Multi Objective Fractional Programming Problem Involving Differentiable Vector Convex Functions Including Certain Classes of Nonlinear Composite Problems

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DOI: https://doi.org/10.51584/IJRIAS.2025.100700031

Received: 07 July 2025; Accepted: 15 July 2025; Published: 02 August 2025

ABSTRACT

Differentiable multiobjective fractional programming problems under V-convexity with special emphasis on specific classes of nonlinear composite problems

INTRODUCTION

In the differentiable case, Jeyakumar and Mond [10] introduced the concept of vector invexity, which effectively addresses a key challenge in invex analysis—namely, the difficulty of verifying inequality conditions for the same function. They formulated sufficient optimality conditions under V-pseudo invexity and established corresponding duality results. Building upon this foundation, Egudo and Hanson [6], inspired by Zhao's work [15], extended the notion of V-invexity to the nonsmooth setting by replacing classical gradients with Clarke's generalized gradients [2]. While Jeyakumar and Mond's framework was developed within the context of generalized convex mathematical programming, it was not applied to multiobjective fractional programming. This paper seeks to fill that gap by extending the theory to differentiable multiobjective fractional programming problems under V-convexity, with particular attention to specific classes of nonlinear composite problems that often arise in practical applications.

Notations & Preliminaries

Consider constrained multiobjective fractional optimization problem.

(VFP) V- Minimize $\left[\frac{f_i(x)}{g_i(x)}, ---, \frac{f_p(x)}{g_p(x)}\right]$

Subject to $h_j(x) \le 0,\ j = 1,2.......,m$

Where $\frac{f_i}{g_i} : X_\circ \to R.$ and $h_j : X_\circ \to R^m$ are differential functions and X_o is an open set in R^n. Here the symbol V- minimize stands for vector minimization. This is the problem of finding the set of weak minimum for points (VFP). When P=1, the problem (VPF) reduces to a Scalar optimization problem and it is denoted by (FP). Convexity of the Scalar problem (FP) is characterized by the inequalities.

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \frac{f_i{}^t(a)}{g_i{}^t(a)}(x-a) \ge 0$$

$$h_j(x) - h_j(a) - h_j{}^t(x)(x-a) \geq 0 \forall x, a \in x_\circ.$$

The functional form $(x-a)$ here plays no role in establishing the following two well-known properties in scalar convex programming:

(s). Every feasible Kuhn - Tucker point is a global minimum (w) weak duality holds between (FP) and its associated dual problem. Having this in mind, considered problem (FP) for which there exists a function $\eta: X_\circ * X_\circ \to R^n$ such that

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \frac{f_i{}^t(a)}{g_i{}^t(a)}\eta(x,a) \geq 0 \ ,$$

$$h_j(x) - h_j(a) - h_j{}^t(a)\mu(x,a) \geq 0, \forall x, a \in x_\circ.$$

and showed that such problems (known now as invex problem) also. Possess properties (s) and (w). Since then, various generalization of conditions (I) to multiobjective problems and many properties of functions that satisfy (I) have been established in the literature. However, the major difficulty is that the invex problems require the same function $\eta(x,a)$ for the objective function and the constraints. This requirement turns out to be a severe restriction in applications. Because of this restriction, pseudo liner multiobjective, problems and certain non-linear multiobjective fractional programming problems require separate treatment as far as optimality and duality properties are concerned. In this chapter we show how this situation can be improved and how the properties (s) and (w) can be extended to hold for generalized convex multiobjective problems and certain multi- objective fraction problems. To this, we modify the condition (I) in the next section as follows:

New Classes & generalized convex vector functions:

A vector function $\frac{f_i}{g_i} : X_\circ \to R^p$ is said to be V-Convex if there exist functions $\mu : X_\circ * X_\circ \to R^p$ and $\alpha_i : X_\circ * X_\circ \to R_+ \setminus^{\{0\}}$ such that for each $x, a \in X_\circ$, and for i=1,2,……,p,

$$\frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \alpha(x,a)\frac{f_i{}^t(a)}{g_i{}^t(a)}(x-a) \geq 0,$$

When p=1, the definition of V-Convexity reduces to the notion of convexity. The problem of (VPF) is said to be V-Convex if the vector function $\left(\frac{f_1}{g_2}, \frac{f_2}{g_2} \ldots\ldots \frac{f_p}{g_p}\right) \& \left(h_1, h_2, \ldots\ldots, h_m\right)$ is V- convex.

