Stochastic Processes in Actuarial Surplus Modelling: A Validation Framework

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ABSTRACT

This study develops a validation framework for actuarial surplus modelling using stochastic processes, focusing on Geometric Brownian Motion (GBM) for assets, the Vasicek model for interest rates, and the Ornstein-Uhlenbeck (OU) process for liabilities. Through Monte Carlo simulations (10,000 paths, 1-year horizon), we evaluate three scenarios: GBM with deterministic liabilities, GBM with OU liabilities, and GBM with Vasicek-discounted liabilities. Analytical derivations of the stochastic differential equations and their solutions provide the mathematical foundation, while Euler-Maruyama discretisation ensures numerical tractability. Results demonstrate that mean-reverting models (OU and Vasicek) reduce surplus volatility (standard deviation: €360,000–€375,000 vs. €400,000) and tail risk (99.5% VaR: -€60,000, –€75,000 vs. -€100,000; CVaR: -€100,000 -€120,000 vs. -€150,000) compared to the deterministic liability model. These findings align with Solvency II's risk-based capital requirements, highlighting the Vasicek model's superior stability for liability valuation and the OU model's suitability for dynamic liability adjustments. The framework supports enhanced solvency assessment and product pricing, with implications for regulatory compliance and risk management in insurance.

Keywords: Stochastic Processes, Surplus Modelling, Mean Reversion, Solvency II, Ornstein-Uhlenbeck, Vasicek model, Geometric Brownian Motion (GBM), Euler-Maruyama

INTRODUCTION

The financial markets and regulatory requirements now require an exponentially increasing level of investment understanding and sophistication, which has created the need to use stochastic processes in all actuarial applications (in particular in modelling surplus evolution during uncertain financial markets or liability conditions). In line with the above, Akhtari (2019) considers surplus, which is the excess of assets over liabilities, to be an important measure of solvency, prices of insurance products, and capital allocation. Held deterministic models cannot usually bring out the random nature of asset returns and liability growth, which requires stochastic models to bring out the actual situation in the real world.

The three stochastic processes commonly applied in financial and actuarial modelling are compared in this study. Geometric Brownian Motion (GBM) is a process that governs the price of assets and characterises continuous compounding and volatility in the market. Vasicek Model: Interest rates are a mean-reverting process necessary to discount liabilities. Ornstein-Uhlenbeck (OU) Process: A mean-reverting model which applies to liability or surplus dynamics.

At historical financial data calibration, we hypothesise that these models could generate accurate and realistic predictions of surplus conduct and promote compliance with solvency regimes such as Solvency II. Mean-reverting stochastic models (OU and Vasicek) minimise excess volatility and tail risk over a deterministic liability-based model. They are therefore more stable and meet the Solvency II Value-at-Risk (VaR)-based Solvency Capital Requirement (SCR) of 99.5 per cent. In modelling assets and liabilities across different economic conditions, we perceive the capability of the models to mimic both the short-run volatility and the long-term stability and the maximum risk situations (Fouque et al., 2000). The findings are used to formulate regulatory policies, capital adequacy checks and insurance products with embedded options or guarantees. This research paper is organised as follows: Section 2 will give a theoretical underpinning to choosing stochastic processes of interest. Section 3 describes the methodology, including its surplus and definition of the simulation framework. Section 4 reports our simulation results, Section 5 comments on their meaning concerning regulation and product design, and Section 6 gives recommendations to the actuarial practice.

LITERATURE REVIEW

Actuarial stochastic processes have expanded significantly due to the increased need to model complicated financial and liability behaviours that face uncertainty(Yang, 2025). This literature review examines the theoretical premise and empirical utilisation of three standard stochastic models (Geometric Brownian Motion (GBM), Vasicek model, and Ornstein-Uhlenbeck (OU) process) in the framework of actuarial surplus modelling processes. Such models have crucial characteristics like volatility, mean reversion and long-term rates of assets, liabilities and interest rates. This modelling type is essential in carrying out solvency tests, capital distributions policies and designing insurance products. The review combines theoretical knowledge and empirical data comprehension to highlight the necessity of these models to analyse surplus and highlight the gap that exists in supporting and fueling the necessity of conducting the current research.

Theoretical Background

Stochastic models provide an effective mathematical model for quantifying uncertainty in financial and actuarial systems. The underlying principles between the Geometric Brownian Motion (GBM), Vasicek, and the Ornstein-Uhlenbeck (OU) processes are firmly based on stochastic calculus, which is a significant branch of activities in financial mathematics and has been broadly applied as a measure of uncertainty in financial markets.

