On The Exponential Diophantine Equation $(2^{2m+1}-1)+(6r+1)^n=z^2$

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Abstract: **Nowadays, scholars are very interested to determine the solution of different Diophantine equations because these equations have many applications in the field of coordinate geometry, cryptography, trigonometry and applied algebra. These equations help us for finding the integer solution of famous Pythagoras theorem and Pell's equation. Finding the solution of Diophantine equations have many challenges for scholars due to absence of generalize methods. In the present paper, authors discussed the existence of the solution of exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$, where m,n,r,z are whole numbers.**

***Keywords*: Positive integer; Diophantine equation; Solution; Congruence; Modulo system.**

***Mathematics Subject Classification*: 11D61, 11D72, 11D45.**

I. INTRODUCTION

Diophantine equations are those equations of theory of numbers which are to be solved in integers. The class of Diophantine equations is classified in two categories, one is linear Diophantine equations and the other one is non-linear Diophantine equations. Both categories of these equations are very important in theory of numbers and have many important applications in solving the puzzle problems. Diophantine equations are very helpful to prove the existence of irrational numbers [4, 6]. Acu [1] studied the Diophantine equation $2^x+5^y=z^2$ and proved that $\{x=3, y=0, z=3\}$ and $\{x=2, y=1, z=3\}$ are the solutions of this equation. Kumar et al. [2] considered the non-linear Diophantine equations $61^x+67^y=z^2$ and $67^x+73^y=z^2$. They showed that these equations have no non-negative integer solution. Kumar et al. [3] studied the non-linear Diophantine equations $31^x+41^y=z^2$ and $61^x+71^y=z^2$. They determined that these equations have no non-negative integer solution.

Rabago [5] discussed the open problem given by B. Sroysang. He determined that the Diophantine equation $8^x+p^y=z^2$, where x,y,z are positive integers has only three solutions namely $\{x=1, y=1, z=5\}$, $\{x=2, y=1, z=9\}$and $\{x=3, y=1, z=23\}$ for $p=17$. The Diophantine equations $8^x+19^y=z^2$ and $8^x+13^y=z^2$ were studied by Sroysang [7-8]. He proved that these equations have a unique solution which is given by $\{x=1, y=0, z=3\}$. Sroysang [9] proved that the exponential Diophantine equation $31^x+32^y=z^2$ has no positive integer solution. Aggarwal et al. [10] discussed the existence of solution of Diophantine equation $181^x+199^y=z^2$.

Aggarwal et al. [11] discussed the Diophantine equation $223^x+241^y=z^2$ for solution. Gupta and Kumar [12] gave the solutions of exponential Diophantine equation $n^x+(n+3m)^y=z^{2k}$. Kumar et al. [13] studied exponential Diophantine equation $601^p+619^q=r^2$ and proved that this equation has no solution in whole number. Mishra et al. [14] studied the existence of solution of Diophantine equation $211^\alpha+229^\beta=\gamma^2$ and proved that the Diophantine equation $211^\alpha+229^\beta=\gamma^2$ has no solution in whole number.

Bhatnagar and Aggarwal [15] examined the exponential Diophantine equation $421^p+439^q=r^2$ and determined that this equation has no solution in whole number. Goel et al. [16] studied the exponential Diophantine equation ${M_5}^p+{M_7}^q=r^2$. Kumar et al. [17] showed that the exponential Diophantine equation $(2^{2m+1}-1)+(6^{r+1}+1)^n=\omega^2$ has no solution in non-negative integers. Kumar et al. [18] proved that the exponential Diophantine equation $[(7^{2m})+(6r+1)^n=z^2]$ has no solution in non-negative integers. Aggarwal and Sharma [19] discussed the existence of the solution of non-linear Diophantine equation$379^x+397^y=z^2$. Aggarwal and Kumar [20] studied the non-linear Diophantine equation$[19]^{2m}+[2^{2r+1}-1]=\rho^2$and showed that this equation has no non-negative integer solution. The exponential Diophantine equation$(19^{2m})+(12\gamma+1)^n=\rho^2$ was studied by Aggarwal and Kumar [21]. Aggarwal [22] examined the exponential Diophantine equation$(2^{2m+1}-$ *1+(13)n=z2* for non-negative integer solution. Aggarwal and Kumar [23] studied the exponential Diophantine equation$(19^{2m})+(6^{\gamma+1}+1)^n=\rho^2$ and determined that this equation is not solvable in non-negative integers.

