Numerical Solution of Convection-Diffusion Equation with the Help of Viscous Burger's Equation

T M A K Azad

Department of Computer Science & Engineering, University of Liberal Arts Bangladesh, Dhaka-1209

DOI: https://doi.org/10.51584/IJRIAS.2025.1002009

Received: 26 January 2025; Accepted: 31 January 2025; Published: 04 March 2025

ABSTRACT

Two finite difference schemes, FTBSCS (forward time backward space and centered space) and FTCS (forward time centered space) have been studied for solving convection-diffusion equation (CDE) with appropriate initial condition and boundary conditions. The convection velocity $u(t, x)$ of CDE is computed by solving viscous Burger's equation using the same schemes. Stability conditions of the schemes are determined and it is analytically shown that FTCS scheme is superior to FTBSCS scheme in terms of time step selection. The conditions of stability are also numerically verified. Some numerical simulation results are presented for various parameters. Error comparisons of both the schemes are presented to estimate accuracy of solutions.

Keywords: Convection-Diffusion Equation, Burger's Equation, Finite Difference Schemes, Stability Conditions.

INTRODUCTION

Historically, Burger's equation was first introduced by Bateman [2] who gave its steady solutions. It was later treated Burger's [4] as a mathematical model for turbulence and after whom such an equation is widely referred to as Burger's equation. Many problems can be modeled by Burger's equation [8]. For example, the Burger's equation can be considered as an approach to the Navier-Stokes equations [1] since both contain nonlinear terms of the type: unknown functions multiplied by a first derivative and both contain higher-order terms multiplied by a small parameter. It is a nonlinear equation for which exact solutions are known and is therefore important as a benchmark problem for numerical methods. The study of the general properties of the Burger's equation has motivated considerable attention due to its applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. The exact solutions of the one-dimensional Burger's equation have been surveyed by Benton and Platzman [3]. Many other authors [5-8] have used a variety of numerical techniques based on finite-difference, finite-element and boundary element methods in attempting to solve the equation particularly for small values of the kinematic viscosity ν which corresponds to steep fronts in the propagation of dynamic waveforms.

Convection-Diffusion Equation (CDE) is one of the most important partial differential equation and observed in a wide range of engineering and industrial applications. This equation reflects physical phenomena where in the diffusion process particles are moving with certain velocity from higher concentration to lower concentration. The analytical/numerical solutions along with an initial condition and two boundary conditions help to understand the contaminant or pollutant concentration distribution behavior through an open medium like air, rivers, lakes and porous medium like aquifer. CDE benefits from wide applications in such different disciplines as environmental engineering, mechanical engineering, heat transfer, soil science and as well in biology. It has wide applications in other disciplines too, like soil physics, petroleum engineering, chemical engineering and biosciences.

Analytical solutions are as useful tools in many areas-

Formerly, the analytical solutions of ADE were obtained by reducing the original ADE into a diffusion equation by omitting the advective term(s). It was done either by introducing moving co-ordinates (Ogata and

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

Banks, 1961[26]; Harleman and Rumer 1963[30]; Bear 1972[14]; Guvanasen and Volker 1983[29]; Aral and Liao, 1996 [11]; Marshal *et al.* 1996[22]; or by introducing another dependent variable (Banks and Ali, 1964[13]; Ogata and Banks 1961[26]; Lai and Jurinak 1971[20] and Al-Niami and Rushton, 1977[9]). In more recent works, analytical solutions of ADE have been obtained using such integral transform techniques as either Laplace or Fourier transform (Lai and Jurinak 1971[20]; Marino 1974 [21] and Kumar et al. 2009[12]). In [10; 15 – 19; 23 – 25; 27; 28; 34] analytical and numerical solutions have been obtained by applying different method.

In (Azad *et al.*, [32]; Azad and Andallah [33]) finite difference schemes are presented for solving the advection diffusion equation (ADE). Numerical solution of the ADE is obtained by using FTBSCS and FTCS techniques for prescribed initial and boundary data. Numerical results for both the schemes are compared in terms of accuracy by error estimation with respect to exact solution of the ADE and also, the numerical features of the rate of convergence are presented graphically.

With the above discussion in view, one-dimensional convection-diffusion equation is solved by using two finite difference schemes FTBSCS and FTCS. The convection velocity *u*(t, x) is computed by solving viscous Burger's equation using the same schemes. Stability conditions for the schemes studied and the condition of stability is also numerically verified.

GOVERNING EQUATION AND NUMERICAL SCHEMES

Governing Equation

In this paper, we consider variable advection velocity *u*(t, x), so that the PDE reads as convection-diffusion equation (CDE) $\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}$ where we have two unknowns $c(t, x)$ and $u(t, x)$. Therefore we have to solve another equation and we select the viscous Burger's equation $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$ to compute the variable velocity $u(t, x)$. Our problem is thus to solve the following system of PDE's simultaneously as an IBVP

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}, \qquad a<x<b, \quad t>0, \tag{1}$$

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial x^2}, \qquad a<x<b, \quad t>0, \tag{2}$$

where $c(t, x)$ represents the solute concentration $[ML^{-3}]$ at x, along longitudinal direction at time t, and $\nu>0$ is the coefficient of kinematic viscosity, D is the solute dispersion, if it is independent of position and time, is called dispersion coefficient $[L^2T^{-1}]$, t = time$[T]$; x = distance$[L]$ and, $u(t, x)$ is the solutions of (1).

Appended with initial condition

$$u(x,0)=f(x); \qquad c(x,0)=f(x) \qquad\qquad a\le x<b$$

and Neumann boundary conditions

$$\frac{\partial}{\partial x}u(t,a)=u_a(t)\,; \qquad \frac{\partial}{\partial x}u(t,b)=u_b(t) \qquad\qquad t_0\le t\le T$$

$$\frac{\partial}{\partial x}c(t,a)=c_a(t)\,; \qquad \frac{\partial}{\partial x}c(t,b)=c_b(t) \qquad\qquad t_0\le t\le T$$

where c_a, c_b, u_a, u_b are constant concentration values.