Geometric Brownian Motion (GBM)

Geometric Brownian Motion denotes a standard model in financial theory, founded in 1973 by Black and Scholes when they published their article on option pricing (Fernandes et al., 2016). It supposes asset prices follow a log-normal distribution, dependent on a fixed drift and volatility. The GBM is defined as follows mathematically:

$$\mathrm{dSt} \ = \ \mu S_t d_t + \sigma \, S_t \mathrm{d}W_t$$

where S_t Is the asset price at time t, μ is the drift coefficient, σ is the volatility, and S_t Denotes a standard Brownian motion. As explained by Peng and Simon (2024), Geometric Brownian Motion has been of great value in continuous-time finance due to its analytical convenience and particularly because it leads to non-negative asset values as an exponential process. The mathematical simplicity, geometric simplicity and more psychologically friendly factor structure have made GBM a natural model for an equity-based portfolio. Petters & Dong (2016) also address their role in developing stochastic calculus in finance.

Nevertheless, GBM has received critical attention because of its simplifying assumptions, especially the assumption that the parameters do not vary and that returns are distributed normally (after taking the log). They overlook empirical characteristics in the financial time series, like fat tails and mean-reversion (Ritschel et al., 2021). Therefore, alternative or complementary models are frequently needed in an actuarial setting when assets, liabilities or interest rates are subject to more complicated dynamics.

Vasicek Model

Vasicek model Vasicek was proposed by Oldrich Vasicek in 1977, as a mean-reverting stochastic process used to describe the dynamics of short-term interest rates (Wu, 2024). It is stated as:

$$\mathrm{d}r_t \ = \ \mathrm{a}(\mathrm{b} - r_t) d_t + \sigma \mathrm{d}W_t$$

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where r_t Denotes the short-term interest rate at time t, a is the speed of mean reversion, b represents the long-term mean level, σ is the volatility, and W_t It is a standard Brownian motion. The mean-reverting characteristics of the model reflect the empirical reality that interest rates tend to fluctuate around some sort of long-term average, which frequently results from macro dynamics or the use of monetary policies.

In the context of term structure modelling, Burgess (2014) offers a detailed discussion of the Vasicek model with its focus on the tractability and the Gaussian distribution of interest rates. These characteristics help price zero-coupon bonds and financing interest rate risk.

Theoretically, the Vasicek model expands the Ornstein-Uhlenbeck (OU) process, incorporating a mean reversion on interest rates, as Guan et al. (2024) explained. Its ability to be solved in closed form and its calibrability are particularly appealing to actuarial use, like discounting future liabilities in regulatory regimes like Solvency II. However, the assumptions of exceptionally stable volatility and normally distributed rates restrict the range of the model to represent extreme fluctuations in rates or the non-negativity property of the interest rate. Such a restriction has necessitated the creation of other varieties, including the Cox-Ingersoll-Ross (CIR) model, which incorporates state-dependent volatility to overcome these issues (A & Shao, 2017).

Ornstein-Uhlenbeck (OU) Process

The Ornstein-Uhlenbeck (OU) process, as a process introduced initially in physics, where, in 1930, Ornstein and Uhlenbeck used the OU process to model mean-reverting quantities, has since been applied to purely financial and actuarial interests (Pham & Chong, 2017). It is stated as:

$$dX_t = \theta(\mu - X_t)d_t + \sigma dW_t$$

Where Xt is the state variable (e.g., surplus deviation or liability), θ is the rate of mean reversion, μ is the long-term mean, σ denotes volatility, and Wt is a standard Brownian motion. Mariani et al. (2023) emphasise the flexibility of the OU process to model variables that fluctuate around some equilibrium, including interest rates, economic indicators, etc. Unlike the GBM, the OU process allows negative values; thus, the process is particularly useful in modelling deviations related to target surplus levels or liability fluctuations. The OU process is structurally related to the Vasicek model, except that the MAIN difference is the application of the model. Lee (2025) explains that it is utilised in mean-reverting spread and stochastic volatility modelling. It has a theoretical basis for using actuarial models that capture behavioural or regulatory liability changes.

Empirical Applications in Actuarial Science

The GBM, Vasicek, and OU have been widely used in actuarial processes, especially surplus modelling, solvency testing, and insurance product design. The numerous empirical applications of these stochastic processes exhibit their versatility and applicability in various areas of insurance and finance.

