On the Diophantine Equation $(5^n)^x + (4^m p + 1)^y = z^2$

Vipawadee Moonchaisook[1], Watakarn Moonchaisook[2] and Khattiya Moonchaisook[3]

[1]*Department of Mathematics Faculty of Science and Technology, Surindra Rajabhat University, Surin 3200, Thailand.*
[2]*Computer Technology, Faculty of Agriculture and Technology, Rajamangala University of Technology Isan, Surin campus, Surin 32000, Thailan*
[3]*Science and Mathematics, Faculty of Agriculture and Technology, Rajamangala University of Technology Isan, Surin campus, Surin 32000, Thailand.*

Abstract: **In this paper, we proved that the Diophantine equation $(5^n)^x + (4^m p + 1)^y = z^2$ has no solution in non-negative integers x, y, z where p is an odd prime and m, n is a natural number.**

Keywords: **Diophantine equations, exponential equations, integer solution.**

I. INTRODUCTION

Diophantine equation is one of the significant problems in elementary number theory and algebraic number theory. The Diophantine equation of the type $a^x + b^y = z^2$has been studied by many authorsfor many years. In 2012, Sroysang [16] proved that the Diophantine equation $3^x + 5^y = z^2$ has a unique non-negative integer solution where x, y and z are non-negative integers. The solution (x, y, z) is (1, 0, 2). In the same year, Sroysang [17] proved that the Diophantine equation $31^x + 32^y = z^2$has no non-negative integer solution.

In 2017, Asthana, S., and Singh, M. M. [3] studies the Diophantine Equation $3^x + 13^y = z^2$and proved that thishas exactly four non-negative integer solutions for x, y and z. The solutions are (1, 0, 2), (1, 1, 4), (3, 2, 14) and (5, 1, 16) respectively.In 2018, Kumar et al. [10] studied the non-linear Diophantine equations $61^x + 67^y = z^2$and $67^x + 73^y = z^2$. They proved that these equations have no non-negative integer solution. Additionally, Kumar et al. [11] studied the non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$.They determined that these equations have no non-negative integer solution. In the same year, BurshteinN.[8] examined the solutions to the Diophantine Equation $M^x + (M+6)^y = z^2$ when M = 6N + 5 and M, M + 6 are primes. They proved that this equation has no solutions.

In 2020, Aggarwal et al. [1] examinedtheDiophantine equation $223^x + 241^y = z^2$, where x, y, z are non-negative integers and determined that this equation has no non-negative integer solution.

Moreover, Aggarwal, S. and Sharma, N.[4]

investigated the non-linear Diophantine equation $379^x + 397^y = z^2$.The results showed that the considered non-linear Diophantine equation has no non-negative integer solution.

Apart from the above claims, Aggarwal et al. [5] studied the existence of solution of Diophantine equation $181^x + 199^y = z^2$ and proved that

this $equation$ has no solution.Similarly, Bhatnagar, K. et al. [7], studied the exponential Diophantine equation and proved that $421^p + 439^q = r,^2$ has no solution.In addition, Mishra, R. et al. [14] studied the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$and proved that this equation has no solution in 2020.In the same year, Kumar, S. et al. [13] investigated the exponential Diophantine equation$(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$and found that this equation has no solution.Kumar, S Kumar, S. et al. [12] also examined the exponential Diophantine equation$(7^{2m}) + (6r + 1)^n = z^2$and proved that it has no solution. Moreover, Goel et al., [9] proved that the exponential Diophantine equation $M_5^p + M_7^q = r^2$ has no solution in whole number.

In 2021, Moonchaisook. V., [15] proved that the non-linear Diophantine equation $p^x + (p + 4^n)^y = z^2$ has no solution. Similarly, Aggarwal, S. [2](2021) studied solutions to the exponential Diophantine equation $(2^{2m+1} - 1) + 13^n = z^2$ where m, n are whole numbers and proved that this equation has no solution in whole number.

Aggarwal, S. et al. [6] investigated the exponential Diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$and found no solution in whole number.

Because of this open problem, the author is therefore interested in studying the Diophantine equation; $(5^n)^x + (4^m p + 1)^y = z^2$has no solution in non-negative integers x, y, z where p is an odd prime and m, n is a natural number.

