tau^3 + \cdots \ldots \ldots \\ = 1 + \tau + \int_0^\tau (\tau - t) \begin{pmatrix} \beta_0 + \beta_1 t \\ +\beta_2 t^2 + \beta_3 t^3 + \cdots \ldots \ldots \end{pmatrix} dt \end{array}\right]$$
$$\Rightarrow \left\{\begin{array}{c} \beta_0 + \beta_1\tau \\ +\beta_2\tau^2 + \beta_3\tau^3 + \cdots \ldots \\ = 1 + \tau + \left[\tau \begin{pmatrix} \beta_0\tau + \beta_1\frac{\tau^2}{2} \\ +\beta_2\frac{\tau^3}{3} \\ +\beta_3\frac{\tau^4}{4} + \cdots \ldots \ldots \end{pmatrix}\right] \\ -\begin{bmatrix} \beta_0\frac{\tau^2}{2} + \beta_1\frac{\tau^3}{3} \\ +\beta_2\frac{\tau^4}{4} \\ +\beta_3\frac{\tau^5}{5} + \cdots \ldots \ldots \end{bmatrix} \end{array}\right\} \qquad (8)$$

Now on simplification, (8) gives a system of following algebraic equations

$$\left.\begin{array}{l} \beta_0 = 1 \\ \beta_1 = 1 \\ \beta_2 = \beta_0 - \frac{\beta_0}{2} \\ \beta_3 = \frac{\beta_1}{2} - \frac{\beta_1}{3} \\ \beta_4 = \frac{\beta_2}{3} - \frac{\beta_2}{4} \\ \beta_5 = \frac{\beta_3}{4} - \frac{\beta_3}{5} \end{array}\right\} \qquad (9)$$

After solving the system (9), we get

$$\left.\begin{array}{l} \beta_0 = 1 \\ \beta_1 = 1 \\ \beta_2 = \frac{1}{2} \\ \beta_3 = \frac{1}{6} \\ \beta_4 = \frac{1}{24} \\ \beta_5 = \frac{1}{120} \end{array}\right\} \qquad (10)$$

Using equation (10) in equation (6), we get the required solution of equation (5) given by

$$\varphi(\tau) = 1 + \tau + \frac{\tau^2}{2} + \frac{\tau^3}{6} + \frac{\tau^4}{24} + \frac{\tau^5}{120} + \cdots \ldots \ldots = e^\tau.$$

Example: 3.2 Consider the following linear second kind non-homogeneous V.I.E.

$$\varphi(\tau) = \tau - \int_0^\tau (\tau - t)\varphi(t)dt \qquad (11)$$

Suppose the solution $\varphi(\tau)$ of equation (11) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau) = \sum_{n=0}^{\infty} \beta_n \tau^n \qquad (12)$$

Use equation (12) in equation (11), we have

$$\sum_{n=0}^{\infty} \beta_n \tau^n = \tau - \int_0^\tau (\tau - t)(\sum_{n=0}^{\infty} \beta_n t^n)dt \qquad (13)$$

Equation (13) can be written as

$$\begin{bmatrix} \beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\cdots\ldots\ldots \\ =\tau-\int_0^\tau(\tau-t)\begin{pmatrix}\beta_0+\beta_1 t \\ +\beta_2 t^2+\beta_3 t^3+\cdots\ldots\ldots\end{pmatrix}dt \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{matrix} \beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\cdots\ldots\ldots \\ =\tau-\left[\tau\begin{pmatrix}\beta_0\tau+\beta_1\frac{\tau^2}{2} \\ +\beta_2\frac{\tau^3}{3}+\beta_3\frac{\tau^4}{4}+\cdots\ldots\ldots\end{pmatrix}\right] \\ +\left[\begin{matrix}\beta_0\frac{\tau^2}{2}+\beta_1\frac{\tau^3}{3} \\ +\beta_2\frac{\tau^4}{4}+\beta_3\frac{\tau^5}{5}+\cdots\ldots\ldots\end{matrix}\right] \end{matrix} \right\} \quad (14)$$

Now on simplification, (14) gives a system of following algebraic equations

$$\left.\begin{matrix} \beta_0=0 \\ \beta_1=1 \\ \beta_2=-\beta_0+\frac{\beta_0}{2} \\ \beta_3=-\frac{\beta_1}{2}+\frac{\beta_1}{3} \\ \beta_4=-\frac{\beta_2}{3}+\frac{\beta_2}{4} \\ \beta_5=-\frac{\beta_3}{4}+\frac{\beta_3}{5} \end{matrix}\right\} \quad (15)$$

After solving the system (15), we get

$$\left.\begin{matrix} \beta_0=0 \\ \beta_1=1 \\ \beta_2=0 \\ \beta_3=-\frac{1}{6} \\ \beta_4=0 \\ \beta_5=\frac{1}{120} \end{matrix}\right\} \quad (16)$$

