

Method of Taylor’s Series for Non-Linear Second Kind Non-Homogeneous Volterra Integral Equations

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Abstract: Integral equations are playing an increasingly important role in obtaining the solution of many scientific and engineering problems such as determination of potentials, seismic travel time, optical fibers and system identification. In this paper, authors have solved non-linear second kind non-homogeneous Volterra integral equations (V.I.E.) using Taylor series method. Authors have been considered two numerical examples for explaining the complete methodology. Results of numerical examples show that Taylor series method is very useful and effective numerical method for handling the problem of obtaining the primitives of non-linear second kind non-homogeneous V.I.E.

Keywords: Taylor series method; Volterra integral equation; Power series.

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I. INTRODUCTION

With the remarkable advancement in different branches of engineering, science, and technology, today more than ever before, the study of integral equations has become essential. For, to have an exhaustive understanding of subjects like waves and electromagnetic, chemistry, fluid dynamics, physics, statistics, mechanics, heat transfer, chemical science, mathematical biology, aerodynamics, electricity the knowledge of determining the solution to integral equations is absolutely necessary. These integral equations may be linear or nonlinear. Finding and interpreting the solutions of these integral equations is therefore a central part of applied mathematics and a thorough understanding of integral equations is essential for any scholars. Aggarwal with others [1-5] used different integral transformations for obtaining the solutions of V.I.E. of second kind. The primitives of first kind V.I.E. were obtained by Aggarwal et al. [6-11] by applying Laplace; Kamal; Mahgoub; Aboodh; Elzaki; Shehu integral transformations on them. Aggarwal and others scholars [12-18] determined the exact solution of famous problem of mechanics (Abel’s problem) by applying Laplace; Kamal; Mohand; Aboodh; Sumudu; Shehu; Sadik integral transformations on it. This problem was a special case of V.I.E.

The goal of this paper is to determine the solutions of non-linear second kind non-homogeneous V.I.E. by applying Taylor series method on them.

II. POWER SERIES (TAYLOR SERIES) OF FREQUENTLY USED FUNCTIONS IN ENGINEERING AND MATHEMATICAL SCIENCE

$$e^\tau = \left[1 + \tau + \frac{\tau^2}{2!} + \frac{\tau^3}{3!} + \frac{\tau^4}{4!} + \frac{\tau^5}{5!} + \dots \dots \dots \right]$$

$$e^{-\tau} = \left[1 - \tau + \frac{\tau^2}{2!} - \frac{\tau^3}{3!} + \frac{\tau^4}{4!} - \frac{\tau^5}{5!} + \dots \dots \dots \right]$$

$$e^{a\tau} = \left[1 + a\tau + \frac{(a\tau)^2}{2!} + \frac{(a\tau)^3}{3!} + \frac{(a\tau)^4}{4!} + \dots \right]$$

$$e^{-a\tau} = \left[1 - a\tau + \frac{(a\tau)^2}{2!} - \frac{(a\tau)^3}{3!} + \frac{(a\tau)^4}{4!} - \dots \dots \right]$$

$$\sin\tau = \left[\tau - \frac{\tau^3}{3!} + \frac{\tau^5}{5!} - \frac{\tau^7}{7!} + \dots \dots \dots \right]$$

$$\cos\tau = \left[1 - \frac{\tau^2}{2!} + \frac{\tau^4}{4!} - \frac{\tau^6}{6!} + \dots \dots \dots \right]$$

$$\tan\tau = \left[\tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \dots \dots \dots \right]$$

$$\sinh\tau = \left[\tau + \frac{\tau^3}{3} + \frac{\tau^5}{5!} + \frac{\tau^7}{7!} + \dots \dots \dots \right]$$

$$\cosh\tau = \left[1 + \frac{\tau^2}{2!} + \frac{\tau^4}{4!} + \frac{\tau^6}{6!} + \dots \dots \dots \right]$$

$$\sin^{-1}\tau = \left[\tau + \frac{1}{2}\left(\frac{\tau^3}{3}\right) + \frac{1.3}{2.4}\left(\frac{\tau^5}{5}\right) + \dots \dots, \tau^2 < 1 \right]$$

$$\tan^{-1}\tau = \left[\tau - \frac{\tau^3}{3} + \frac{\tau^5}{5} - \dots \dots \dots \right]$$

$$\log(1 + \tau) = \left[\tau - \frac{\tau^2}{2} + \frac{\tau^3}{3} - \frac{\tau^4}{4} + \dots, -1 < \tau \leq 1 \right]$$