II. PRELIMINARIES

Lemma 1. For every integer n $\geq \mathbf{1}$and M, N are natural number. Then $(4^m p + 1)^n = 4N + 1$

*Proof:*Let $p(n)$ $be\ the\ proposition\ that$

$(4^m p + 1)^n = 4N + 1$ forinteger n $\geq \mathbf{1}$.   (1)

1. P (1) is true. For n = 1, Then$4^m p + 1 = 4^m p + 1$.
2. Show that (1) holds for n = k, Assume (1) holds for n = k,

that is $(4^m p+1)^k = 4N+1$is true. (2)

We considerP(k+1),

$(4^m p+1)^{k+1} = (4^m p+1)(4^m p+1)^k$

$= (4^m p+1)(4N+1)$

$= 4(4^m pN + 4^{m-1}p + N) + 1$

Where $4^m pN + 4^{m-1}p + N$ be natural number.

Hence. Byinduction P(n) is true for integer n $\geq \mathbf{1}$.

*Lemma 2.*The Diophantine equation

$(5^n)^x + 1 = z^2$has no solution in nonnegative integer x, z where p is an odd prime and, n is a natural number.

*Proof:*Suppose that $(5^n)^x + 1 = z^2$

$\rightarrow (5^n)^x = z^2 - 1 = (z-1)(z+1)$.

Thus.we can fine two non-negative integers α and β

Such that $(5^n)^\alpha = z-1$ and $(5^n)^\beta = z+1$ with $\alpha < \beta$ and $\alpha + \beta = x$

Now $(5^n)^\alpha\big((5^n)^{\beta-\alpha} - 1\big) = 2$

This implies $\alpha = 0$ $and (5^n)^{\beta-\alpha} - 1 = 2$

$\rightarrow (5^n)^\beta = 3$ which is impossible.

Hence the Diophantine equation

$(5^n)^x + 1 = z^2$has no solution.

*Lemma 3.*The exponential Diophantine equation$1 + (4^m p+1)^y = z^2$has no solution in nonnegative integer y, zwhere p is an odd prime and, m is a natural number.

*Proof:*Suppose that $1 + (4^m p+1)^y = z^2$

$\rightarrow (4^m p+1)^y = z^2 - 1 = (z-1)(z+1)$.

Thus. we can fine two non-negative integers α and β

Such that $(4^m p+1)^\alpha = z-1$ and

$(4^m p+1)^\beta = z+1$with$\alpha < \beta$ and $\alpha + \beta = y$

Now $(4^m p+1)^\alpha\big((4^m p+1)^{\beta-\alpha} - 1\big) = 2$

This implies $\alpha = 0$,$\beta = 1$ and

$(4^m p+1)^{\beta-\alpha} - 1 = 2$

$\rightarrow 4^m p + 1 = 3$ (m$\geq$ 1)

$\rightarrow 4^m p = 2$ which is impossible.

Hence the Diophantine equation

$1 + (4^m p+1)^y = z^2$has no solution.

III. MAIN THEOREM

*Theorem1.*The Diophantine equation

$(5^n)^x + (4^m p+1)^y = z^2$has no solution in non-negative integer x, y, zwhere p is an odd prime and m, n are natural numbers.

*Proof.*Suppose that $(5^n)^x + (4^m p+1)^y = z^2$

when x, y and $z$are non-negative integers,

m and n are natural number.

we consider 4 cases including x = 0 and $x \geq 1$.

Case 1. Suppose that x = 0, y= 0.

Thus$z^2 = 2$ which is impossible.

*Case 2.*Suppose that x = 0, $y \geq 1$

The Diophantine equation $1 + (4^m p+1)^y = z^2$

has no solution in nonnegative integer y, z where p is an odd prime and m is a natural number.

By lemma 3.

*Case 3*Suppose that$x \geq 1$, and $y = 0$,

The Diophantine equation $(5^n)^x + 1 = z^2$has no solution in nonnegative integer solution x, z where p is an odd prime andn is a natural number.

By lemma 2.