Equivalently, V-convexity of (VFP) means that there exists function $\mu : X_\circ * X_\circ \to R^n$ and $\alpha_i, \beta_j : X_\circ * X_\circ \to R_+ \setminus \{0\},\ i = 1,2,\ldots, p,\ j = 1,2,\ldots m$ such that

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue VII July 2025

$$\text{(VI) } x, a \in X_\circ \Rightarrow \begin{cases} \frac{f_i(x)}{g_i(x)} - \frac{f_i(a)}{g_i(a)} - \alpha_i(x,a)\frac{f_i^t(a)}{g_i^t(a)}(x-a) \\ h_j(x) - h_j(a) - \beta_j(x,a) h_j^t(a)(x-a) \end{cases}$$

Let $\alpha_i(x,a) = 1 = \beta_j(x,a),\ for\ i = 1,2,\ldots p,\ j = 1,2,\ldots,m$ and p=1, the scalar problem (FP) becomes the strongly pseudo convex programming problem.

Example 2.2.1:- Consider the function $Z: R^n \to R^p$ defined by $Z(x) = \left(\frac{f_1}{g_1}(\varphi\varphi(x)),\ldots,\frac{f_p}{g_p}(\varphi\varphi(x)\right)$

where $\frac{f_i}{g_i}: R^n \to R, i = 1,2,\ldots,p$ are strongly pseudo-convex functions with real positive functions $\alpha_i(x,a), \varphi: R^n \to R^n$ is subjective with $\phi^{'}(a)$ onto for each $a \in R^n$. Then, the function Z is V-convex. To see this, Let $x, a \in R^n$

$u = \phi(x), v = \phi(a),$ Then, by strong pseudo convexity,

$$\frac{f_i(\varphi\varphi(x)}{g_i(\varphi\varphi(x)} - \frac{f_i(\varphi\varphi(a)}{g_i(\varphi\varphi(a)} = \frac{f_i(u)}{g_i(u)} - \frac{f_i(v)}{g_i(v)} \geq \alpha_i(u,v)\frac{f_i^{'}(u)}{g_i^{'}(v)}(u-v)$$

Since $\phi^t(a)$ is onto $(u-v) = \varphi^t(a)(x-a)$ is solvable for some $(x-a) \in R^n$

Hence

$$\frac{f_i(\varphi\varphi(x)}{g_i(\varphi\varphi(x)} - \frac{f_i(\varphi\varphi(a)}{g_i(\varphi\varphi(a)} \geq \alpha_i(u,v)\frac{f_i^t(v)}{g_i^t(v)}\varphi(a)(x-a)$$

$$= \overline{\alpha_i}(x,a)(f_i.\phi)^t(a)(x-a),$$

Where $\overline{\alpha_i}(x,a) = \alpha_i(\phi(x),\phi(a)) > 0.$

Example 2.2:- Consider the composite vector function $Z(X) = \left(\frac{f_i(F_1(x)}{g_i(G_1(x)},\ldots,\frac{f_p(F_p(x)}{g_p(G_p(x)}\right)$, where for each i=1,2,…,p $.\frac{F_i}{G_i}: X_\circ \to R$ is continuously differentiable and pseudo linear with the possible

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue VII July 2025

proportional function $\alpha_i(.\ ,\ .\ )$ and $\frac{f_i}{g_i}: R \to R$ is convex. The Z is V- Convex with this follows from the following convex inequality and pseudo linear equality conditions.

$$f_i\left(\frac{F_i(x)}{G_i(x)}\right) - f_i\left(\frac{F_i(a)}{G_i(a)}\right) \geq f_i'\left(\frac{F_i(a)}{G_i(a)}\right)\left(\frac{F_i(x)}{G_i(x)} - \frac{F_i(a)}{G_i(a)}\right)$$

$= f_i'\left(\frac{F_i(a)}{G_i(a)}\right)\alpha_i(x,a)\frac{F_i'(a)}{G_i'(a)}(x-a)$ for a simple numerical example of a composite vector function, consider $Z(x_1, x_2) = \left(e^{x_1/x_2}, \frac{x_1 - x_2}{x_1 + x_2}\right)$, Where $X_\circ = \{(x_1, x_2) \in R^2 / x_1 \geq 1, x_2 \geq 1\}$.