Analytic solution

The exact solution of the advection-diffusion equation as IVP with initial condition

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

c(x, 0) = $f(x)$ is given [31]

$$c(x,t) = \frac{M}{A\sqrt{4\pi D \mathrm{t}}} exp\left(-\frac{(x - (x_0 + ut))^2}{4D\mathrm{t}}\right) \tag{3}$$

where M = mass of tracer

A = uniformly cross section area at the point x = 0, at time t = 0.

Finite Difference Scheme

We consider the one–dimensional CDE as an initial and boundary value problem.

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2},$$

with initial condition $c(t_0, x) = c_0(x);$ $\quad a \le x \le b$

and Neumann boundary conditions

$$\frac{\partial}{\partial x}c(t,a) = c_a(t)\,; \qquad t_0 \le t \le T$$

$$\frac{\partial}{\partial x}c(t,b) = c_b(t)\,; \qquad t_0 \le t \le T$$

FDMs are the efficient approach to numerical solutions of partial differential equations. A finite difference method proceeds by replacing the derivatives in the differential equation by the finite difference approximations. This gives a large algebraic system of equation to be developing a computer programming code.

Explicit Finite Difference Scheme

For the numerical solution of the one –dimensional linear convection- diffusion equation, we consider the IBVP

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2},$$

with initial condition $c(x,0) = 0.02 \times e^{-10x},$ $\quad 0 \le x < l$

and Neumann boundary conditions

$$\frac{\partial}{\partial x}c(t, x = 0) = 0, \qquad 0 < t \le T$$

$$\frac{\partial}{\partial x}c(t, x = l) = 0 \qquad 0 < t \le T$$

In order to develop the schemes, we discretize the x-t plane by choosing a spatial grid size $h \equiv \Delta$x and temporal grid size $k \equiv \Delta$t. Then we can define the discrete grid points

$x_i = a + ih,\ i = 0, 1, 2, 3, \dots\dots\dots, M$ and $t_n = nk, n = 0, 1, 2, \dots\dots\dots, N$ where $M = (b - a)$/h and $N = T/k$. Now we present two finite difference schemes as follows-

FINITE DIFFERENCE FORMULAE

Derivatives in the convection- diffusion equation are approximated by truncated Taylor Series expansions, which are follows-

$$\frac{\partial c}{\partial t} = \frac{c_i^{n+1} - c_i^n}{\Delta t} \text{ (1st order forward difference in time)} \quad (4)$$

$$\frac{\partial c}{\partial x} = \frac{c_i^n - c_{i-1}^n}{\Delta x} \text{(1st order backward space difference formula)} \quad (5)$$

$$\frac{\partial c}{\partial x} = \frac{c_{i+1}^n - c_{i-1}^n}{2\Delta x} \text{ (1st order centered space difference formula)} \quad (6)$$

and

$$\frac{\partial^2 c}{\partial x^2} = \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{\Delta x^2} \text{ (2nd order centered space difference formula)} \quad (7)$$

Finite Difference (FTBSCS) Scheme

Substituting equations (4), (5), (7) into equation (1), (2) and rearranging according the time level,

(1) tends to

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2},$$

$$\Rightarrow u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n) + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n),$$

$$\Rightarrow u_i^{n+1} = u_i^n - \gamma(u_i^n - u_{i-1}^n) + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$We\ get, u_i^{n+1} = (\gamma + r)u_{i-1}^n + (1 - \gamma - 2\mathrm{r})u_i^n + r u_{i+1}^n, \quad (8)$$

$$where, \quad \gamma = \frac{\Delta t}{\Delta x} u_i^n, \quad r = \frac{\nu \Delta t}{\Delta x^2}$$

(2) tends to

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} + u_i^n \frac{c_i^n - c_{i-1}^n}{\Delta x} = D \frac{c_{i+1}^n - 2c_i^n + c_{i-1}^n}{\Delta x^2},$$

$$\Rightarrow c_i^{n+1} = c_i^n - \frac{\Delta t}{\Delta x} u_i^n (c_i^n - c_{i-1}^n) + \frac{D \Delta t}{\Delta x^2}(c_{i+1}^n - 2c_i^n + c_{i-1}^n),$$

$$\Rightarrow c_i^{n+1} = c_i^n - \gamma(c_i^n - c_{i-1}^n) + \frac{D \Delta t}{\Delta x^2}(c_{i+1}^n - 2c_i^n + c_{i-1}^n)$$

$$We\ get, c_i^{n+1} = (\gamma + \lambda)c_{i-1}^n + (1 - \gamma - 2\lambda)c_i^n + \lambda c_{i+1}^n, \quad (9)$$

$$where, \quad \gamma = \frac{\Delta t}{\Delta x} u_i^n, \quad \lambda = \frac{D \Delta t}{\Delta x^2}$$

Finite Difference (FTCS) Scheme

Substituting equations (4), (6), (7) into equation (1), (2) and rearranging according the time level,

