Exact Solution of Linear Volterra integro-differential Equation of First Kind Using Abaoub-Shkheam Transform

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Abstract: **We employ Abaoub - Shkheam transformation to solve linear Volterra integro-differential equation of the first kind, we considered the kernel of that equation is a deference type kernel. Moreover, we prove the existence and uniqueness of solutions of the equation under some conditions in the Banach space and fixed-point theory. Finally, some examples are included to demonstrate the validity and applicability of the proposed technique.**

Keywords: **Volterra integro-differential equation, Abaoub-Shkheam transform, Fixed-point method, Contraction mapping, convolution theorem.**

I. INTRODUCTION

In recent years, many researchers have used integro-differential equations which is the combination of differential and integral equations as model of many problems of science and theoretical physics such as engineering, biological models, electrostatics, control theory of industrial [5]. This type of equations was termed as Volterra integro-differential equations, given in the form

$$f(x) = \lambda_1 \int_a^x k_1(x,t)\, u(t)dt + \lambda_2 \int_a^x k_2(x,t) u^{(n)}(t)dt \,, \qquad (1)$$

where $k_2(x,t) \neq 0$, and $u^{(n)}(x) = \frac{d^n u}{dx^n}$, it is necessary to define initial conditions $u^{(m)}(a)$, $0 \leq m \leq n-1$ for the determination of the particular solution $u(x)$ of the linear Volterra integro-differential equation (1).

$$u^{(m)}(a) = a_m (2)$$

There are various numerical and analytical methods to solve such problems, for example, the homotopy perturbation method [4], and the Laplace transform method [4].

We can rewrite Eq. (1) in the integral operator form T as following

$$T(u) = f(x) - \lambda_1 \int_a^x k_1(x,t)\, u(t)dt - \lambda_2 \int_a^x k_2(x,t) u^{(n)}(t)dt \,, \quad (3)$$

The main aim of this paper is to determine the analytical solution of first kind Volterra integro-differential equation using the Abaoub-Shkheam transform.

II. ABAOUB – SHKHEAM TRANSFORM.[1]

The Abaoub-Shkheam transform of the function $f(t)$for all $t \geq 0$, isdefined as

$$Q[f(t)] = \int_0^\infty f(vt) e^{-\frac{t}{s}} dt \qquad (4)$$

Provided the integral exists for some s, where$-t_1 < s < t_2$. Here Q is called the Abaoub-Shkheam transform operator.

The original function $f(t)$ in (4) is called the inverse transform, and is denoted by Q^{-1}.

III. SOME PROPERTIES OF ABOUB-SHKHEAM TRANSFORM.

In this part, we present some properties of the Abaoub-Shkheam transform.

i- $Q[a\, f(t) + b\, g(t)] = a\, Q[f(t)] + b\, Q[g(t)]$, where a and b are constants.

ii- $Q\left[f^{(n)}(t)\right] = \frac{Q[f]}{v^n s^n} - \frac{1}{u} \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{(vs)^{n-k-1}}$

iii- $Q[f(t) * g(t)] = v\, Q[f(t)] Q[g(t)]$.

IV. DEFINITION AND THEOREM [7, 3]

In this section we introduce definition and theorem which are used in this paper

Definition I:

Let $(H, \|.\|)$ be a Banach space, and $T: H \to H$ a bounded operator on H. Then T is said to be a contractive operator if there is a positive constant $\alpha < 1$ such that for each $u, v \in H$ we have

$$\|T(u) - T(v)\| \leq \alpha \|u - v\|.$$

Theorem I:

Let $(H, \|.\|)$ be a Banach space and T: $H \to H$ be a contractive operator on H. Then T has a unique fixed point. Moreover, for any $u_0 \in H$ the sequence of iterates $u_n = T(u_{n-1})$ for $n \geq 1$ converges to a fixed point of T.

V. MAIN RESULT.

In this section, we shall give some results as following:

i. We Applicate the Abaoub-Shkheam transform to solve the linear Volterra integro-differential equation of first kind.

ii. We prove the operator T which defend in (3) as contractions mapping.

iii. We prove the existence and uniqueness of solution of first kind linear Volterra integro-differential equation.

Description the Method

Several methods have been used to solve the linear Volterra integro-differential equation of first kind such as the Laplace transform method; in this section, as a new method, we will use the Abaoub-Shkheam transform for solving the linear Volterra integro-differential equations of first kind (1).