The main aim of this article is to discuss the existence of the solution of exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$, where m,n,r,z are whole numbers.

II. PRELIMINARIES

Lemma: 1 The exponential Diophantine equation $(2^{2m+1}-$ *1+1=z2, where m,z are the whole numbesr, has no solution in whole number.*

*Proof***:** Since 2^{2m+1} is an even number so $(2^{2m+1}-1)$ is an odd number for all whole number m.

$\Rightarrow (2^{2m+1}-1)+1=z^2$ is an even number for all whole number m.

$\Rightarrow z$ is an even number.

$\Rightarrow z^2 \equiv 0(mod3)$ or $z^2 \equiv 1(mod3)$ (1)

Now, $2^{2m+1} \equiv 2(mod3)$, for all whole number m.

$\Rightarrow (2^{2m+1}-1) \equiv 1(mod3)$, for all whole number m.

$\Rightarrow (2^{2m+1}-1)+1 \equiv 2(mod3)$, for all whole number m.

$\Rightarrow z^2 \equiv 2(mod3)$ (2)

Equation (2) contradicts equation (1).

Hence the exponential Diophantine equation $(2^{2m+1}-1)+1=z^2$, where $m$and z are whole numbers, has no solution in whole number.

Lemma: 2 The exponential Diophantine equation$1+(6r+1)^n=z^2$, *where* r,n,z *are whole numbers, has no solution in whole number.*

Proof: Since $(6r+1)$ is an odd number for all whole number r so $(6r+1)^n$ is an odd number for all whole numbers r and n.

$\Rightarrow 1+(6r+1)^n=z^2$ is an even number for all whole numbers r and n.

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod3)$ or $z^2 \equiv 1(mod3)$ (3)

Now, $(6r+1) \equiv 1(mod3)$, for all whole number r.

$\Rightarrow (6r+1)^n \equiv 1(mod3)$, for all whole numbers r and n.

$\Rightarrow 1+(6r+1)^n \equiv 2(mod3)$, for all whole numbers r and n.

$\Rightarrow z^2 \equiv 2(mod3)$ (4)

Equation (4) contradicts equation (3).

Hence the exponential Diophantine equation $1+(6r+1)^n=z^2$, where r,n,z are whole numbers, has no solution in whole number.

III. MAIN THEOREM

The exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$, where m,n,r,z are whole numbers, has no solution in whole number.

Proof: There are four cases:

Case: 1 If $m=0$ then the exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$ becomes$1+(6r+1)^n=z^2$, which has no whole number solution by lemma 2.

Case: 2 If $n=0$ then the exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$ becomes $(2^{2m+1}-1)+1=z^2$, which has no whole number solution by lemma 1.

Case: 3 If m,n are positive integers, then $(2^{2m+1}-$ *1,6r+1n* are odd numbers.

$\Rightarrow (2^{2m+1}-1)+(6r+1)^n=z^2$ is an even number

$\Rightarrow z$ is an even number

$\Rightarrow z^2 \equiv 0(mod3)\ or\ z^2 \equiv 1(mod3)$ (5)

Now, $2^{2m+1} \equiv 2(mod3)$

$$\Rightarrow \begin{bmatrix} (2^{2m+1}-1) \equiv 1(mod3) \\ and\ (6r+1) \equiv 1(mod3) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (2^{2m+1}-1) \equiv 1(mod3) \\ and\ (6r+1)^n \equiv 1(mod3) \end{bmatrix}$$

$\Rightarrow (2^{2m+1}-1)+(6r+1)^n \equiv 2(mod3)$

$\Rightarrow z^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5).

Hence the exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$, where m,n are positive integers and r is whole number, has no solution in whole number.

Case: 4 If $m,n=0$, then $(2^{2m+1}-1)+(6r+1)^n=1+1=2=z^2$, which is impossible because z is a whole number. Hence exponential Diophantine equation $(2^{2m+1}-$ *1+6r+1n=z2,* where *m,n=0* and r is whole number, has no solution in whole number.

IV. CONCLUSION

In this article, authors successfully discussed the existence of the solution of exponential Diophantine equation$(2^{2m+1}-$ *1+6r+1n=z2,* where *m,n,r,z* are whole numbers. Authors determined that the exponential Diophantine equation $(2^{2m+1}-1)+(6r+1)^n=z^2$, where m,n,r,z are whole numbers, has no solution in whole number.