$$\log(1 - \tau) = \left[-\tau - \frac{\tau^2}{2} - \frac{\tau^3}{3} - \frac{\tau^4}{4} - \dots, -1 \leq \tau < 1 \right]$$

$$\frac{1}{(1 - \tau)} = [1 + \tau + \tau^2 + \tau^3 + \dots \dots \dots, |\tau| < 1]$$

$$\frac{1}{(1 + \tau)} = [1 - \tau + \tau^2 - \tau^3 + \dots \dots \dots, |\tau| < 1]$$

$$\frac{1}{(1 - \tau)^2} = [1 + 2\tau + 3\tau^2 + 4\tau^3 + \dots \dots \dots, |\tau| < 1]$$

$$\frac{1}{(1 - \tau)^3} = [1 + 3\tau + 6\tau^2 + 10\tau^3 + \dots \dots, |\tau| < 1]$$

$$(1 + \tau)^{\frac{1}{2}} = \left[1 + \frac{\tau}{2} - \frac{\tau^2}{8} + \frac{\tau^3}{16} - \dots \dots \dots, |\tau| < 1 \right]$$

$$(1 + \tau)^{-\frac{1}{2}} = \left[1 - \frac{\tau}{2} + \frac{3\tau^2}{8} - \frac{5\tau^3}{16} + \dots \dots \dots, |\tau| < 1 \right]$$

III. METHOD OF TAYLOR'S SERIES FOR NON-LINEAR SECOND KIND NON-HOMOGENEOUS V.I.E.

The second kind non-homogeneous non-linear Volterra integral equation is given by [19, 21]

$$\varphi(\tau) = f(\tau) + \delta \int_0^\tau K(\tau, t)(\varphi(t))^m dt \tag{1}$$

where

$$\left. \begin{aligned} \varphi(t) &= \text{unknown function} \\ f(\tau) &= \text{known function (perturbation function)} \\ \delta &= \text{non - zero parameter} \\ K(\tau, t) &= \text{kernel of integral equation} \\ m &= \text{positive integer} > 1 \end{aligned} \right\}$$

Suppose the solution $\varphi(\tau)$ of equation (1) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau) = \sum_{n=0}^{\infty} \beta_n \tau^n \tag{2}$$

Use equation (2) in equation (1), we have

$$\sum_{n=0}^{\infty} \beta_n \tau^n = T(f(\tau)) + \delta \int_0^\tau K(\tau, t) (\sum_{n=0}^{\infty} \beta_n t^n)^m dt \tag{3}$$

where $T(f(\tau))$ is the Taylor series expansion of the function $f(\tau)$.

Equation (35) can be written as

$$\left[\begin{aligned} &\beta_0 + \beta_1 \tau + \beta_2 \tau^2 \\ &+ \beta_3 \tau^3 + \dots \dots \dots \\ &= T(f(\tau)) \\ + \delta \int_0^\tau K(\tau, t) &\left(\begin{aligned} &\beta_0 + \beta_1 t \\ &+ \beta_2 t^2 + \beta_3 t^3 + \dots \dots \dots \end{aligned} \right)^m dt \end{aligned} \right] \tag{4}$$

On simplification, (4) gives a system of algebraic equations in terms of $(\beta_0, \beta_1, \beta_2, \beta_3, \dots \dots \dots)$. After solving this system, we get a chain of coefficients namely $(\beta_0, \beta_1, \beta_2, \beta_3, \dots \dots \dots)$. The required solution of equation (1) may be obtained by using these coefficients in equation (2).

Example: 3.1 Consider the following non-linear second kind non-homogeneous V.I.E.

$$\varphi(\tau) = \tau + \int_0^\tau (\varphi(t))^2 dt \tag{5}$$

Suppose the solution $\varphi(\tau)$ of equation (5) is analytic so it can be represent in the form of Taylor's series as

$$\varphi(\tau) = \sum_{n=0}^{\infty} \beta_n \tau^n \tag{6}$$

Use equation (6) in equation (5), we have

$$\sum_{n=0}^{\infty} \beta_n \tau^n = \tau + \int_0^\tau (\sum_{n=0}^{\infty} \beta_n t^n)^2 dt \tag{7}$$