*Case 4.*Suppose that x ≥ 1, y ≥ 1

Sinethe Diophantine equation

$$(5^n)^x + (4^m p+1)^y = z^2$$

(a) If x= 2t and y ≥ 1
(b) If y = 2s and x ≥ 1
(c) If x= 2t+1 and y = 2s+1

(a) If x= 2t (t > 0 integer) and y ≥ 1

Suppose that$(5^n)^x + (4^m p+1)^y = z^2$

$\rightarrow (4^m p+1)^y = z^2 - (5^{nt})^2$

$\rightarrow (4^m p+1)^y (z - 5^{nt})(z + 5^{nt})$

Thus. we can fine two non-negative integers α and β

Such that $(4^m p+1)^\alpha = z - 5^{nt}$ and

$(4^m p+1)^\beta = z + 5^{nt}$ with$\alpha < \beta$ and $\alpha + \beta = y$

$\rightarrow (4^m p+1)^\beta = (4^m p+1)^\alpha + 2(5^{nt})$

This implies $(4^m p+1)|2(5^{nt})$ which is impossible.
Hence,the Diophantine equation

$(5^n)^x + (4^m p+3)^y = z^2$has no solution in nonnegative integer solution x, y, z where p is an odd prime and n is a natural number.

(b) If y = 2s(s > 0 integer) and x ≥ 1

Suppose that$(5^n)^x + (4^m p+1)^y = z^2$

$$\rightarrow\ (5^n)^x = z^2 - (4^m p + 1)^{2s}$$

$\rightarrow\ (5^n)^x = (z - (4^m p + 1)^s)(z + (4^m p + 1)^s)$

Thus. we can fine two non-negative integers α and β

Such that $(5^n)^\alpha = z - (4^m p + 1)^s\ and$

$(5^n)^\beta = z + (4^m p + 1)^s$with $\alpha < \beta$, $\alpha + \beta = x$

$$\rightarrow (5^n)^\beta = (5^n)^\alpha + 2(4^m p + 1)^{2s}$$

This implies $(5^n)|2(4^m p + 1)$ whichis impossible.
Hence,the Diophantine equation

$(5^n)^x + (4^m p + 3)^y = z^2$has no solution in non-negative integer solution x, y, z where p is an odd prime and n is a natural number.

c) If x = 2t+1 (t≥ 0integer) and y = 2s+1 ($s \geq 0$ integer) Suppose that$(5^n)^x + (4^m p + 1)^y = z^2$

$\rightarrow\ (5^n)^x + (4N + 1) = z^2$, by lemma 1.

$\rightarrow\ (5^n)^x + 4N$ = (z+1) (z−1)

Thus. we can fine two non-negative integers α and β

Such that $((5^n)^x + 4N)^\alpha = z - 1\ and$

$((5^n)^x + 4N)^\beta = z + 1$ with$\alpha < \beta$, $\alpha + \beta = 1$

$$\rightarrow ((5^n)^x + 4N)^\alpha \left[((5^n)^x + 4N)^{\beta-\alpha}\right] = 2$$

This implies $\alpha = 0\ and\ \beta = 1$

Thus $(5^n)^x + 4N = 2$ which is impossible.

Hence,the Diophantine equation

$(5^n)^x + (4^m p + 3)^y = z^2$has no solution in nonnegative integer solution x, y, z where p is an odd prime and n is a natural number.

Corollary 1. The Diophantine equation

$(5^n)^x + (4^m p + 1)^y = u^{2n}$

has no solution, in non-negative integer x, y, u and m, n arenatural number.

Proof. Let $u^n = z$then $(5^n)^x + (4^m p + 1)^y = z^2$, which has no solution by Theorem 1.

Corollary 2. The Diophantine equation

$(5^n)^x + (4^m p + 1)^y = u^{2n+2}$ has no solution, in non-negative integer x, y, u and m, narenatural number.

Proof. Let $u^{n+1} = z$

then$(5^n)^x + (4^m p + 1)^y = u^{2n+2} = z^2$, which has no solution by Theorem 1.

IV. CONCLUSION

The main focus of this paper is to study the solvability of the class of Diophantine equation $(5^n)^x + (4^m p + 1)^y = z^2$which p is an odd prime.