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

(1) tends to

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+u_i^n\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=\nu\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{\Delta x^2},$$

$$\Rightarrow u_i^{n+1}=u_i^n-\frac{\Delta t}{2\Delta x}u_i^n(u_{i+1}^n-u_{i-1}^n)+\frac{\nu\Delta t}{\Delta x^2}(u_{i+1}^n-2u_i^n+u_{i-1}^n),$$

$$\Rightarrow u_i^{n+1}=u_i^n-\frac{\gamma}{2}(u_{i+1}^n-u_{i-1}^n)+r(u_{i+1}^n-2u_i^n+u_{i-1}^n),$$

$$We\ get,\ \ u_i^{n+1}=\left(r+\frac{\gamma}{2}\right)u_{i-1}^n+(1-2r)u_i^n+\left(\mathrm{r}-\frac{\gamma}{2}\right)u_{i+1}^n, \qquad (10)$$

$$where,\quad \gamma=\frac{\Delta t}{\Delta x}u_i^n, \qquad r=\frac{\nu\Delta t}{\Delta x^2}$$

(2) tends to

$$\frac{c_i^{n+1}-c_i^n}{\Delta t}+u_i^n\frac{c_{i+1}^n-c_{i-1}^n}{2\Delta x}=D\frac{c_{i+1}^n-2c_i^n+c_{i-1}^n}{\Delta x^2},$$

$$\Rightarrow c_i^{n+1}=c_i^n-\frac{\Delta t}{2\Delta x}u_i^n(c_{i+1}^n-c_{i-1}^n)+\frac{D\Delta t}{\Delta x^2}(c_{i+1}^n-2c_i^n+c_{i-1}^n),$$

$$\Rightarrow c_i^{n+1}=c_i^n-\frac{\gamma}{2}(c_{i+1}^n-c_{i-1}^n)+\lambda(c_{i+1}^n-2c_i^n+c_{i-1}^n)$$

$$We\ get,\quad c_i^{n+1}=\left(\lambda+\frac{\gamma}{2}\right)c_{i-1}^n+(1-2\lambda)c_i^n+\left(\lambda-\frac{\gamma}{2}u_i^n\right)c_{i+1}^n, \qquad (11)$$

$$where,\quad \gamma=\frac{\Delta t}{\Delta x}u_i^n, \qquad \lambda=\frac{D\Delta t}{\Delta x^2}$$

It is seen that the truncation errors for the forward and backward differences are of first order; whereas the centered difference yields a second order truncation error (using by Taylor Series expansions). Therefore, both the schemes outlined above are consistent.

STABILITY ANALYSIS

After surveying the relevant literature on the subject, we discover that no practical stability criterion exists for the schemes. We have developed stability conditions for both the schemes in the following two propositions and maintaining the criteria we verify the results of the schemes numerically in the next sections.

proposition 1

Statement: The stability conditions for the FTBSCS scheme are

$$0\le\frac{D\Delta t}{\Delta x^2}\le 1 \text{ and } -\frac{D\Delta t}{\Delta x^2}\le\frac{\Delta t}{\Delta x}u_i^n\le 1-2\frac{D\Delta t}{\Delta x^2}$$

This is guaranteed by the simultaneous inequalities

$$0\le\frac{D\Delta t}{\Delta x^2}\le 1 \text{ and } -\frac{D}{\Delta x}\le\max(u_i^0)\le\frac{\Delta x}{\Delta t}-2\frac{D}{\Delta x}$$

Proof:

The explicit centered difference scheme using by FTBSCS for CDE (2) is given by

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

$$c_i^{n+1} = (\gamma + \lambda)c_{i-1}^n + (1 - \gamma - 2\lambda)c_i^n + \lambda c_{i+1}^n, \qquad (12)$$

where $\gamma = \frac{\Delta t}{\Delta x} u_i^n$, $\lambda = \frac{D\Delta t}{\Delta x^2}$

The equation (12) implies that if

$$0 \le \gamma + \lambda \le 1 \qquad \text{(i)}$$

$$0 \le 1 - \gamma - 2\lambda \le 1 \qquad \text{(ii)}$$

$$0 \le \lambda \le 1 \qquad \text{(iii)}$$

then the new solution is a convex combination of the two previous solutions. That is the solution at new time-step (n+1) at a spatial node i is an average of the solutions at the previous time-step at the spatial-nodes i-1, i and i+1. This means that the extreme value of the new solution is the average of the extreme values of the previous two solutions at the three consecutive nodes. Therefore, the new solution continuously depends on the initial value $c_i^0,\ i = 1, 2, 3, \dots \dots \dots, M$.

$$\text{(ii) implies } \gamma \le 1 - 2\lambda \le 1 + \gamma \qquad \text{(iv)}$$

$$\text{(i) implies } -\lambda \le \gamma \le 1 - \lambda$$

$$\therefore -\lambda \le \gamma \le 1 - 2\lambda \text{ by (iv)}$$

Therefore, the conditions are $0 \le \lambda \le 1$ and $-\lambda \le \gamma \le 1 - 2\lambda$

That is $0 \le \frac{D\Delta t}{\Delta x^2} \le 1$ and $-\frac{D\Delta t}{\Delta x^2} \le \frac{\Delta t}{\Delta x} u_i^n \le 1 - 2\frac{D\Delta t}{\Delta x^2}$

This is guaranteed by the simultaneous inequality

$$0 \le \frac{D\Delta t}{\Delta x^2} \le 1 \text{ and } -\frac{D}{\Delta x} \le \max(u_i^0) \le \frac{\Delta x}{\Delta t} - 2\frac{D}{\Delta x}$$

Proposition 2

Statement: The stability conditions for the FTCS scheme are

$$0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2} \text{ and } -2\frac{D\Delta t}{\Delta x^2} \le \frac{\Delta t}{\Delta x} u_i^n \le 2\left(1 - \frac{D\Delta t}{\Delta x^2}\right).$$

This is guaranteed by the conditions $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$ and $-\frac{2D}{\Delta x} \le \max(u_i^0) \le 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$.