Applying the Abaoub-Shkheam transform of both sides in (1), we have

$$Q[f(x)] = Q\left[\lambda_1 \int_a^x k_1(x-t)u(t)dt + \lambda_2 \int_a^x k_2(x-t)u^{(n)}(t)dt\right]. \quad (5)$$

Using linearity property of the Abaoub-Shkheam transform in (5), we get

$$Q[f(x)] = \lambda_1 Q\left[\int_a^x k_1(x-t)u(t)dt\right] + \lambda_2 Q\left[\int_a^x k_2(x-t)u^{(n)}(t)dt\right]. \quad (6)$$

Using convolution theorem of the Abaoub-Shkheam transform in (6), we obtain

$$Q[f(x)] = v\lambda_1 Q[k_1(x)]Q[u(x)] + v\lambda_2 Q[k_2(x)]Q\left[u^{(n)}(x)\right]. \quad (7)$$

Applying the property "Abaoub-Shkheam transform of derivative of functions" on (7), we get

$$Q[f(x)] = v\lambda_1 Q[k_1(x)]Q[u(x)] + v\lambda_2 Q[k_2(x)]\left[\frac{Q[u(x)]}{v^n s^n} - \frac{1}{v}\left(\frac{u(a)}{(vs)^{n-1}} + \frac{u'(a)}{(vs)^{n-2}} + \frac{u''(a)}{(vs)^{n-3}} + \cdots u^{(n-1)}(a)\right)\right],$$

$$Q[f(x)] = v\lambda_1 Q[k_1(x)]Q[u(x)] + \lambda_2 Q[k_2(x)]\left[\frac{Q[u(x)]}{v^{n-1}s^n} - \frac{u(a)}{(vs)^{n-1}} - \frac{u'(a)}{(vs)^{n-2}} - \frac{u''(a)}{(vs)^{n-3}} - \cdots - u^{(n-1)}(a)\right]. \quad (8)$$

Now using (2) in (8), we get

$$Q[f] = v\lambda_1 Q[k_1]Q[u] + \lambda_2 Q[k_2]\left[\frac{Q[u]}{v^{n-1}s^n} - \frac{a_0}{(vs)^{n-1}} - \frac{a_1}{(vs)^{n-2}} - \frac{a_2}{(vs)^{n-3}} - \cdots - a_{n-1}\right]$$

$$Q[u]\left[v\lambda_1 Q[k_1] + \frac{\lambda_2 Q[k_2]}{v^{n-1}s^n}\right] = Q[f] + \lambda_2 Q[k_2]\left[\frac{a_0}{(vs)^{n-1}} + \frac{a_1}{(vs)^{n-2}} + \frac{a_2}{(vs)^{n-3}} + \cdots + a_{n-1}\right]$$

$$Q[u] = \frac{Q[f] + \lambda_2 Q[k_2]\left[\frac{a_0}{(vs)^{n-1}} + \frac{a_1}{(vs)^{n-2}} + \frac{a_2}{(vs)^{n-3}} + \cdots + a_{n-1}\right]}{\lambda_1 v Q[k_1] + \lambda_2 Q[k_2]v^{1-n}s^{-n}}, \quad (9)$$

where

$$v\lambda_1 Q[k_1] + \lambda_2 Q[k_2]v^{1-n}s^{-n} \neq 0.$$

Now we use the inverse of the Q - transform in (9), we get

$$u(x) = Q^{-1}\left[\frac{Q[f] + \lambda_2 Q[k_2]\left[\frac{a_0}{(vs)^{n-1}} + \frac{a_1}{(vs)^{n-2}} + \frac{a_2}{(vs)^{n-3}} + \cdots + a_{n-1}\right]}{\lambda_1 v Q[k_1] + \lambda_2 Q[k_2]v^{1-n}s^{-n}}\right]. \quad (10)$$

This is the required solution of (1) with initial conditions (2).

The Application of Contraction Mapping

We consider the linear Volterra integro-differential equation of first kind (1) and the operator T which defined in (3).

We note, the solution of equation (1) is the fixed-point of operator T.

Before starting and proving the operator $T$is contractive operator, we introduce the following hypotheses:

(A1) We suppose that$k_1, k_2 \in C([a,b] \times [a,b])$ are continues such that $|k_1(x,t)| \leq M_1$, and$|k_2(x,t)| \leq M_2$.

(A2) There exist a constants $\gamma > 0$ such that, for any $u, v \in C^n[a,b]$

$$\left\|u^{(n)}(x) - v^{(n)}(x)\right\|_\infty \leq \gamma \ \|u - v\|_\infty,$$

we suppose that $u^{(n)}(x)$ is Lipschitz continuous.

(A3) We assume that the function $f(x)$ is continues for all $x \in [a,b]$.

Theorem II

Suppose that (A1) – (A3) holds. If additionally the following inequality

$$|\lambda| < \frac{1}{(M_1 + M_2\gamma)(b-a)}, \qquad (11)$$

where $|\lambda| = \max \{|\lambda_1|, |\lambda_2|\}$, satisfied, then the operator T that defined in equation (3) becomes a contractive.