Using equation (16) in equation (2), we get the required solution of equation (11) given by

$$\varphi(\tau)=0+1.\tau+0.\tau^2-\frac{\tau^3}{6}+0.\tau^4+\frac{\tau^5}{120}-\cdots\ldots\ldots$$

$$=\tau-\frac{\tau^3}{6}+\frac{\tau^5}{120}-\cdots\ldots\ldots\ldots=sin\tau.$$

Example: 3.3 Consider the following linear second kind non-homogeneous V.I.E.

$$\varphi(\tau)=2+2\tau-\int_0^\tau\varphi(t)dt \quad (17)$$

Suppose the solution $\varphi(\tau)$ of equation (17) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau)=\sum_{n=0}^{\infty}\beta_n\tau^n \quad (18)$$

Use equation (18) in equation (17), we have

$$\sum_{n=0}^{\infty}\beta_n\tau^n=2+2\tau-\int_0^\tau(\sum_{n=0}^{\infty}\beta_n t^n)dt \quad (19)$$

Equation (19) can be written as

$$\begin{bmatrix} \beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\cdots\ldots\ldots \\ =2+2\tau-\int_0^\tau\begin{pmatrix}\beta_0+\beta_1 t \\ +\beta_2 t^2+\beta_3 t^3+\cdots\ldots\ldots\end{pmatrix}dt \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{matrix} \beta_0+\beta_1\tau+\beta_2\tau^2+\beta_3\tau^3+\cdots\ldots\ldots \\ =2+2\tau-\left[\left(\begin{matrix}\beta_0\tau+\beta_1\frac{\tau^2}{2} \\ +\beta_2\frac{\tau^3}{3}+\beta_3\frac{\tau^4}{4}+\cdots\ldots\ldots\end{matrix}\right)\right] \end{matrix} \right\} \quad (20)$$

Now on simplification, (20) gives a system of following algebraic equations

$$\left.\begin{matrix} \beta_0=2 \\ \beta_1=2-\beta_0 \\ \beta_2=-\frac{\beta_1}{2} \\ \beta_3=-\frac{\beta_2}{3} \\ \beta_4=-\frac{\beta_3}{4} \\ \beta_5=-\frac{\beta_4}{5} \end{matrix}\right\} \quad (21)$$

After solving the system (21), we get

$$\left.\begin{matrix} \beta_0=2 \\ \beta_1=0 \\ \beta_2=0 \\ \beta_3=0 \\ \beta_4=0 \\ \beta_5=0 \end{matrix}\right\} \quad (22)$$

Using equation (22) in equation (18), we get the required solution of equation (17) given by

$$\varphi(\tau)=\begin{bmatrix}2+0.\tau+0.\tau^2 \\ +0.\tau^3+0.\tau^4+0.\tau^5+\cdots\ldots\ldots\end{bmatrix}=2.$$

IV. CONCLUSIONS

In the present paper, authors fruitfully discussed the Taylor series method for determining the primitives of linear second kind non-homogeneous V.I.E. The complete methodology explained by taking numerical examples. Results of numerical examples depict that Taylor series method is very effective method for determining the primitives of linear second kind non-homogeneous V.I.E. without large computational work.

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AUTHORS PROFILE

Dr. Sudhanshu Aggarwal has received his Ph.D. degree from C.C.S. University, Meerut in 2019. He has also qualified CSIR NET examination (June-2010, June-2012, June-2013, June-2014 and June-2015) in Mathematical Sciences. He is working as an Assistant Professor in National P.G. College Barhalganj, Gorakhpur. He is equipped with an extraordinary caliber and appreciable academic potency. His fields of interest include Integral Transform Methods, Differential and Partial Differential Equations, Integral Equations, Vibration of Plates, Theory of Elasticity and Number Theory. He has published many research papers in national and international journals.

Dr. Kavita Bhatnagar has been associated with the NIET, Greater Noida for the last three years as a competent, successful and responsible faculty member in Mathematics. She is working as an Assistant Professor in the Department of Applied Sciences there. She is awarded with the degree of Ph.D. by Rajasthan University, Jaipur in 2015. She has published ten research papers in National and International Journals. She has also participated in various workshops. She has around 12 years of teaching experience in the Engineering Colleges.

Dr. Arti Dua is working as an Assistant Professor in Applied Science and Humanities Department at I.T.S. Engineering College, Greater Noida. She has completed her Ph.D. degree from Singhania University, Rajasthan in 2014. She has published many research papers in reputed journals and also a member of different Mathematical Societies. Her fields of interest include Integral Transform Methods and Fluid Mechanics. She has around 13 years of teaching experience in the Engineering Colleges.