Proof:

The explicit centered difference scheme using by FTCS for CDE (2) is given by

$$c_i^{n+1} = \left(\lambda + \frac{\gamma}{2}\right) c_{i-1}^n + (1 - 2\lambda)c_i^n + \left(\lambda - \frac{\gamma}{2} u_i^n\right) c_{i+1}^n, \qquad (13)$$

where $\gamma = \frac{\Delta t}{\Delta x} u_i^n$, $\lambda = \frac{D\Delta t}{\Delta x^2}$

The equation (13) implies that if

$$0 \le \lambda + \frac{\gamma}{2} \le 1 \qquad \text{(i)}$$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

$$0 \leq 1 - 2\lambda \leq 1 \qquad \text{(ii)}$$

$$0 \leq \lambda - \frac{\gamma}{2} \leq 1 \qquad \text{(iii)}$$

then the new solution is a convex combination of the two previous solutions. That is, the solution at new time-step (n+1) at a spatial node i is an average of the solutions at the previous time-step at the spatial-nodes i-1, i and i+1. This means that the extreme value of the new solution is the average of the extreme values of the previous two solutions at the three consecutive nodes. Therefore, the new solution continuously depends on the initial value $c_i^0,\ i = 1, 2, 3, \dots \dots \dots . M$.

$$\text{(ii) implies } 0 \leq \lambda \leq \frac{1}{2} \qquad \text{(iv)}$$

$$\text{(iii) implies } \lambda - 1 \leq \frac{\gamma}{2} \qquad \text{(v)}$$

$$\text{(i) implies } -\lambda \leq \frac{\gamma}{2} \leq 1 - \lambda \qquad \text{(vi)}$$

$$\text{From (v) \& (vi), it follows that } -\lambda \leq \frac{\gamma}{2} \leq 1 - \lambda$$

$$\therefore -2\lambda \leq \gamma \leq 2(1 - \lambda)$$

Therefore, (from (v), (vi)) the conditions are $0 \leq \lambda \leq \frac{1}{2}$ and $-2\lambda \leq \gamma \leq 2(1 - \lambda)$

That is $\qquad 0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$ and $-2\frac{D\Delta t}{\Delta x^2} \leq \frac{\Delta t}{\Delta x} u_i^n \leq 2\left(1 - \frac{D\Delta t}{\Delta x^2}\right)$.

This is guaranteed by the conditions $0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$ and $-\frac{2D}{\Delta x} \leq \max(u_i^0) \leq 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$

NUMERICAL SIMULATION AND RESULTS DISCUSSIONS

Various finite difference equations were used to represent the parabolic model equation (2). It is extremely important to experiment with the application of these numerical techniques. It is hoped that by writing computer codes and analyzing the results, additional insights into the solution procedures are gained. Therefore, this section proposes an example and presents solutions by the described schemes.

Numerical verification of Stability Conditions:

In this study, we assume that the length of spatial domain, l = 6 meters at all time, t = 1minute to t = 6 minutes with viscosity, $\nu = 0.01 m^2/s = 36\ m^2/h$ and diffusion coefficient, $D = 0.01 m^2/s = 36\ m^2/h$.

The convection-diffusion equation for this problem is $\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}$. Various values of spatial nodes size and time steps are to be used to investigate the numerical schemes and the effect of steps on stability.

An attempt is made to solve the stated problem subject to the imposed initial and Neumann boundary conditions by the following:

The FTBSCS and FTCS schemes with

I. Spatial step size, Δx = 0.05 m Temporal step size, Δt = 0.033s, Time, T = 60×2 sec

II. Spatial step size, Δx = 0.05 m Temporal step size, Δt = 0.067s, Time, T = 60×4 sec

III. Spatial step size, Δx = 0.05 m Temporal step size, Δt = 0.01s, Time, T = 60×6 sec

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

IV. Spatial step size, $\Delta x = 0.05$ m   Temporal step size, $\Delta t = 0.1192$s, Time, $T = 60\times 7.152$ sec

Solutions:

Case *I*. When the step sizes are $\Delta x = 0.05$, $\Delta t = 0.033$.

In this case, both the schemes are to be used as stated previously.

The stability conditions of FTBSCS is determined by equation (12) as

$$0 \le \lambda \le 1 \text{ and } -\lambda \le \gamma \le 1 - 2\lambda$$

This is guaranteed by the simultaneous inequalities

$$0 \le \frac{D\Delta t}{\Delta x^2} \le 1 \; and - \frac{D}{\Delta x} \le \max(u_i^0) \le \frac{\Delta x}{\Delta t} - 2\frac{D}{\Delta x}$$

The stability conditions of FTCS is determined by equation (13) as

$$0 \le \lambda \le \frac{1}{2} \; and - 2\lambda \le \gamma \le 2(1-\lambda)$$

This is guaranteed by the conditions $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2} \; and - \frac{2D}{\Delta x} \le \max(u_i^0) \le 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$

$$where, \; \gamma = \frac{\Delta t}{\Delta x}\max(u_i^0), \qquad \lambda = \frac{D\Delta t}{\Delta x^2}$$

For this particular application,

The value of $\max(u_i^0) = 0.02$ which satisfies the guaranteed inequality in both the schemes.

$$\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0) = \frac{0.033}{0.05}\times 0.02 = 0.0132 \; and \; \lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01\times 0.033}{(0.05)^2} = 0.132$$

FTBSCS $\Rightarrow 0 \le 0.132 \le 1 \; and - 0.132 \le 0.0132 \le 1 - 2\times 0.132 = 0.736$

and

FTCS $\Rightarrow 0 \le 0.132 \le \frac{1}{2} \; and - 0.264 \le 0.0132 \le 1.736$

Therefore, the stability conditions for both the schemes are satisfied and a stable solution is expected. The concentration profiles are to be obtained up to $t = 2$ minutes are shown in figure 5.1.