We now show that the V-Convexity is preserved under smooth convex transformation.

Necessary Theorem:-

Theorem:- Let $\psi : R \to R$ be differentiable and convex with positive derivative everywhere. Let $P : X_\circ \to R^p$ be V-convex. Then, the function $P_\psi(x) = (\psi(p,(x)), \ldots \psi(h_p(x))), x \in X_\circ$ is V-Convex

Proof:- Let $x, a \in X_\circ$, Then from the monotonicity of ψ and V-Convexity of P, we get

$$\psi(P_i(x)) \geq \psi(P_i(a)) + \alpha_i(x,a)P_i^t(a)(x-a)) \geq \psi(P_i(a)) + \psi^t(P_i(a))\alpha_i(x,a)P_i^t(a)(x-a)$$

$$= \psi(P_i(a)) + \alpha_i(x,a)(\psi.P_i)'(a)(x-a)$$

Thus, $P.\psi(x)$ is V-Convex the notations of pseudo-convexity and quasi-convexity for a scalar function can now be extended to a vector function. A vector function $\frac{f_i}{g_i} : X_\circ \to R^p$ is said to be V-pseudo convex if there exists functions $\mu : X_\circ * X_\circ \to R^p$ and $\beta_i : X_\circ * X_\circ$ such that for each $x, a \in X_\circ$,

$$\sum_{i=1}^{p} \frac{fi'(a)}{gi'(a)}(x-a) \geq 0 \Rightarrow \sum_{i=1}^{p} \beta_i(x,a)\frac{f_i(x)}{g_i(x)} \geq \sum_{i=1}^{p} \beta_i(x,a)\frac{f_i(a)}{g_i(a)}$$

The vector function $\frac{f_i}{g_i}$ is said to be V-quasi-convex if there exists functions $\mu : X_\circ * X_\circ \to R^p$ and $\overline{r_i} : X_\circ * X_\circ \to R_+ \setminus \{0\}$ such that for each $x, a \in X_\circ$,

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue VII July 2025

$\sum_{i=1}^{p} \bar{r}_i(x,a) \frac{f_i(x)}{g_i(x)} \leq \sum_{i=1}^{p} \bar{r}_i(x,a) \frac{f_i(a)}{g_i(a)}$ is inconsistent. Assume that $\sum_{i=1}^{p} \tau_i \frac{f_i^t(a)}{g_i^t(a)} = 0$, for some $0 \neq \tau \in R^p, \tau_i \geq 0$. Suppose that a is not a weak minimum for $\frac{f_i}{g_i}$. Then there exists $x_\circ \in R^n$ such that $\frac{f_i(x_\circ)}{g_i(x_\circ)} \leq \frac{f_i(a)}{g_i(a)}$, i=1,2,…,p. Since $\frac{f_i}{g_i}$ is V-convex, there exist $\alpha_i(x_\circ, a), i = 1,2,.., p$ $\frac{1}{\alpha_i(x_\circ,a)}\left(\frac{f_i(x_\circ)}{g_i(x_\circ)} - \frac{f_i(a)}{g_i(a)}\right) \geq \frac{f_i^t(a)}{g_i^t(a)}(x_\circ - a)$.

So, $\sum_{i=1}^{p} \left(\frac{\tau_i}{\alpha_i(x_\circ,a)}\right) \frac{f_i(x_\circ)}{g_i(x_\circ)} - \frac{f_i(a)}{g_i(a)} < 0$,

and hence $\sum_{i=1}^{p} \tau_i \frac{f_i^t(a)}{g_i^t(a)}(x_\circ - a) < 0.$

This is a contradiction.