GBM in Asset Modelling

Geometric Brownian Motions are commonly used to describe insurers' asset portfolios. Garayeta et al. (2022) applied GBM to model equity returns, as part of Solvency Capital Requirement (SCR) calculations, under Solvency II. Their study confirmed that GBM could measure short-term asset volatility, thus helping insurers determine capital requirements under stress conditions. Chen et al. (2008) used GBM in the pricing and risk management of equity-linked type insurance products, particularly variable annuities, demonstrating how the framework may generate plausible historical growth paths of assets. Nevertheless, the limitations of GBM are also indicated in the empirical studies. According to Berger and Missong (2014), GBM performs poorly in capturing tail risks during an economic crisis, raising concerns about its efficacy in modelling solvency where high but infrequent losses are essential. This result highlights the necessity of jumps or stochastic volatility models, especially in actuarial applications.

Vasicek Model in Liability Discounting

The Vasicek model has gained considerable popularity in the actuarial aspects of interest rate modelling and interest rate liability discounting. Georges (2003) estimated the term structure of interest rates to value pension liabilities using the Vasicek model, reflecting the past mean-reverting rate dynamics. EIOPA (2025), under Solvency II, advises using stochastic interest rate models such as Vasicek to calculate the technical provisions valuation, owing to their ability to capture the changes in the volatility of the present value calculations.

Also, Tarashev (2005) published empirical findings on applying the Vasicek model to regulatory reporting, demonstrating that it can produce robust predictions of risk margins when calibrated to public data concerning bond markets. Nevertheless, the normal distribution assumption of rates can result in too high an estimate of the probability of extreme interest rate movements, or the possibility of these models underestimating it, as Hallelman (2017) points out, the use of models like CIR and Hull-White to deal with these factors.

OU Process in Liability and Surplus Dynamics

There is an increasing empirical application of the OU process in modelling insurance liabilities and surplus dynamics. The OU process was applied by Liu and Liu (2024) to model non-life claim reserves by considering the mean reversion property of the claim payment due to the influence of economic and regulatory factors. In life insurance, Bouknegt and Pelsser (2002) employed the OU process to model policyholder liabilities in participating contracts, where liabilities adjust over time due to profit-sharing or market changes.

Moreover, Lee (2024) demonstrated the effectiveness of the OU process in modelling surplus deviations under Solvency II, particularly for insurers offering products with mean-reverting liability patterns. Its ability to model negative deviations makes it suitable for representing reserve margins, as shown in Taylor & McGuire (2018), who applied the OU process in stochastic reserving models. However, practical challenges remain, particularly in calibrating the speed of mean reversion from historical data, a concern echoed by Georgios Moysiadis et al. (2019).

Gaps and Motivation for Current Research

Despite the strong theoretical and empirical foundations of GBM, Vasicek, and OU models in actuarial science, several gaps remain unaddressed. Firstly, most studies examine these models in isolation—focusing either on asset modelling, interest rates, or liabilities—without integrating them into a unified framework for surplus modelling (Lee, 2025). As a result, the joint interaction between assets and liabilities under realistic market conditions is often overlooked, particularly within the Solvency II regime.

Secondly, parameter calibration using real-world actuarial and financial data is often underexplored. Stylised assumptions represent many studies, and the applicability of model output is limited (Akhtari, 2019). Thirdly, these models have not been adequately tested in comparative terms concerning their capacity to capture the tail risk and mean-reverting behaviour of these extremes of market conditions.

This research attempts to fill those gaps by integrating GBM, Vasicek and OU processes into a common surplus modelling. Monte Carlo simulations on a large scale will be performed to check the effectiveness of the models in real-world financial conditions. Comparing the vocal strength of each model to the risk-based capital requirements of Solvency II, with risk-based capital measures such as the value-at-Risk (VaR) and the Conditional value-at-Risk (CVaR).

This study provides a valuable addition to the literature on actuarial science by proposing a rigorous theoretical premise on modelling surplus, making the approach compliant with regulatory requirements, and helping design products based on data (Ramachandran, 2025).

METHODS

Introduction

In this paper, in response to the necessity to model surplus with uncertainty as a critical application of knowledge in actuarial science, the mathematical foundations of three stochastic processes have been derived; namely, the Geometric Brownian Motion (GBM), the Vasicek Model, and the Ornstein-Uhlenbeck (OU) Process. These models show the surplus equation's asset dynamics, interest rates, and liability changes. $S_t = A_t - L_t$. Each section gives the stochastic differential equation (SDE), its analytical solution, and its application to actuarial surplus modelling, with special consideration to the Solvency II requirements. Another chapter speaks of numerical implementation via the Euler-Maruyama method for simulation.

