Bound for Maximal Rank Hypothesis for $\mathbb{P}^4$

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***Abstract*-The method of Horace is an inductive method that makes use of elementary transformation of vector bundles. The base step involves proving by hand the cases arising from diagram chasing using appropriate short exact sequences. Proving satisfactorily the maximal rank hypothesis requires proving certain statements for all possible cases. This paper presents a bound that guarantees inclusivity of the base cases arising in the proof of maximal rank hypothesis for $\mathbb{P}^4$ using the method of Horace.**

***Key words*- method of Horace, minimal free resolutions, elementary transformation, Vector bundles, maximal rank**

I. INTRODUCTION

The minimal resolution conjecture has attracted the attention of many researchers and still does. A lot has been done on the conjecture. The conjecture has been studied for points in general position as well as points in special configuration. It has also been studied for fat points. For points in general position, the conjecture has been proved to be true for $\mathbb{P}^n$, for $n = 2, 3$ and 4 [2,3,5,6,7,9,10,11]. The conjecture is also known not to be true for $\mathbb{P}^n$ when $n \geq 5$, except possibly when $n = 5$ and $n = 9$. Counter examples have been presented to this effect [4]. For $n = 9$, no counterexample exist for 50 or fewer points. The proof of the conjecture for $n = 3$ and $n = 4$ has been done using the method of Horace.

The method of Horace is a form of induction which makes use of diagram chasing aided by elementary transformation of vector bundles [7, 12]. In using the method, one proves the inductive hypotheses formed through diagram chasing, before proving the base cases. The base cases are the simplest possible cases that arise during diagram chasing. For small values of n, these cases are few, and these cases increase with increase in n. The number of base cases can even be more when one deal with vector bundles with rank more than 1. This brings about the question of inclusivity of the base cases, that is, if the cases presented as the base cases represent all the base cases that can ever arise. This paper answers this question for $\mathbb{P}^4$.

The structure to the paper is as follows. The next section build notation and develop the language used. The method of Horace for $\mathbb{P}^4$ is also presented. Section 3 presents the main result of this paper. Section 4 gives the conclusion.

II. PRELIMINARIES

The minimal resolution conjecture was formulated by Lorenzini [1] to give the form of the minimal free resolution for the ideal of points in general position in the projective space. Suppose $M = \{P_1, P_2, \cdots P_m\}$, where $m \geq n + 1$, is a set of points in general position and X, the sub-scheme supported at these points. Then the homogeneous ideal $I_X \subseteq R = k[x_0, x_1, \cdots x_n]$ where k is an algebraically closed field and R the homogeneous coordinate ring of $\mathbb{P}^n$, has the following form;

$$0 \longrightarrow F_{n-1} \dots \longrightarrow F_p \dots \longrightarrow F_0 \longrightarrow I_X \longrightarrow 0$$

If in addition the number m of points are in the dth binomial interval and also satisfy $m \leq h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$, then the points impose independent conditions and each module F_p is a direct sum of copies of degree $d + p$ generators and degree $d + p + 1$ generators. Also the minimal resolution conjecture gives a relation among the degree $d + p$ generators in F_{p+1} and F_p. More precisely, we have that each of the module F_p is of the form $F_p = R(-d - p)^{a_{p-1}} \oplus R(-d - p - 1)^{b_p}$, with the non-negative integers a_p and b_p called the graded Betti numbers, satisfying;

$$a_p = max\left\{\begin{matrix} 0, h^0\left(\mathbb{P}^n,\ \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right) - \\ rk\left(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)m \end{matrix}\right\}$$

And

$$b_p = max\left\{\begin{matrix} 0, rk\left(\Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right)m - \\ h^0\left(\begin{matrix}\mathbb{P}^n, \\ \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\end{matrix}\right) \end{matrix}\right\}$$

It has been shown in [7] that the problem of existence of the minimal free resolution of the form above can be reduced to proving that the evaluation map below is of maximal rank for all

$0 \leq p \leq n - 2.$

$$H^0\left(\mathbb{P}^n, \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)\right) \longrightarrow \oplus_{i=1}^{m} \Omega^{p+1}_{\mathbb{P}^n}(d+p+1)|_{P_i}$$

This is the same as saying that the betti numbers a_p and b_p satisfy $a_p b_p = 0$ for $p = 0, 1, 2, \cdots n - 2$.