REFERENCES

[1] Acu, D. (2007) On a Diophantine equation $2^x+5^y=z^2$, General Mathematics, 15(4), 145-148.

[2] Kumar, S., Gupta, S. and Kishan, H. (2018) On the non-linear Diophantine equations $61^x+67^y=z^2$ and $67^x+73^y=z^2$, Annals of Pure and Applied Mathematics, 18(1), 91-94.

[3] Kumar, S., Gupta, D. and Kishan, H. (2018) On the non-linear Diophantine equations $31^x+41^y=z^2$ and $61^x+71^y=z^2$, Annals of Pure and Applied Mathematics, 18(2), 185-188.

[4] Mordell, L.J. (1969) Diophantine equations, Academic Press, London, New York.

[5] Rabago, J.F.T. (2013) On an open problem by B. Sroysang, Konuralp Journal of Mathematics, 1(2), 30-32.

[6] Sierpinski, W. (1988) Elementary theory of numbers, 2nd edition, North-Holland, Amsterdam.

[7] Sroysang, B. (2012) More on the Diophantine equation $8^x+19^y=z^2$, International Journal of Pure and Applied Mathematics, 81(4), 601-604.

[8] Sroysang, B. (2014) On the Diophantine equation $8^x+13^y=z^2$, International Journal of Pure and Applied Mathematics, 90(1), 69-72.

[9] Sroysang, B. (2012) On the Diophantine equation $31^x+32^y=z^2$, International Journal of Pure and Applied Mathematics, 81(4), 609-612.

[10] Aggarwal, S., Sharma, S.D. and Vyas, A. (2020) On the existence of solution of Diophantine equation $181^x + 199^y = z^2$, International Journal of Latest Technology in Engineering, Management & Applied Science, 9 (8), 85-86.

[11] Aggarwal, S., Sharma, S.D. and Singhal, H. (2020) On the Diophantine equation $223^x + 241^y = z^2$, International Journal of Research and Innovation in Applied Science, 5 (8), 155-156.

[12] Gupta, D. and Kumar, S. (2020) On the solutions of exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, International Journal of Interdisciplinary Global Studies, 14(4), 74-77.

[13] Kumar, A., Chaudhary, L. and Aggarwal, S. (2020) On the exponential Diophantine equation $601^p + 619^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 29-30.

[14] Mishra, R., Aggarwal, S. And Kumar, A. (2020) On the existence of solution of Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.

[15] Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation $421^p + 439^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 128-129.

[16] Goel, P., Bhatnagar, K. and Aggarwal, S. (2020) On the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$, International Journal of Interdisciplinary Global Studies, 14(4), 170-171.

[17] Kumar, S., Bhatnagar, K., Kumar, A. and Aggarwal, S. (2020) On the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$, International Journal of Interdisciplinary Global Studies, 14(4), 183-184.

[18] Kumar, S., Bhatnagar, K., Kumar, N. and Aggarwal, S. (2020) On the exponential Diophantine equation $\left[(7^{2m}) + (6r + 1)^n = z^2\right]$, International Journal of Interdisciplinary Global Studies, 14(4), 181-182.

[19] Aggarwal, S. and Sharma, N. (2020) On the non-linear Diophantine equation $379^x + 397^y = z^2$, Open Journal of Mathematical Sciences, 4(1), 397-399. DOI: 10.30538/oms2020.0129

[20] Aggarwal, S. and Kumar, S. (2021) On the non-linear Diophantine equation $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$, International Journal of Latest Technology in Engineering, Management & Applied Science, 10 (2), 14-16.

[21] Aggarwal, S. and Kumar, S. (2021) On the exponential Diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$, International Journal of Research and Innovation in Applied Science, 6 (3), 14-16.

[22] Aggarwal, S. (2021) On the exponential Diophantine equation $(2^{2m+1} - 1) + (13)^n = z^2$, Engineering and Applied Science Letters, 4(1), 77-79.

[23] Aggarwal, S. and Kumar, S. (2021) On the exponential Diophantine equation $(19^{2m}) + (6^{\gamma+1} + 1)^n = \rho^2$, International Journal of Research and Innovation in Applied Science, 6 (2), 112-114.