Geometric Brownian Motion (GBM)

Stochastic Differential Equation (SDE)

GBM is used to model asset price evolution, assuming the asset price A_t Follows a process with constant drift and volatility. The SDE is:

$$dA_t = \mu A_t dt + \sigma A_t dW_t$$

Where:

- A_t Asset value at time t.
- μ : Expected return (drift).
- σ : Volatility.
- W_t Standard Brownian motion (Wiener process).
- dt : Infinitesimal time increment.
- dW_t : Increment of Brownian motion, $dW_t \sim N(0, dt)$.

This SDE implies that the relative change in asset price (dA_t/A_t) consists of a deterministic drift component (μdt) and a stochastic component ($\sigma d\, W_t$).

Derivation of the Analytical Solution

To solve the SDE, we find the explicit form of A_t. Since the SDE is multiplicative (the noise term is proportional to A_t We apply Itô's lemma. Consider the transformation $Y_t = \ln A_t$. The differential dY_t using Itô's lemma for $f(A_t) = \ln A_t$ Is:

$$df = \frac{\partial f}{\partial A_t} dA_t + \frac{1}{2}\frac{\partial^2 f}{\partial A_t^2}(dA_t)^2$$

Compute the partial derivatives:

- $\frac{\partial f}{\partial A_t} = \frac{1}{A_t}$.
- $\frac{\partial^2 f}{\partial A_t^2} = -\frac{1}{A_t^2}$.

Substitute $dA_t = \mu A_t dt + \sigma A_t dW_t$. The term $(dA_t)^2$ is calculated using Itô's rules $((dt)^2 = 0, dt \cdot dW_t = 0, (dW_t)^2 = dt)$:

$$(dA_t)^2 = (\mu A_t dt + \sigma A_t dW_t)^2 = \sigma^2 A_t^2 (dW_t)^2 = \sigma^2 A_t^2 dt$$

Thus:

$$dY_t = \frac{1}{A_t}(\mu A_t dt + \sigma A_t dW_t) + \frac{1}{2}\left(-\frac{1}{A_t^2}\right)(\sigma^2 A_t^2 dt)$$
$$= \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

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The SDE for $Y_t = \ln A_t$ Is:

$$dY_t = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dW_t$$

This resembles arithmetic Brownian motion. Integrate from 0 to :

$$Y_t - Y_0 = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right)ds + \int_0^t \sigma dW_s$$
$$\ln A_t - \ln A_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$

Exponentiate:

$$\ln \frac{A_t}{A_0} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t$$
$$A_t = A_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

This solution shows that. A_t follows a log-normal distribution, as $\ln A_t$ It is normally distributed.

Relevance to Actuarial Surplus Modelling

In surplus modelling ($S_t = A_t - L_t$), GBM simulates asset dynamics (A_t). Its exponential form ensures positive asset values, aligning with real-world constraints for equity or portfolio values (Wekwete et al., 2023). The model captures continuous compounding and market volatility, making it suitable for projecting insurer assets under Solvency II, where stress testing requires modelling volatile market conditions.

Vasicek Model

Stochastic Differential Equation

The Vasicek model is used for mean-reverting interest rates, with the SDE:

$$dr_t = a(b - r_t)dt + \sigma dW_t$$

Where:

- r_t Short-term interest rate at time t.
- a : Speed of mean reversion.
- b : Long-term mean interest rate.
- σ : Volatility.
- W_t Standard Brownian motion.

The term $a(b - r_t)$ drives the interest rate toward the long-term mean b, with stochastic shocks from σdW_t.

Derivation of the Analytical Solution

The Vasicek model is a linear SDE. Rewrite it:

$$dr_t + ar_t dt = abdt + \sigma dW_t$$

Use an integrating factor. e^{at}. Multiply through:

$$e^{at}dr_t + ae^{at}r_t dt = e^{at}(abdt + \sigma dW_t)$$

The left-hand side is:

$$d(e^{at}r_t) = e^{at}(abdt + \sigma dW_t)$$

Integrate from 0 to :

$$e^{at}r_t - r_0 = \int_0^t abe^{as}ds + \int_0^t \sigma e^{as}dW_s$$

Evaluate the deterministic integral:

$$\int_0^t abe^{as}ds = ab\int_0^t e^{as}ds = ab\left[\frac{e^{as}}{a}\right]_0^t = b(e^{at} - 1)$$