Thecase$(5,4p + 1) = (5,13)$ was not considered in this work, but through a brief investigation it might be misunderstood that $5^{2s+1} + 13^{2t} = z^2$ is an even. Thus $z^2 \equiv 0 (\text{mod } 3)$ has a solution when x is an odd number and y is an even number.But if we proved by using theorem1 as stated earlier, we will find that$5^{2s+1} + 13^{2t} = z^2$has nosolution.

However, there are still some further points to be considered. There might be other solutions in solving positive integers that need to be investigated.

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REFERENCES

[1]. Aggarwal, S., Sharma, S.D. and Singhal, H., (2020) On the Diophantine equation $223^x + 241^y = z^2$, International Journal of Research and Innovation in Applied Science, 5 (8), 155-156.

[2]. Aggarwal, S., (2021) On the exponential Diophantine equation $(2^{2m+1} - 1) + 13^n = z^2$.Eng. Appl. Sci. Lett., 4(1), 77-79

[3]. Asthana, S., and Singh, M. M. (2017). On the Diophantine equation $3^x + 13^y = z^2$. International Journal of Pure and Applied Mathematics, 114, 301 - 304.

[4]. Aggarwal, S. and Sharma, N., (2020) On the non-linear Diophantine equation $379^x + 397^y = z^2$, Open Journal of mathematical Sciences, 4(1), 397-399.

[5]. Aggarwal, S., Sharma, S.D. and Vyas, A., (2020). On the existence of solution of Diophantine equation $181^x + 199^y = z^2$, International of Latest Technology in Engineering, Management& Applied Science, 9(8), 85-86.

[6]. Aggarwal, S. and Kumar, S., (2021) On the exponential Diophantine equation $(19^{2m}) + (12\gamma + 1)^n = \rho^2$, International Journal of Research and Innovation in Applied Science, 5(3), 14-16.

[7]. Bhatnagar, K. and Aggarwal, S., (2020) On the exponential Diophantine equation $421^p + 439^q = r,^2$ International Journal of Interdisciplinary Global Studies, 14(4), 128-129.

[8]. Burshtein, N., (2018) On Solutions to the Diophantine Equation$M^x + (M + 6)^y = z^2$when M = 6N + 5, Annals of Pure and Applied Mathematics, 18 (2), 193– 200.

[9]. Goel, P., Bhatnagar, K. and Aggarwal, S., (2020) On the exponential Diophantine equation $M_5^p + M_7^q = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.

[10]. Kumar, S., Gupta, S., & Kishan, H. (2018) On the non-linear Diophantine equations $61^x + 67^y = z^2$and $67^x + 73^y = z^2$. Annals of Pure and Applied Mathematics, 18(1), 91-94.

[11]. Kumar, S., Gupta, D. and Kishan, H.,(2018) On the non-linear Diophantine equations$31^x + 41^y = z^2$and $61^x + 71^y = z^2$. Annals of Pureand Applied Mathematics, 18(2), 185-188.

[12]. Kumar, S., Bhatnagar, K., Kumar, N. and Aggarwal, S., (2020) On the exponential Diophantine equation$(7^{2m}) + (6r + 1)^n = z^2$, International Journal of Interdisciplinary Global Studies, 14(4), 181-182.

[13]. Kumar, S., Bhatnagar, K., Kumar, A. and Aggarwal, S., (2020) On the exponential Diophantine equation$(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$, International Journal of Interdisciplinary Global Studies, 14(4), 183-184.

[14]. Mishra, R., Aggarwal, S. and Kumar, A., (2020) On the existence of solution of Diophantine equation $211^{\alpha} + 229^{\beta} = \gamma^2$, International Journal of Interdisciplinary Global Studies, 14(4), 78-79.
[15]. Moonchaisook, V., (2021) On the Non-Linear Diophantine Equation $p^x + (p + 4^n)^y = z^2$ where p and $p + 4^n$ are primes, Annals of Pure and Applied Mathematics, 23(2), 117-121.
[16]. Sroysang, B., (2012) On the Diophantine equation $31^x + 32^y = z^2$. International Journal of Pure and Applied Mathematics, 81(4), 609-612.
[17]. Sroysang, B., (2012)On the Diophantine equation $3^x + 5^y = z^2$. International Journal of Pure and Applied Mathematics, 81(4), 605-608.