Equation (7) can be written as

$$\left[\begin{aligned} &\beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3 + \beta_4 \tau^4 + \beta_5 \tau^5 + \dots \dots \\ &= \tau + \int_0^\tau \left(\begin{aligned} &\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 \\ &+ \beta_4 t^4 + \beta_5 t^5 + \dots \dots \end{aligned} \right)^2 dt \end{aligned} \right] \\ \Rightarrow \beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3 + \beta_4 \tau^4 + \beta_5 \tau^5 + \dots \dots \dots \\ = \tau + \int_0^\tau \left[\begin{aligned} &\beta_0^2 + 2\beta_0 \beta_1 t + (\beta_1^2 + 2\beta_0 \beta_2) t^2 \\ &+ (2\beta_1 \beta_2 + 2\beta_0 \beta_3) t^3 \\ &+ (\beta_2^2 + 2\beta_1 \beta_3 + 2\beta_0 \beta_4) t^4 \\ &+ (2\beta_2 \beta_3 + 2\beta_1 \beta_4 + 2\beta_0 \beta_5) t^5 + \dots \dots \dots \end{aligned} \right] dt \end{aligned}$$

$$\Rightarrow \beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3 + \beta_4 \tau^4 + \beta_5 \tau^5 + \dots \dots \dots \\ = \tau + \left[\begin{aligned} &\beta_0^2 \tau + \beta_0 \beta_1 \tau^2 + (\beta_1^2 + 2\beta_0 \beta_2) \frac{\tau^3}{3} \\ &+ (2\beta_1 \beta_2 + 2\beta_0 \beta_3) \frac{\tau^4}{4} \\ &+ (\beta_2^2 + 2\beta_1 \beta_3 + 2\beta_0 \beta_4) \frac{\tau^5}{5} \\ &+ (2\beta_2 \beta_3 + 2\beta_1 \beta_4 + 2\beta_0 \beta_5) \frac{\tau^6}{6} + \dots \dots \dots \end{aligned} \right] \tag{8}$$

Now on simplification, (8) gives a system of following algebraic equations

$$\left. \begin{aligned} \beta_0 &= 0 \\ \beta_1 &= 1 + \beta_0^2 \\ \beta_2 &= \beta_0 \beta_1 \\ \beta_3 &= \frac{(\beta_1^2 + 2\beta_0 \beta_2)}{3} \\ \beta_4 &= \frac{(2\beta_1 \beta_2 + 2\beta_0 \beta_3)}{4} \\ \beta_5 &= \frac{(\beta_2^2 + 2\beta_1 \beta_3 + 2\beta_0 \beta_4)}{5} \end{aligned} \right\} \tag{9}$$

After solving the system (9), we get

$$\left. \begin{aligned} \beta_0 &= 0 \\ \beta_1 &= 1 \\ \beta_2 &= 0 \\ \beta_3 &= \frac{1}{3} \\ \beta_4 &= 0 \\ \beta_5 &= \frac{2}{15} \end{aligned} \right\} \tag{10}$$

Using equation (10) in equation (6), we get the required solution of equation (5) given by

$$\varphi(\tau) = 0 + 1. \tau + 0. \tau^2 + \left(\frac{1}{3}\right) \tau^3 + 0. \tau^4 + \left(\frac{2}{15}\right) \tau^5 + \dots$$

$$= \tau + \frac{\tau^3}{3} + \frac{2\tau^5}{15} + \dots \dots = \tan\tau.$$

Example: 3.2 Consider the following non-linear second kind non-homogeneous V.I.E.

$$\varphi(\tau) = \tau + \frac{\tau^4}{4} - \int_0^\tau (\varphi(t))^3 dt \tag{11}$$

Suppose the solution $\varphi(\tau)$ of equation (11) is analytic so it can be represent in the form of Taylor’s series as

$$\varphi(\tau) = \sum_{n=0}^\infty \beta_n \tau^n \tag{12}$$

Use equation (12) in equation (11), we have

$$\sum_{n=0}^\infty \beta_n \tau^n = \tau + \frac{\tau^4}{4} - \int_0^\tau (\sum_{n=0}^\infty \beta_n t^n)^3 dt \tag{13}$$