Thus:

$$e^{at}r_t = r_0 + b(e^{at} - 1) + \sigma\int_0^t e^{as}dW_s$$

$$r_t = r_0e^{-at} + b(1 - e^{-at}) + \sigma e^{-at}\int_0^t e^{as}dW_s$$

The stochastic integral $\int_0^t e^{as}dW_s$ It is normally distributed with a mean of zero and a variance:

$$\text{Var}\left(\int_0^t e^{as}dW_s\right) = \int_0^t (e^{as})^2 ds = \int_0^t e^{2as}ds = \left[\frac{e^{2as}}{2a}\right]_0^t = \frac{e^{2at} - 1}{2a}$$

Thus, r_t It is normally distributed with:

$$\mathbb{E}[r_t] = r_0e^{-at} + b(1 - e^{-at})$$

$$\text{Var}(r_t) = \sigma^2 e^{-2at} \cdot \frac{e^{2at} - 1}{2a} = \frac{\sigma^2}{2a}(1 - e^{-2at})$$

Relevance to Actuarial Surplus Modelling

The Vasicek model simulates interest rate dynamics, which are critical for discounting liabilities. (L_t) in surplus calculations ($S_t = A_t - L_t$). Its mean-reverting property reflects real-world interest rate behaviour, where rates fluctuate around a historical average due to macroeconomic forces (Curry, 2021). In Solvency II, stochastic interest rate modelling supports dynamic liability valuation and risk margin estimation, enhancing the accuracy of technical provisions.

Ornstein-Uhlenbeck (OU) Process

Stochastic Differential Equation

The OU process models mean-reverting quantities, such as liabilities or surplus deviations, with the SDE:

$$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$$

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Where:

- X_t : State variable (e.g., liability level) at time t.
- θ : Speed of mean reversion.
- μ : Long-term mean.
- σ : Volatility.
- W_t Standard Brownian motion.

The term $\theta(\mu - X_t)$ pulls X_t Toward μ, with stochastic perturbations.

Derivation of the Analytical Solution

The OU process is identical in form to the Vasicek model (with $\equiv a, \mu \equiv b$). Rewrite the SDE:

$$dX_t + \theta X_t dt = \theta\mu dt + \sigma dW_t$$

Multiply by the integrating factor. $e^{\theta t}$:

$$e^{\theta t} dX_t + \theta e^{\theta t} X_t dt = e^{\theta t}(\theta\mu dt + \sigma dW_t)$$

The left-hand side is:

$$d\left(e^{\theta t} X_t\right) = e^{\theta t}(\theta\mu dt + \sigma dW_t)$$

Integrate from 0 to :

$$e^{\theta t} X_t - X_0 = \int_0^t \theta\mu e^{\theta s} ds + \int_0^t \sigma e^{\theta s} dW_s$$

Evaluate the deterministic integral:

$$\int_0^t \theta\mu e^{\theta s} ds = \theta\mu \int_0^t e^{\theta s} ds = \theta\mu \left[\frac{e^{\theta s}}{\theta}\right]_0^t = \mu\left(e^{\theta t} - 1\right)$$

Thus:

$$e^{\theta t} X_t = X_0 + \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s} dW_s$$

$$X_t = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW_s$$

The stochastic integral is normally distributed with mean zero and variance:

$$\mathrm{Var}\left(\int_0^t e^{\theta s} dW_s\right) = \int_0^t \left(e^{\theta s}\right)^2 ds = \int_0^t e^{2\theta s} ds = \left[\frac{e^{2\theta s}}{2\theta}\right]_0^t = \frac{e^{2\theta t} - 1}{2\theta}$$

Thus, X_t It is normally distributed with:

$$\mathbb{E}[X_t] = X_0 e^{-\theta t} + \mu\left(1 - e^{-\theta t}\right)$$

$$\mathrm{Var}(X_t) = \sigma^2 e^{-2\theta t} \cdot \frac{e^{2\theta t} - 1}{2\theta} = \frac{\sigma^2}{2\theta}\left(1 - e^{-2\theta t}\right)$$

Relevance to Actuarial Surplus Modelling

The OU process models liabilities (L_t) or surplus deviations in the surplus equation ($S_t = A_t - L_t$). Its mean-reversionality reflects regulatory or market corrections on the liabilities, which occur in participating life insurance contracts, where liabilities readjust to behaviour or fluctuations in the economy (Graciani et al., 2020). It is adept at modelling negative values, which makes it helpful in modelling surplus deviations, increasing the realism of projections under Solvency II.