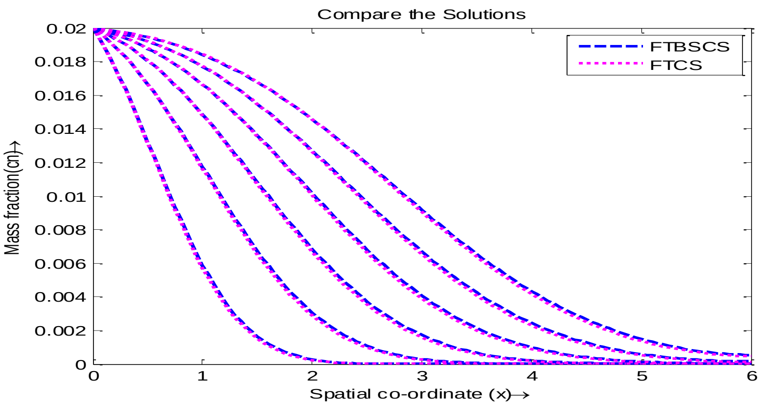

Figure 5.1: Concentration profiles with $\Delta x = 0.05$, $\Delta t = 0.033$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

Case *II*. When the step sizes are $\Delta x = 0.05$, $\Delta t = 0.067$.

In this case, both the schemes are to be used as stated previously.

The stability conditions of FTBSCS is determined by equation (12) as

$$0 \le \lambda \le 1 \text{ and } -\lambda \le \gamma \le 1 - 2\lambda$$

This is guaranteed by the simultaneous inequalities

$$0 \le \frac{D\Delta t}{\Delta x^2} \le 1 \text{ and } -\frac{D}{\Delta x} \le \max(u_i^0) \le \frac{\Delta x}{\Delta t} - 2\frac{D}{\Delta x}$$

The stability conditions of FTCS is determined by equation (13) as

$$0 \le \lambda \le \frac{1}{2} \text{ and } -2\lambda \le \gamma \le 2(1-\lambda)$$

This is guaranteed by the conditions $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$ and $-\frac{2D}{\Delta x} \le \max(u_i^0) \le 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$

where $\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0)$, $\lambda = \frac{D\Delta t}{\Delta x^2}$

For this particular application,

The value of $\max(u_i^0) = 0.02$ which satisfies the guaranteed inequality in both the schemes.

$$\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0) = \frac{0.067}{0.05} \times 0.02 = 0.0268 \text{ and } \lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01 \times 0.067}{(0.05)^2} = 0.268$$

$$\text{FTBSCS} \Rightarrow 0 \le 0.268 \le 1 \text{ and } -0.268 \le 0.0268 \le 1 - 2 \times 0.268 = 0.464$$

and

$$\text{FTCS} \Rightarrow 0 \le 0.268 \le \frac{1}{2} \text{ and } -0.536 \le 0.0268 \le 1.464$$

Therefore, the stability conditions for both the schemes are satisfied and a stable solution is expected. The concentration profiles are to be obtained up to $t = 4$ minutes are shown in figure 5.2.

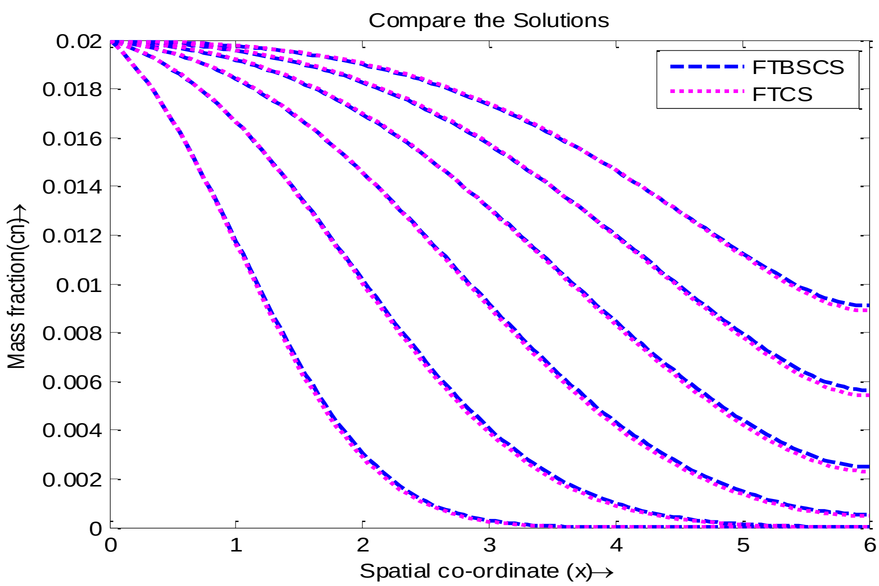

Figure 5.2: Concentration profiles with $\Delta x = 0.05$, $\Delta t = 0.067$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

Case *III*. When the step sizes are $\Delta x = 0.05$, $\Delta t = 0.1$.

In this case, both the schemes are to be used as stated previously.

The stability conditions of FTBSCS is determined by equation (12) as

$$0 \le \lambda \le 1 \text{ and } -\lambda \le \gamma \le 1 - 2\lambda$$

This is guaranteed by the simultaneous inequalities

$$0 \le \frac{D\Delta t}{\Delta x^2} \le 1 \text{ and } -\frac{D}{\Delta x} \le \max(u_i^0) \le \frac{\Delta x}{\Delta t} - 2\frac{D}{\Delta x}$$

The stability conditions of FTCS is determined by equation (13) as

$$0 \le \lambda \le \frac{1}{2} \text{ and } -2\lambda \le \gamma \le 2(1-\lambda)$$

This is guaranteed by the conditions $0 \le \frac{D\Delta t}{\Delta x^2} \le \frac{1}{2}$ and $-\frac{2D}{\Delta x} \le \max(u_i^0) \le 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$

where $\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0)$, $\lambda = \frac{D\Delta t}{\Delta x^2}$