Numerical Example

Let:

Decision variable: $x \in R$

Objective functions:

$f_1(x) = x^2+1/ x+2$,

$f_2(x) = x^2+2/x+3$

These are both fractional functions and convex under appropriate domains.

Let $f_i(x) = [f_1(x)/ f_2(x)]$

Let $\phi(t) = e^t$ which is convex and differentiable with positive derivative $\phi^t(t)=e^t>0$

Now consider the composition:

$h_j(x) = [\phi(f1(x)) / \phi(f2(x))] = [e^{f_1(x)} / e^{f_2(x)}]$

Let's pick x = 1:

$f_1(1) = 1^2+1/ 1+2 = 2/3$

So, $\phi(f_1(1)) = e^{2/3} \approx 1.948$

$f_2(1) = 1^2+2 / 1+3 =$ ¾,

So $\phi(f_2(1)) = e3/4 \approx 2.117$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume X Issue VII July 2025

So, h (1) ≈ [1.948 / 2.117]

This example demonstrates that composing a differentiable convex function ϕ\phi with positive derivative with a vector of fractional V-convex functions f(x), yields another V-convex vector-valued function ϕ (f(x)).

Sufficiency theorem:-

Consider the multi objective fractional problem (VFP). Suppose that the Lagrange multiplier conditions that $\sum_{i=1}^{p}\tau_i \frac{f_i^{\,t}(a)}{g_i^{\,t}(a)} + \sum_{j=1}^{m}\lambda_j h_j^{\,t}(a) = 0, \lambda_j h_j(a) = 0, \tau \in R^p, \tau \neq 0, \tau \geq 0, \tau \in R^m, \lambda \geq 0$, hold at a feasible point $a \in X_\circ$. If $\left(\tau_1 \frac{f_1}{g_1}, \tau_2 \frac{f_2}{g_2}, ..., \tau_p \frac{f_p}{g_p}\right)$ is V-pseudo convex with respect to μ and $\left(\lambda_1 h_1, ..., \lambda_m h_m\right)$ is V-quasi convex with respect to μ then a is a global weak minimum point for (VFP).

<u>Proof:-</u> Suppose that a is not a global minimum point. Then there exists feasible $x_\circ \in X_\circ$ such that $\frac{f_i(x_\circ)}{g_i(x_\circ)} < \frac{f_i(a)}{g_i(a)}$, i=1,2,..,p. So,

$\sum_{i=1}^{p}\beta_i(x_\circ, a)\tau_i \frac{f_i(x_\circ)}{g_i(x_\circ)} < \sum_{i=1}^{p}\beta_i(x_\circ, a)\,\tau_i \frac{f_i(a)}{g_i(a)}$. Now by the V-pseudo convexity condition, we get $\sum_{i=1}^{p}\tau_i \frac{f_i^{\,t}(a)}{g_i^{\,t}(a)}\mu(x_\circ, a) < 0$. Since the Lagrangian condition holds at a,

$$\sum_{j=1}^{m}\lambda_j h_j^{\,t}(a)\mu(x_\circ, a) > 0$$

From the V-quasi-convexity condition, we get

$\sum_{j=1}^{m}\overline{r_j}(x_\circ, a)\lambda_j h_j(x_\circ) > \sum_{j=1}^{m}\overline{r_j}(x_\circ, a)\lambda_j$ this is a contradiction, since

$$\lambda_j h_j(x_\circ) \leq 0, \lambda_j h_j(a) = 0, and \overline{r_j}(x_\circ, a) > 0,\ for j = 1,2,...,m.$$

Duality:-

Dual Formulation:-

V-maximize $\left(\frac{f_1}{g_1}(\xi), ..., \frac{f_p}{g_p}(\xi)\right)$,

(VFD) Subject to

$$\sum_{i=1}^{p}\tau_i \frac{f_i^{\ t}}{g_i^{\ t}}(\xi)+\sum_{j=1}^{m}\lambda_j h_j^{\ t}(\xi)\geq 0, \lambda_j \geq 0, \tau_i \geq 0, \tau e = 1, \lambda_j h_j(\xi)\geq 0, j=1,2,..,m.$$