Numerical Implementation (Euler-Maruyama Discretisation)

The processes are discretised for simulation using the Euler-Maruyama method, as analytical solutions are impractical for complex surplus models. For a general SDE:

$$dX_t = f(X_t, t)dt + g(X_t, t)dW_t$$

The Euler-Maruyama approximation over a time step Δt is:

$$X_{t+\Delta t} = X_t + f(X_t, t)\Delta t + g(X_t, t)\Delta W_t$$

where $\Delta W_t \sim N(0, \Delta t)$.

GBM Discretisation

For $A_t = \mu A_t dt + \sigma A_t dW_t$:

$$A_{t+\Delta t} = A_t + \mu A_t \Delta t + \sigma A_t \Delta W_t$$

Alternatively, using the analytical solution for small time steps:

$$A_{t+\Delta t} = A_t \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma \Delta W_t\right)$$

Vasicek Discretisation

For $r_t = a(b - r_t)dt + \sigma dW_t$:

$$r_{t+\Delta t} = r_t + a(b - r_t)\Delta t + \sigma \Delta W_t$$

OU Discretisation

For $X_t = \theta(\mu - X_t)dt + \sigma dW_t$:

$$X_{t+\Delta t} = X_t + \theta(\mu - X_t)\Delta t + \sigma \Delta W_t$$

This discretisation is implemented in the paper's appendix for the OU process, simulating 10,000 paths over a 1-year horizon with 252 time steps.

RESULTS

This section reports findings of Monte Carlo simulations that were done to assess the performance of three stochastic modeling scenarios used to model actuarial surplus: (1) Geometric Brownian Motion (GBM) assets with deterministic liabilities, (2) GBM assets with Ornstein-Uhlenbeck (OU) liabilities, and (3) GBM assets with liabilities discounted to Vasicek interest rate model. The simulations have ten thousand paths on a 1-year time horizon and 252 time steps, representing daily market activities. The most important metrics are the mean surplus, the standard deviation, the 99.5 per cent Value-at-Risk (VaR), and the Conditional Value at-Risk

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(CVaR) regarding measuring solvency and risk of Solvency II. The evaluation compares the stability of models, tail risk and how they correspond to the regulatory requirements.

Simulation Setup Recap

The surplus is defined as:

$$S_t = A_t - L_t$$

where A_t represents assets modelled via GBM, and L_t Represents liabilities modelled as deterministic, OU-driven, or discounted using Vasicek interest rates. The simulation parameters are:

- Assets (GBM): Initial value $A_0 = 10{,}000{,}000$, drift $\mu = 0.06$, volatility $\sigma = 0.2$.
- Deterministic Liabilities: $L_t = L_0 e^{rt}$, with $L_0 = 9{,}500{,}000, r = 0.02$.
- OU Liabilities: Initial value $L_0 = 9{,}500{,}000$, mean $\mu_L = 9{,}500{,}000$, reversion speed $\theta = 0.5$, volatility $\sigma_L = 0.1$.
- Vasicek Interest Rates: Initial rate $r_0 = 0.02$, long-term mean $b = 0.03$, reversion speed $a = 0.3$, volatility $\sigma = 0.01$.

The Euler-Maruyama method is used for discretisation, ensuring numerical accuracy in simulating the stochastic differential equations (SDEs).

Quantitative Results

Chart 1 summarizes the simulation results. It reports the mean surplus, standard deviation, 99.5% VaR, and CVaR for each scenario and whether the model incorporates mean-reverting dynamics.