Proof

Let $u(x), v(x) \in C^n[a,b]$, then

$$\|T(u) - T(v)\|_\infty = \max_{x\in[a,b]}\left|T(u(x)) - T(v(x))\right| =$$

$$= \max_{x\in[a,b]}\left|\lambda_1 \int_a^x k_1(x,t)[u(t) - v(t)]d + \lambda_2 \int_a^x k_2(x,t)\left[u^{(n)}(t) - v^{(n)}(t)\right]dt\right|$$

$$\leq \max_{x\in[a,b]}\left|\lambda_1 \int_a^x k_1(x,t)[u(t) - v(t)]dt\right| + \max_{x\in[a,b]}\left|\lambda_2 \int_a^x k_2(x,t)\left[u^{(n)}(t) - v^{(n)}(t)\right]dt\right|$$

$$\leq |\lambda_1| \max_{x\in[a,b]} \int_a^x |k_1(x,t)||u(t) - v(t)|dt + |\lambda_2| \max_{x\in[a,b]} \int_a^x |k_2(x,t)|\left|u^{(n)}(t) - v^{(n)}(t)\right|dt$$

$$\leq |\lambda_1| \max_{x\in[a,b]} \int_a^x |k_1(x,t)| \max_{x\in[a,b]}|u(t) - v(t)|dt + |\lambda_2| \max_{x\in[a,b]} \int_a^x |k_2(x,t)| \max_{x\in[a,b]}\left|u^{(n)}(t) - v^{(n)}(t)\right|dt$$

$$\leq |\lambda_1| \|u(x) - v(x)\|_\infty \max_{x\in[a,b]} \int_a^x |k_1(x,t)|dt + |\lambda_2| \left\|u^{(n)}(x) - v^{(n)}(x)\right\|_\infty \max_{x\in[a,b]} \int_a^x |k_2(x,t)|dt$$

$$\leq |\lambda_1| M_1(b-a)\|u(x) - v(x)\|_\infty + |\lambda_2| M_2(b-a)\left\|u^{(n)}(x) - v^{(n)}(x)\right\|_\infty$$

By (A2) we obtain

$$\|T(u) - T(v)\|_\infty \leq |\lambda_1| M_1(b-a)\|u(x) - v(x)\|_\infty + |\lambda_2|\gamma\, M_2(b-a)\|u(x) - v(x)\|_\infty$$

$$\|T(u) - T(v)\|_\infty \leq |\lambda|(M_1 + \gamma\, M_2)(b-a)\|u(x) - v(x)\|_\infty$$

where $|\lambda| = max \{|\lambda_1|, |\lambda_2|\}$,then by the condition (11) we get

$$\psi = |\lambda|(M_1 + \gamma\, M_2)(b-a) < 1.$$

Consequently, the operator T of the linear Volterra integro-differential equation becomes acontractive.

Existence and Uniqueness of the Solution

In this section, we shall give an existence and uniqueness of the solution of Eq. (1), with the initial condition (2) and prove it.

To show that (1) has a solution we shall set up an iteration procedure.

Select any initial function $u_0(x) \in C^n[a,b]$, and then construct a sequence $\{u_n(x)\}_{n=0}^{\infty}$defined by

$$u_{n+1}(x) = T(u_n) = f(x) - \lambda_1 \int_a^x k_1(x,t)u_n(t)dt - \lambda_2 \int_a^x k_2(x,t)u_n^{(n)}(t)dt.$$

We shall show that this sequence is a Cauchy sequence, and then that its limit is indeed a solution of (1).

That it has a limit will follow from the fact that a Cauchy sequence must have a unique limit in a Banach space. The limit will be independent of the initial choice $u_0(x)$, since it will be a solution of (1), which must be unique.

Theorem III

Let $(C^n[a,b], \|.\|_\infty)$ be a Banach space and T be a contractive operator of linear Volterra integro-differential equation of first kind defined in (3). Then equation (1) has a unique solution u in$C^n[a,b]$. Such a solution is said to be a fixed point of T.

Proof:

First we prove that T has a unique fixed-point $u \in C^n[a,b]$.By taking the limit of both sides of $u_{n+1}(x) = T(u_n)$

$$\lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty} Tu_n(x)$$

Since T is contraction operator, then it is continuous, so

$$\lim_{n\to\infty} u_{n+1}(x) = T \lim_{n\to\infty} u_n(x).$$

Thus$u = T(u)$. Hence T has a fixed-point.