For this particular application,

The value of $\max(u_i^0) = 0.02$ which satisfies the guaranteed inequality in both the schemes.

$$\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0) = \frac{0.1}{0.05} \times 0.02 = 0.04 \text{ and } \lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01 \times 0.1}{(0.05)^2} = 0.4$$

FTBSCS $\Rightarrow$ $0 \le 0.4 \le 1$ and $-0.4 \le 0.04 \le 1 - 2 \times 0.4 = 0.2$

and

FTCS $\Rightarrow$ $0 \le 0.4 \le \frac{1}{2}$ and $-0.8 \le 0.04 \le 1.2$

Therefore, the stability conditions for both the schemes are satisfied and a stable solution is expected. The concentration profiles are to be obtained up to *t* = 6 minutes are shown in figure 5.3.

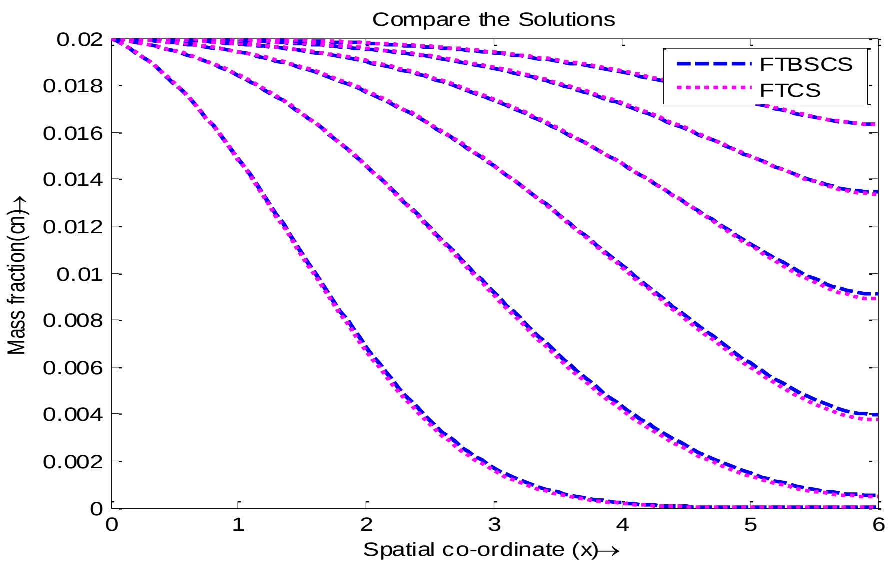

Figure 5.3: Concentration profiles with $\Delta x = 0.05$, $\Delta t = 0.1$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

Case *IV*. When the step sizes are increased to $\Delta x = 0.05$, $\Delta t = 0.1192$,

The stability conditions of FTBSCS is determined by equation (12) as

$$0 \leq \lambda \leq 1 \text{ and } -\lambda \leq \gamma \leq 1 - 2\lambda$$

This is guaranteed by the simultaneous inequalities

$$0 \leq \frac{D\Delta t}{\Delta x^2} \leq 1 \text{ and } -\frac{D}{\Delta x} \leq \max(u_i^0) \leq \frac{\Delta x}{\Delta t} - 2\frac{D}{\Delta x}$$

The stability conditions of FTCS is determined by equation (13) as

$$0 \leq \lambda \leq \frac{1}{2} \text{ and } -2\lambda \leq \gamma \leq 2(1-\lambda)$$

This is guaranteed by the conditions $0 \leq \frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$ and $-\frac{2D}{\Delta x} \leq \max(u_i^0) \leq 2\left(\frac{\Delta x}{\Delta t} - \frac{D}{\Delta x}\right)$

where $\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0)$, $\lambda = \frac{D\Delta t}{\Delta x^2}$

For this particular application,

The value of $\max(u_i^0) = 0.02$ which satisfies the guaranteed inequality in both the schemes.

$$\gamma = \frac{\Delta t}{\Delta x}\max(u_i^0) = \frac{0.1192}{0.05} \times 0.02 = 0.04768 \text{ and } \lambda = \frac{D\Delta t}{\Delta x^2} = \frac{0.01 \times 0.1192}{(0.05)^2} = 0.4768$$

FTBSCS $\Rightarrow 0 \leq 0.4768 \leq 1$ and $-0.4768 \leq 0.04768 \leq 0.0464$ which does not satisfy the stability condition of FTBSCS scheme, and

FTCS $\Rightarrow 0 \leq 0.4768 \leq \frac{1}{2}$ and $-0.9536 \leq 0.04768 \leq 1.0464$

In this case, FTBSCS of the CDE shows an instability which is shown in the following figure 5.4.