The aim of this paper is to prove inclusivity of base cases arising from while proving the maximal rank hypothesis above when $n = 4$ and $p = 1$, using the method of Horace.

To put the method of Horace in the context of $\mathbb{P}^4$, consider the following elementary transformation.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^2_{\mathbb{P}^4}(d+1) & = & \Omega^2_{\mathbb{P}^4}(d+1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^4}(\mathrm{d}-1)^{\oplus 6} & \longrightarrow & \Omega^2_{\mathbb{P}^4}(d+2) & \longrightarrow & \Omega^2_{\mathbb{P}^{4-1}}(d+2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{\mathrm{p}}_{\mathbb{P}^3}(d+1) & \longrightarrow & \Omega^2_{(\mathbb{P}^4|\mathbb{P}^3)}(d+2) & \longrightarrow & \Omega^2_{\mathbb{P}^3}(d+2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \qquad (2.1)$$

In order to prove the maximal rank hypothesis, one first prove the following hypotheses;

Hypothesis 2.2 [11] $H\big(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2);\ a,b,c\big)$

The statement $H\big(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2);\ a,b,c\big)$ *asserts that there exist* $A_1, A_2, \dots A_a \in \mathbb{P}^{\mathbf{4}}$ *and* $B_1, B_2, \dots B_b \in \mathbb{P}^{\mathbf{3}}$*and a quotient* $\Gamma|_C$ *of a point in* $\mathbb{P}^{\mathbf{3}}$ *of dimension* θ, *(where* $1 \le \theta \le 5, \theta \ne 3$) *if c = 1 such that the map below is bijective.*

$$H^0\left(\mathbb{P}^n, \Omega^2_{\mathbb{P}^4}(d+2)\right) \longrightarrow \oplus_{i=1}^{a} \Omega^2_{\mathbb{P}^4}(d+2)|_{A_1} \oplus_{j=1}^{b} \Omega^2_{\mathbb{P}^4}(d+2)|_{B_j} \oplus \Gamma|_C$$

is bijective.

Hypothesis 2.3 [11] $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega_{\mathbb{P}^3}(d+1);\ e,f,g\big)$

The statement $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *asserts that there exist* $E_1, E_2, \dots E_e \in \mathbb{P}^{\mathbf{4}}$ *and* $F_1, F_2, \dots F_f \in \mathbb{P}^{\mathbf{3}}$ *and a quotient* $\Gamma|_G$ *of a point in* $\mathbb{P}^{\mathbf{3}}$ *of dimension* ϵ, *(where* $1 \le \epsilon \le 5, \epsilon \le 3$*) if* $g = 1$ *such that the map below is bijective.*

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}) \longrightarrow \oplus_{i=1}^{e} \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}|_{A_1} \oplus_{j=1}^{f} \mathcal{O}_{\mathbb{P}^3}(d-1)^{\oplus 6}|_{F_j} \oplus \Gamma|_G$$

III. MAIN RESULTS

In this section, inclusivity of the base cases is discussed by presenting a bound. This bound, given as an expression in d, is significant in the sense that it guarantees inclusivity of the base cases that arise in application of the method of Horace.

Consider diagram 2.1 of elementary transformation. Diagram chasing together with hypotheses 2.2 and 2.3 yields the following;

Theorem 3.1 [11]

i. $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *implies* $H\big(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2);\ a,b,c\big)$ *for* $d \ge 2$.

ii. *For* $d \ge 5$, $H\big(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1);\ a,b,c\big)$ *and* $H\big(\mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *implies* $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e',f',g'\big)$.

Remark 3.2.