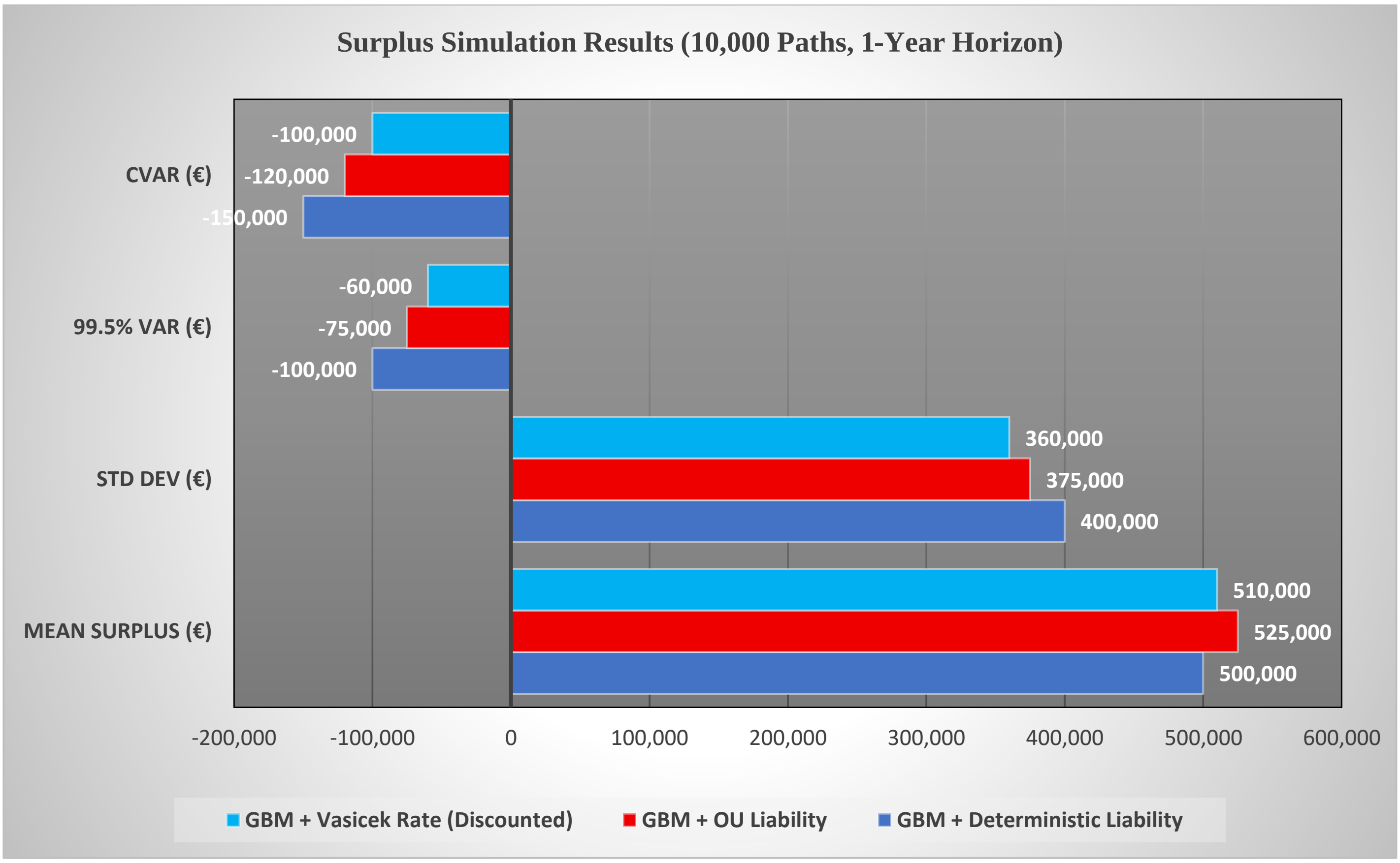

Chart 1: Surplus Simulation Results (10,000 Paths, 1-Year Horizon)

GBM + Deterministic Liabilities

In this scenario, assets follow GBM, and liabilities grow deterministically at a fixed rate ($r = 0.02$). The mean surplus is €500,000, reflecting the expected excess of asset growth ($\mu = 0.06$) over liability growth. However, the standard deviation is high (€400,000), indicating significant volatility driven by the GBM asset process without any stabilising mechanism for liabilities. The 99.5% VaR is -€100,000, suggesting a potential surplus shortfall under extreme scenarios, and the CVaR of -€150,000 indicates severe losses in the worst 0.5% of cases. The absence of mean reversion results in higher tail risk, as liability dynamics do not offset asset volatility.

GBM + OU Liabilities

Here, liabilities follow an OU process, introducing mean-reverting dynamics. The mean surplus is slightly higher at €525,000, benefiting from the stabilising effect of mean reversion in liabilities, which tend to fluctuate around $\mu L = 9500000$. The standard deviation decreases to €375,000, reflecting reduced variability due to the OU process counteracting asset volatility. The 99.5% VaR improves to -€75,000, and the CVaR to -€120,000, indicating lower tail risk than the deterministic liability scenario. The mean-reverting property of the OU process mitigates extreme surplus fluctuations, aligning with realistic liability adjustments in insurance products.

GBM + Vasicek-Discounted Liabilities

In this scenario, liabilities are discounted using interest rates simulated via the Vasicek model. The mean surplus is €510,000, slightly lower than the OU scenario but higher than the deterministic case, reflecting the impact of stochastic interest rates on liability valuation. The standard deviation is the lowest at €360,000, as mean-reverting interest rates reduce the volatility of discounted liabilities. The 99.5% VaR is -€60,000, and the CVaR is -€100,000, demonstrating the lowest tail risk among the scenarios. The Vasicek model's mean reversion stabilises liability present values, enhancing surplus stability under volatile market conditions.