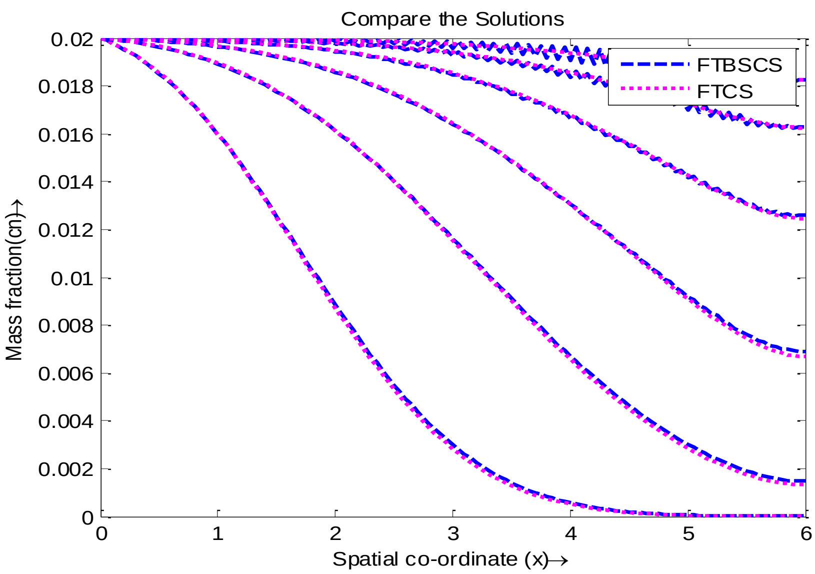

Figure 5.4: Concentration profiles with $\Delta x = 0.05$, $\Delta t = 0.1192$

Error Estimation and Convergence

We compute the relative error in L_1-norm which is defined as

INTERNATIONAL JOURNAL OF RESEARCH AND INNOVATION IN APPLIED SCIENCE (IJRIAS)
ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue II February 2025

$$err = \frac{\| c_e - c_n \|_1}{\| c_e \|_1} \quad (14)$$

where, c_e is the exact solution, and c_n is the numerical solution computed by the finite difference schemes for time $t \in [0, 6]$. The following **figure 6.1** shows the convergence of relative error by the scheme FTBSCS.

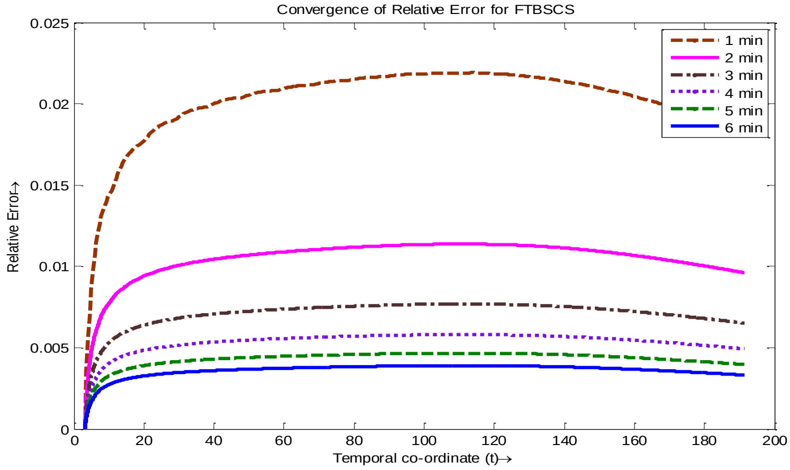

Figure 6.1 Rate of Numerical feature of Convergence

The following **figure 6.2** shows the convergence of relative error by the scheme FTCS.

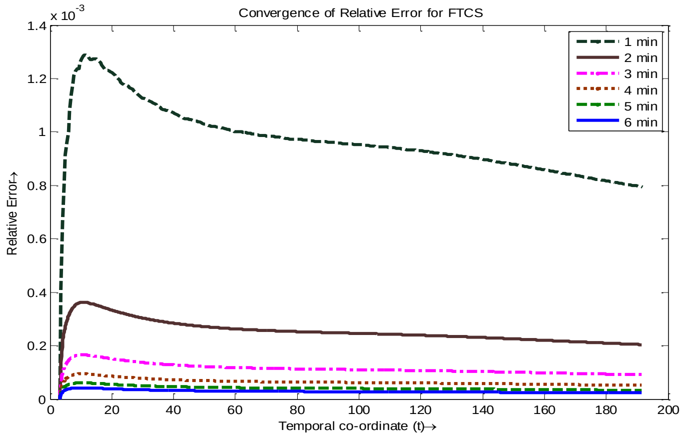

Figure 6.2 Rate of Numerical feature of Convergence

The following **figure 6.3** shows the comparison of relative errors for the both schemes.

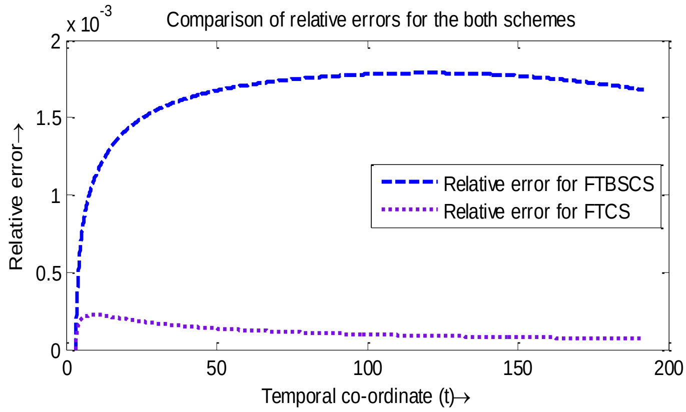

Figure 6.3 Comparison of relative errors for the both schemes

CONCLUSION

We have developed stability conditions and numerical solutions by using FTBSCS and FTCS schemes for convection diffusion equation with an initial condition and Neumann boundary conditions. The solution of Burger's equation is used as convection term in the CDE. Some numerical experiment is presented graphically.

In the Figure 5.1, 5.2, 5.3, it has been found that FTCS scheme gives better pointwise solution than FTBSCS scheme. In figure 5.4, an unstable solution is appeared by using FTBSCS scheme however, the solutions by using FTCS scheme is stable at the increased time step size $\Delta t = 0.1192$ and it is numerically shown that FTCS scheme is superior to FTBSCS scheme interms of time step selection.

The analytical result is used for code validation and for error comparison of both schemes. In addition, it is used to study the effect of step size on the accuracy of solutions. The results shown in Figure 6.1 – 6.3 are the error terms as defined above at time level [1, 6]. Two points to emphasize with regard to Figure 6.1-6.3 are: (1) for this application, the FTCS scheme has minimum error in comparison with FTBSCS scheme, and the amount of error is decreased for the both schemes as the solution is marched in time. This error reduction is due to a decrease in the influence of the initial data.