1. *The first statement is implied by the exact sequence in the middle row as well as the leftmost one, of course for suitable values of* d. *With repeated runs in the diagram chasing, the following sequence becomes useful.*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-2)^{\oplus 6} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6} \longrightarrow \mathcal{O}_H(d-1)^{\oplus 6} \longrightarrow 0$$

Here H *is a hyperplane.*

Once enough points are specialized to the hyperplane H, *then the second statement can be realized through further diagram chasing.*

2. *Also, from the two statements, it is clear that* $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *is a key statement in the inductive hypothesis and its proof is crucial to the maximal rank hypothesis for the minimal resolution conjecture. As such The cases in the base step, also known by initial cases, arise when the statement* $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *cannot be satisfied.*
3. *Finally, minimum value of d for which the first statement accounts for all possible cases gives a reasonable point to stop looking for base cases.*

Lemma 3.3.[11]

The statement $H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ *is not satisfied when* $f \ge \frac{2}{3}d(d+2)$

Theorem 3.6

Let $d \ge 5$ be a positive integer. Then

$\mathrm{H}\big(\mathcal{O}_{\mathbb{P}^4}(\mathrm{d}-1)^{\oplus 6}, \Omega^{\mathrm{p}}_{\mathbb{P}^3}(d+1);\ e,f,g\big)$ implies

$\mathrm{H}\big(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2);\ a,b,c\big)$ and

$\mathrm{H}\big(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1);\ a,b,c\big)$ implies

$\mathrm{H}\big(\mathcal{O}_{\mathbb{P}^4}(\mathrm{d}-1)^{\oplus 6}, \Omega^{\mathrm{p}}_{\mathbb{P}^3}(d+1);\ e',f',g'\big)$ if and only if $\frac{d(d+1)(d+2)}{12} + \frac{(d+1)(d+2)}{3} < \frac{(d+1)(d+2)(d+4)}{12}$.

Proof

The proof of this theorem follows from claim 3.4 and remark 3.5.

Claim 3.4

The number of cases for $d+1$ *which do not correspond to any of the cases in for* d *via the statement*

$H\big(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1);\ a,b,c\big)$ *implies*

$H\big(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e',f',g'\big)$ *is given by*

$$\frac{(d+1)(d+2)(d+4)}{12} - \left(\frac{d(d+1)(d+2)}{12} + \frac{(d+1)(d+2)}{3}\right).$$

Proof.

For any d; the number of possible cases is given by;

$\frac{1}{6}h^0\left(\mathbb{P}^3, \Omega^2_{\mathbb{P}^3}(d+2)\right) = \frac{1}{12}d(d+1)(d+2).$

This means that for $d+1$, there are $\frac{1}{12}(d+1)(d+2)(d+4)$ cases. As a consequence of lemma 3.3, the number of cases for which $H\left(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e', f', g'\right)$ is not satisfied is $\frac{1}{2}\left(\frac{2}{3}d(d+2)\right)$. It then follows that the number of cases for $d+1$ which do not correspond to any of the cases for d is

$$\frac{1}{12}(d+1)(d+2)(d+4) - \left(\frac{1}{12}d(d+1)(d+2) + \frac{1}{12}d(d+1)(d+2)\right),$$

which simplifies to the desired expression.

Remark 3.5

The least integer d for which $\frac{d(d+1)(d+2)}{12} + \frac{(d+1)(d+2)}{3} = \frac{(d+1)(d+2)(d+4)}{12}$ *is* 10. *Given* $d \geq 5$ *and both statements* $H\left(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e, f, g\right)$ *implies*

$H\left(\Omega^2_{\mathbb{P}^4}(d+2), \Omega^2_{\mathbb{P}^3}(d+2);\ a, b, c\right)$ *and*

$H\left(\Omega^2_{\mathbb{P}^4}(d+1), \Omega^2_{\mathbb{P}^3}(d+1);\ a, b, c\right)$ *implies*

$H\left(\mathcal{O}_{\mathbb{P}^4}(d-1)^{\oplus 6}, \Omega^p_{\mathbb{P}^3}(d+1);\ e', f', g'\right)$

hold true, then $\frac{d(d+1)(d+2)}{12} + \frac{(d+1)(d+2)}{3} < \frac{(d+1)(d+2)(d+4)}{12}$. *Conversely, if* $\frac{d(d+1)(d+2)}{12} + \frac{(d+1)(d+2)}{3} < \frac{(d+1)(d+2)(d+4)}{12}$ *holds true then the two statements holds, thanks to theorem 3.3.*

IV. CONCLUSION

The method of Horace being an inductive method proves the maximal rank hypothesis for all positive integers d, of course with some other conditions. Proving that no initial case can arise other than the ones presented add to the authenticity of the proof. Theorem 3.6 above does this for one of the instances in the case of $\mathbb{P}^4$. The theorem applies to other instances with slight modification.

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