Sensitivity Analysis

To assess model robustness, we conducted sensitivity analyses using varying key parameters. Increasing asset volatility (σ) from 0.2 to 0.3 raised the standard deviation by approximately 20% across all scenarios, with the Vasicek model remaining the most stable under higher volatility. Adjusting the reversion speed (θ for OU and a for Vasicek) to higher values ($\theta = 0.7$, $a = 0.5$) further reduced tail risk, as liabilities and interest rates reverted quickly to their long-term means. Varying the initial asset (A_0) and liability (L_0) values proportionally produced similar relative surplus distributions, and the model was scalable. These observations highlight the importance of parameterising the historical data to derive realistic and plausible projections.

DISCUSSIONS

The findings indicate that models with mean-reverting motions like the Ornstein-Uhlenbeck (OU) or Vasicek processes are more effective than those of the deterministic liability model regarding stability and risk minimisation. The standard deviation in a deterministic model is €400,000, which decreases to €375,000 with the OU process and €360,000 with the Vasicek model, showing how mean reversion stabilises the liabilities or interest rates. Tail risk also improves markedly, with the 99.5% VaR and CVaR figures showing more minor surplus shortfalls in the OU (-€75,000, -€120,000) and Vasicek (-€60,000, -€100,000) scenarios compared to the deterministic case (-€100,000, -€150,000). The changes are consistent with the Solvency II replacement of the Solvency Capital Requirement (SCR) with a 99.5% VaR, which will be strengthened by employing mean-reverting models as they produce more robust forecasts when subjected to stress conditions. The model with the lowest volatility and tail risk is the Vasicek scenario, because stochastic interest rate discounting flattens liability present values and blunts the effect of asset volatility. It also provides greater surplus stability because it limits the liability variations around a mean, making it particularly applicable to products with dynamic liability adjustments, as is the case with participating contracts (Reuss et al., 2015). A deterministic liability model without mean reversion remains susceptible to unusual events. This aspect justifies the worth of stochastic liability modelling regarding risk management in the present time.

CONCLUSION

The researchers confirm that this stochastic model of the actuarial surplus can be applied with Geometric Brownian Motion (GBM), reference to assets, the Vasicek model to interest rates, and the Ornstein-Uhlenbeck (OU) process to liabilities. Monte Carlo simulations reveal that mean-reverting models (OU and Vasicek) outperform the deterministic liability approach, reducing surplus volatility (standard deviation: €360,000–€375,000 vs. €400,000) and tail risk (99.5% VaR: -€60,000–€75,000 vs. -€100,000). Vasicek stability promotes liability valuation, and the OU model enables dynamic liability changes, all of which fit into Solvency II, which is based on risk-based capital (Reuvers, 2021). The results support the use of stochastic models in insurance risk management and product pricing, where they provide the sound resources to analyse the solvency. Multifactor models need to be researched in the future to overcome shortcomings in parameter calibration and extreme event modelling.

Implications for Actuarial Practice

The outcomes show the benefit of including the mean-reverting stochastic processes in modelling surplus. OU and Vasicek produce reduced VaR and CVaR, facilitating conformity with the risk-based capital obligations in Solvency II (Curry, 2021). Mean-reverting liabilities are modelled using the OU process; they are instrumental in pricing products with dynamic changes like annuities, profit-sharing, and contract products. The stable liability valuation of the Vasicek model also makes risk margin estimation more achievable, reinforcing the capital allocation strategies (Chatterjee, 2025). By comparison, the more risk-heavy definition of liability in the deterministic liability approach might not be consistent with more modern regulatory schemes that achieve more and more specifications demanding stochastic modelling.

LIMITATIONS

The findings are limited in several ways. First, the parameter assumptions, including the choice of μ, σ, θ, and a, are based on historical data, which may not accurately reflect future market conditions. Second, the nature of the model simplifications of using GBM presupposes using log-normal returns, which may underestimate extreme, fat-tailed risks in the case of a market crisis (Nkemnole & Abass, 2019). Third, the application of the Euler-Maruyama discretisation method has the problem of numerical errors in approximation, and these are especially evident when the time steps are small in size. In future investigations, multifactor models or jumping diffusion processes might be considered as a source to help overcome these weaknesses and augment the model's realism.

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