Equation (13) can be written as

$$\left[\begin{array}{l} \beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3 \\ + \beta_4 \tau^4 + \beta_5 \tau^5 + \beta_6 \tau^6 + \dots \dots \dots \\ = \tau + \frac{\tau^4}{4} - \int_0^\tau \left(\begin{array}{l} \beta_0 + \beta_1 t + \beta_2 t^2 \\ + \beta_3 t^3 + \beta_4 t^4 + \beta_5 t^5 + \dots \dots \dots \end{array} \right)^3 dt \end{array} \right]$$

$$\left[\begin{array}{l} \Rightarrow \beta_0 + \beta_1 \tau + \beta_2 \tau^2 + \beta_3 \tau^3 \\ + \beta_4 \tau^4 + \beta_5 \tau^5 + \beta_6 \tau^6 + \dots \dots \dots \\ = \tau + \frac{\tau^4}{4} \\ - \int_0^\tau \left\{ \begin{array}{l} \beta_0^3 + 3\beta_0^2 \beta_1 t \\ + (3\beta_0^2 \beta_2 + 3\beta_1^2 \beta_0) t^2 \\ + (\beta_1^3 + 3\beta_0^2 \beta_3) t^3 \\ + 6\beta_0 \beta_1 \beta_2 \\ + (3\beta_0^2 \beta_4 + 3\beta_1^2 \beta_2) t^4 \\ + (3\beta_0^2 \beta_5 + 3\beta_1^2 \beta_3) \\ + (3\beta_2^2 \beta_0 + 6\beta_0 \beta_1 \beta_3) t^5 + \dots \dots \end{array} \right\} dt \end{array} \right]$$

$$\left[\begin{array}{l} \Rightarrow \beta_0 + \beta_1 \tau + \beta_2 \tau^2 \\ + \beta_3 \tau^3 + \beta_4 \tau^4 \\ + \beta_5 \tau^5 + \beta_6 \tau^6 + \dots \dots \dots \\ = \tau + \frac{\tau^4}{4} - \left\{ \begin{array}{l} \beta_0^3 \tau + \frac{3}{2} \beta_0^2 \beta_1 \tau^2 \\ + \frac{1}{3} (3\beta_0^2 \beta_2 + 3\beta_1^2 \beta_0) \tau^3 \\ + \frac{1}{4} (\beta_1^3 + 3\beta_0^2 \beta_3) \tau^4 \\ + (3\beta_0^2 \beta_4 + 3\beta_1^2 \beta_2) \frac{\tau^5}{5} \\ + (3\beta_0^2 \beta_5 + 3\beta_1^2 \beta_3) \frac{\tau^6}{6} + \dots \dots \end{array} \right\} \end{array} \right] \tag{14}$$

Now on simplification, (14) gives a system of following algebraic equations

$$\left. \begin{array}{l} \beta_0 = 0 \\ \beta_1 = 1 - \beta_0^3 \\ \beta_2 = -\frac{3}{2} \beta_0^2 \beta_1 \\ \beta_3 = -\frac{1}{3} (3\beta_0^2 \beta_2 + 3\beta_1^2 \beta_0) \\ \beta_4 = \frac{1}{4} - \frac{1}{4} (\beta_1^3 + 3\beta_0^2 \beta_3 + 6\beta_0 \beta_1 \beta_2) \\ \beta_5 = -\frac{1}{5} \left(\begin{array}{l} 3\beta_0^2 \beta_4 + 3\beta_1^2 \beta_2 \\ + 3\beta_2^2 \beta_0 + 6\beta_0 \beta_1 \beta_3 \end{array} \right) \\ \beta_6 = -\frac{1}{6} \left(\begin{array}{l} 3\beta_0^2 \beta_5 + 3\beta_1^2 \beta_3 \\ + 3\beta_2^2 \beta_1 + 6\beta_0 \beta_1 \beta_4 \\ + 6\beta_0 \beta_2 \beta_3 \end{array} \right) \end{array} \right\} \tag{15}$$

After solving the system (15), we get

$$\left. \begin{array}{l} \beta_0 = 0 \\ \beta_1 = 1 \\ \beta_2 = 0 \\ \beta_3 = 0 \\ \beta_4 = 0 \\ \beta_5 = 0 \\ \beta_6 = 0 \end{array} \right\} \tag{16}$$

Using equation (16) in equation (12), we get the required solution of equation (11) given by

$$\varphi(\tau) = \left[\begin{array}{l} 0 + 1. \tau + 0. \tau^2 + 0. \tau^3 \\ + 0. \tau^4 + 0. \tau^5 + 0. \tau^6 + \dots \dots \dots \end{array} \right]$$

$$= \tau.$$

IV. CONCLUSIONS

In the present paper, authors fruitfully discussed the Taylor series method for determining the primitives of non-linear second kind non-homogeneous V.I.E. The complete methodology explained by taking two numerical examples. Results of numerical examples depict that Taylor series method is very effective method for determining the primitives of non-linear second kind non-homogeneous V.I.E. without large computational work.

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