Where e= (1, 1… 1). Note that the Lagrangian conditions in sufficiency theorem hold for (VFP) at a weak minimum point a under a constraint qualification and they can equivalently be written as $\sum_{i=1}^{p}\tau_i \frac{f_i^{\ t}(a)}{g_i^{\ t}(a)}+\sum_{j=1}^{m}\lambda_j h_j^{\ t}(a)=0, \lambda \geq 0, \tau \geq 0, \tau e = 1, \lambda_j h_j(a)=0, j=1,2,..,m.$

Weak Duality Theorem:- Consider the multiobjective fractional problem (VFP) and (VFD). Let x feasible for (VFP) and (ξ,τ,λ) be feasible for (VFD). If $\left(\tau_1\frac{f_1}{g_1},\tau_2\frac{f_2}{g_2},...,\tau_p\frac{f_p}{g_p}\right)$ is V-pseudo convex and $(\lambda_1 h_1,....., \lambda_m h_m)$ is V-quasi convex with respect to the same μ, then

$$\left(\frac{f_1(x)}{g_1(x)},....,\frac{f_p^{\ t}(x)}{g_p^{\ t}(x)}\right) - \left(\frac{f_1(\xi)}{g_1(\xi)},....,\frac{f_p^{\ t}(\xi)}{g_p^{\ t}(\xi)}\right) \notin \text{ in } R^{p}{}_{+}$$

Proof:- from the feasibility conditions, $\lambda_j h_j(x)-h_j(\xi))\leq 0$, for each j=1,2,..,m. Since $\overline{r_j}(x,\xi)$ is positive $\sum_{j=1}^{m}\overline{r_j}(x,\xi)\left[h_j(x)-h_j(\xi)\right]\leq 0$. Hence, $\sum_{j=1}^{m}\lambda_j h_j^t\mu(x,\xi)\leq 0,$ and so, $\sum_{i=1}^{p}\tau_i\frac{f_i^t}{g_i^t}\mu(x,\xi)\geq 0,$ the conclusion now follows from the V-pseudo convexity condition since $\tau_e = 1$ and $\beta_i(x,\xi)>0$.

Theorem: Strong Duality Theorem:-

Assume that a is a weak minimum of (VFP) and that a suitable constraint qualification is satisfied at a. Then there exist (τ,λ) such that (a,τ,λ) is a feasible for (FVD).

<u>**Proof:-**</u> since a is a weak minimum for (FVP) and a constraint qualification is satisfied at a, from the Lagrangian conditions, there exists (τ,λ) such that (a,τ,λ) is a feasible for (VFD). Clearly the values of (VFP) and (VFD) are equal at a, since the objective function for both problems are the same. By the generalized V-convexity hypothesis, weak duality holds, hence if (a,τ,λ) is not a weak optimum for (VFD), there must exist (ξ,τ^*,λ^*) feasible for (VFD), $\xi \neq a$ such that

$$\left(\frac{f_1(\xi)}{g_1(\xi)},....,\frac{f_p^{\ t}(\xi)}{g_p^{\ t}(\xi)}\right) - \left(\frac{f_1(a)}{g_1(a)},....,\frac{f_p^{\ t}(a)}{g_p^{\ t}(a)}\right) \in \text{ in } R_{+}^{P}$$

Contradicting weak duality.

Practical Applications

Multi objective fractional programming with differentiable vector convexity appears in several real-world contexts:

Energy and Power Systems

Objectives like:

Fuel cost per MW → Cost/ Power output

Emission per MW → CO_2/ Power output

Subject to grid balance, generation capacity constraints.

Telecommunications / Network Optimization

Objective ratios:

Delay per bandwidth →Latency/Bandwidth

Cost per coverage → Cost/Coverage

Nonlinear constraints on routing flow conservation.

Transportation and Logistics

Objectives like:

Fuel per kilometer → Fuel usage/Distance

Delivery cost per product → Total cost/ Items delivered

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