REFERENCES

1. W.F. Ames, Nonlinear Partial Differential Equation in Engineering, Academic Press, New York, 1965.
2. H. Bateman, Some recent researches on the motion of the fluids, Monthly Weather Rev. **43**(1915): 163-170.
3. E. Benton, G.W. Platzman, A table of solutions of the one-dimensional Burger's equation, Quart. Appl. Math. **30** (1972): 195-212.
4. J.M. Burger, A Mathematical Model Illustrating the Theory of Turbulence, Adv. in Appl. Mech. I, Academic Press, New York, 1948, pp 171-199.
5. J. Caldwell, P. Wanless, A.E. Cook, A finite element approach to Burger's equation, Appl. Math. Modelling **5**(1981): 189-193.
6. D.J. Evans, A.R. Abdullah, The group explicit method for the solution of Burger's equation, Computing **32** (1984) :239-253.
7. Y.C. Hon, X.Z. Mao, An efficient numerical scheme for Burger's equation, Appl. Math. Comput. **95**(1998): 37-50.
8. C.A. Fletcher, Burger's equation: a model for all reasons, in: J. Noye (Ed.), Numerical Solutions of partial Differential Equations, North-Holland, Amsterdam, 1982.
9. Al-Niami A N S and Ruston K R 1977, "Analysis of flow against dispersion in porous media", J.Hydrol. **33**:87-97.
10. Anderson, Mary P., and William W. Woessner, Applied Groundwater Modeling, Simulation of Flow and Advective Transport, Academic Press Inc., San Diego, 1992. Pages 6-68.
11. Aral M M and Liao B 1996, Analytical solutions for two-dimensional transport equation with time-dependent dispersion coefficients, J. Hydrol. Engg. **1**(1): 20-32.
12. Atul Kumar, Dilip Kumar Jaiswal and Naveen Kumar, Analytical solution of one dimensional Advection diffusion equation with variable coefficients in a finite domain, J. Earth Syst. Sci. **118**, No.5, pp. 539-549, October 2009.
13. Banks R B and Ali J 1964, Dispersion and adsorption in porous media flow, J. Hydraul. Div. **90**: 13-31.
14. Bear J 1972, Dynamics of fluids in porous media (New York: Amr. Elsev. Co.)
15. Bender, Edward A, An Introduction to Mathematical Modeling, John Wiley & Sons Inc. New York, 1978. Pages 1-140.
16. Changiun Zhu, Liping Wa and Sha Li,A numerical Simulation of Hybrid Finite Analytic Methods For Ground Water Pollution, Advanced Materials Research Vol. 121-122(2010), pp 48-51.
17. Changjun Zhu and Shuwen Li, A numerical Simulation of River Water Pollution using Grey Differential Model, Journal of Computer, No.9, September 2010.

18. Donea J, Giulinai S, Laval H. Quartapelle, Time accurate solution of advection-diffusion problems by finite elements, Comput Methods Appl Mech Eng 1984; **45**: 123-45.
19. F.B. Agusto and O. M. Bamingbola, Numerical Treatment of the Mathematical Models for Water Pollution, Research Journal of Applied Sciences **2(5):** 548-556, 2007.
20. Lai S H and Jurinak J J 1971, Numerical approximationof cation exchange in miscible displacement through soil columns; Soil Sci.Soc. Am. Proc. **35**:894-899.
21. Marino M A 1974, Distribution of contaminants in porous media flow; Water resour. Res. **10**: 1013-1018.
22. Marshal T J, Holmes J W and Rose C W 1996, Soil Physics (Cambridge University Press, 3rd edn.)
23. L.F. Leon, P.M. Austria, Stability Criterion for Explicit Scheme on the solution of Advection-Diffusion Equation, Maxican Institute of Water Technology.
24. M. Thongmoon and R. Mckibbin, A comparison of some numerical methods for the advection-diffusion equation, Int. Math. Sci. 2006, Vol. 10, pp. 49-52.
25. Nicholas J.Higham, Accuracy and stability of Numerical Algorithms, Society of Industrial and Applied Mathematics, Philadelphia, 1996.
26. Ogata A, Banks RB, A Solution of the differential equation of longitudinal dispersion in porous media, US Geological Survey, Paper 1961; 411-A; 1961.
27. Randall J. Leveque, Numerical methods for conservation laws, second edition, 1992, Springer.
28. Romao, Silva and Moura, Error analysis in the numerical solution of 3D convection diffusion equation by finite difference methods, Thermal technology, vol-08, pp12-17, 2009.
29. Guvanasen V and Volker R E 1983, Experimental investigations of unconfined aquifer pollution from recharge basins; Water resour, Res. **19(3):** 707-717.
30. Harleman D R F and Rumer R R 1963, Longitudinal and lateral dispersion in an isotropic porous medium; J. Fluid Mech. 385-394.
31. Scott A. Socolofsky & Gerhard H. Jirka, Environmental Fluid Mechanics, part 1, 2nd Edition, 2002.
32. T M A K Azad, M Begum and L S Andallah, An explicit finite difference scheme for advection diffusion equation, jahangirnagar journal of mathematics and mathematical sciences, Vol.-24, 2015, ISSN 2219-5823.
33. T M A K Azad and L S Andallah, An Explicit Finite Difference Method, Ulab Journal of Science and Engr. Vol. **6**, 2015 ISSN 2079-4398 (PRINT), ISSN 2414-102X (ONLINE)
34. Young-San Park, Jong-Jin Baik, Analytical solution of the advection diffusion equation for a ground level finite area source, Atmospheric Environment **42**: 9063-9069, 2008.

====