Now, suppose that v is also a fixed-point of T, that is

$$v = T(v)$$

$$\|u - v\| = \|T(u) - T(v)\| \le \alpha \|u - v\|.$$

Since $0 \le \alpha < 1$, hence

$$\|u - v\| \le \alpha \|u - v\|.$$

This means that $\alpha \ge 1$, which is contradicts our assumption, therefore

$$u = v .$$

Now, we need to prove $\{u_n(x)\}_{n=0}^{\infty}$is a Cauchy sequence.

First we note that

$$\|u_{n+1} - u_n\| = \|T(u_n) - T(u_{n-1})\| \le \alpha \|u_n - u_{n-1}\|$$

By a successive application of the above we have

$$\|u_{n+1} - u_n\| \le \alpha \|u_n - u_{n-1}\| \le \alpha^2 \|u_{n-1} - u_{n-2}\| \le \cdots \le \alpha^n \|u_1 - u_0\|.$$

More generally we have, if $n > m$,

$$\|u_n - u_m\| = \|(u_n - u_{n-1}) + (u_{n-1} - u_{n-2}) + \cdots + (u_{m+1} - u_m)\|$$

$$\le \|u_n - u_{n-1}\| + \|u_{n-1} - u_{n-2}\| + \cdots + \|u_{m-1} - u_m\|$$

$$\le (\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m)\|u_1 - u_0\| = \frac{\alpha^m}{1-\alpha}\|u_1 - u_0\|.$$

Taking the limit of both sides produces

$$\lim_{n,m\to\infty} \|u_n - u_m\| \le \lim_{m\to\infty} \frac{\alpha^m}{1-\alpha}\|u_1 - u_0\| = 0$$

Because 0 ≤ α < 1. Hence

$$\lim_{n,m\to\infty} \|u_n - u_m\| = 0.$$

It follows that $\{u_n(x)\}_{n=0}^{\infty}$ is a Cauchy sequence, and we denote its limit by u .

VI. NUMERICAL PROBLMES

In this section, we will solve some problems to demonstrate the effectiveness of the Abaoub-Shkheam transform for solving linear Volterra integro-differential equations of the first kind.

I. Solve the following linear Volterra integro-differential equation of the first kind

$$\int_0^x (x-t)u(t)dt + \int_0^x (x-t)^2 u'(t)dt = 3x - 3\sin x,$$

with

$$u(0) = 0.$$

Substituting in Eq. (10), we get

$$u(x) = Q^{-1}\left[\frac{Q[3x - 3\sin x]}{[vQ[x] + Q[x^2]s^{-1}]}\right].$$

$$u(x) = Q^{-1}\left[\frac{3Q[x] - 3\,Q[\sin x]}{[vQ[x] + Q[x^2]s^{-1}]}\right].$$

$$u(x) = Q^{-1}\left[\frac{3vs^2 - \frac{3vs^2}{1+v^2s^2}}{[v^2s^2 + 2v^2s^2]}\right].$$

$$u(x) = Q^{-1}\left[\frac{3v^3s^4}{(1+v^2s^2)3v^2s^2}\right].$$

$$u(x) = Q^{-1}\left[\frac{v\ \ s^2}{1+v^2s^2}\right].$$

$$u(x) = \sin x.$$

II. Consider the linear Volterra integro-differential equation of first kind

$$\int_0^x (x-t)u(t)dt + \frac{1}{4}\int_0^x (x-t-1)u''(t)dt = \frac{1}{2}\sin 2x,$$

with the initial condition

$$u(0) = 1, \qquad u'(0) = 0$$

Substituting in Eq. (10), we get

$$u(x) = Q^{-1}\left[\frac{Q\left[\frac{1}{2}\sin 2x\right] + \frac{1}{4}Q[x-1]\left(\frac{1}{vs} + \frac{0}{1}\right)}{\left[vQ[x] + \frac{1}{4}Q[x-1]v^{-1}s^{-2}\right]}\right]$$

$$= Q^{-1}\left[\frac{\frac{1}{2}.\frac{2vs^2}{1+4v^2s^2} + \frac{1}{4vs}(vs^2 - s)}{v^2s^2 + \frac{v}{4}(vs^2 - s)v^{-1}s^{-2}}\right]$$

$$= Q^{-1}\left[\frac{\frac{vs^2}{1+4v^2s^2} + \frac{1}{4vs}(vs^2 - s)}{v^2s^2 + \frac{1}{4}(vs^2 - s)v^{-1}s^{-2}}\right]$$

$$= Q^{-1}\left[\frac{s}{1+4v^2s^2}\right] = \cos 2x$$

VII. CONCLUSION

In this work, we used the Abaoub-Shkheam transform to solve linear Volterra integro-differential equation of first kind successfully, and the existence and uniqueness of solution for that equation is proved and verified, the result is obtained by the help of some extensions of contraction mapping in Banach space. The fixed-point method introduced to solve the mentioned equation. Two examples are presented for illustration and good exact results are found.

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