The Weighted Inverse Weibull Distribution

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***Abstract*: This paper introduces the Weighted Inverse Weibull distribution as inverse weighting of the Inverse Weibull distribution. Its various basic statistical properties were explicitly derived and the method of maximum likelihood estimation was used in estimating the model parameters. The model was applied to two real life data sets and its performance and flexibility was assessed with respect to existing distribution using the log-likelihood and Akaike Information Criteria as basis for judgment.**

***Keywords*: Exponential distribution, Generalization, Inversion, Statistical Properties, Weighted distributions, Azzalini.**

I. INTRODUCTION

The exponential distribution has been considered in literature to be effective to analyse lifetime data as a result of its analytical tractability. Although, one-parameter exponential distribution has a lot of interesting properties such as memoryless; one of the major disadvantages of this distribution is that it has a constant hazard function. Moreover, the graph of its probability density function (PDF) is a decreasing function. As a result of this reason several generalizations and weighting of the exponential and Weibull distributions have been developed in the literature. For instance, generalized exponential (GE) distribution as considered by Gupta and Kundu, (2000) is different extension from the exponential distribution. The generalized exponential distribution has increasing or unimodal PDFs, and monotone hazard functions Kanpur, (2015).

Weighted distribution theory gives unified approach to dealing with problem of specifying an appropriate and effective distribution, when the existing distribution is not suitable to capture the entire behaviour of a data set. The concept of weighted distribution was introduced by Fisher (1934) and latter put in unifying form by Rao (1965). Let X denote a non negative continuous random variable with its probability density function $f(x)$, then the probability density function of the weight random variable $f_w(x)$ is given by

$$f_w(x) = \frac{w(x)\, f(x)}{w_D}$$

where $w(x)$ is the weight function and

$$w_D = \int_0^\infty w(x)\, f(x) dx$$

A random variable x is said to have an Inverse Weibull distribution with parameters $\alpha \,\&\, \beta$ if its PDF and CDF are given respectively by:

$$F_{IW}(x);= e^{-\alpha x^{-\beta}} \quad x>0, \alpha>0, \beta>0 \qquad 1$$

$$f_{IW}(x) = \alpha\beta x^{-\beta-1} e^{-\alpha x^{-\beta}}; x>0, \alpha>0, \beta>0 \qquad 2$$

II. THE WEIGHTED INVERSE WEIBULL (WIW) DISTRIBUTION

Let X denote a continuous random variable, considering the weight function $w(x) = x^{-1}$ and the two-parameter Inverse Weibull distribution as given in equation 1 and 2, then the pdf and cdf of the Weighted Inverted Generalized Exponential distribution are:

$$f_w(x) = \frac{\beta \alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)} x^{-(\beta+2)} e^{-\alpha x^{-\beta}} \quad x>0, \alpha>0, \beta>0 \qquad 3$$

and

$$F_w(x) = \frac{\Gamma\left(1+\frac{1}{\beta}, \alpha x^{-\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)} \quad x>0, \alpha>0, \beta>0 \qquad 4$$

and β is a scale parameter and α is the shape parameter

Derivative of WIW Distribution

$$f_w(x)=\frac{w(x)f(x)}{w_D}$$

where $f(x)$ is pdf of IW and $w_D=\int_0^\infty w(x)f(x)dx$

$$w_D=\int_0^\infty \beta\alpha^{\left(1+\frac{1}{\beta}\right)}x^{-(\beta+2)}e^{-\alpha x^{-\beta}}dx \qquad 5$$

Let $t=\alpha x^{-\beta}$ then $x=\alpha^{\frac{1}{\beta}}t^{-\frac{1}{\beta}}$ and $dx=-\alpha^{\frac{1}{\beta}}\beta^{-1}t^{-\frac{1}{\beta}-1}dt$

$w_D=\alpha^{-\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)$ Therefore; $f_w(x)=\frac{w(x)f(x)}{w_D}$

$$f_w(x)=\frac{x^{-1}\beta\alpha x^{-(\beta+1)}e^{-\alpha x^{-\beta}}}{\alpha^{-\frac{1}{\beta}}\Gamma\left(1+\frac{1}{\beta}\right)}$$

$$f_w(x)=\frac{\beta\alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)}x^{-(\beta+2)}e^{-\alpha x^{-\beta}} \qquad 6$$

Equation 6 is the pdf of the Weighted Inverse Weibull distribution.

Its associated CDF is obtained as follows:

$$F(x)=\int_0^x f(y)dy$$

$$F_w(x)=\frac{\beta\alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)}\int_0^x x^{-(\beta+2)}e^{-\alpha x^{-\beta}}dx \qquad 7$$

Let $t=\alpha x^{-\beta}$ then $x=\alpha^{\frac{1}{\beta}}t^{-\frac{1}{\beta}}$ and $dx=-\alpha^{\frac{1}{\beta}}\beta^{-1}t^{-\frac{1}{\beta}-1}dt$

$$F_w(x)=\frac{1}{\Gamma\left(1+\frac{1}{\beta}\right)}\int_{\alpha x^{-\beta}}^{\infty} t^{\frac{1}{\beta}}e^{-t}dt$$

$$F_w(x)=\frac{\Gamma\left(1+\frac{1}{\beta},\alpha x^{-\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)} \qquad 8$$

Proof of validity of WIW Distribution

For the PDF to be valid, it suffices that; $\int_0^\infty f_w(x)dx=1$

$$\frac{\beta\alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)}\int_0^\infty x^{-(\beta+2)}e^{-\alpha x^{-\beta}}dx=1 \qquad 9$$

Let $t=\alpha x^{-\beta}$ then $x=\alpha^{\frac{1}{\beta}}t^{-\frac{1}{\beta}}$ and $dx=-\alpha^{\frac{1}{\beta}}\beta^{-1}t^{-\frac{1}{\beta}-1}dt$

$$\frac{\beta\alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)}\int_0^\infty\left(\alpha^{\frac{1}{\beta}}\mathrm{t}^{-\frac{1}{\beta}}\right)^{-(\beta+2)}e^{-t}(-1)\beta^{-1}\alpha^{\frac{1}{\beta}}\mathrm{t}^{-\frac{1}{\beta}-1}$$

$$\frac{1}{\Gamma\left(1+\frac{1}{\beta}\right)}\int_0^\infty t^{\frac{1}{\beta}}e^{-t}dt \quad \frac{1}{\Gamma\left(1+\frac{1}{\beta}\right)}\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]=1$$

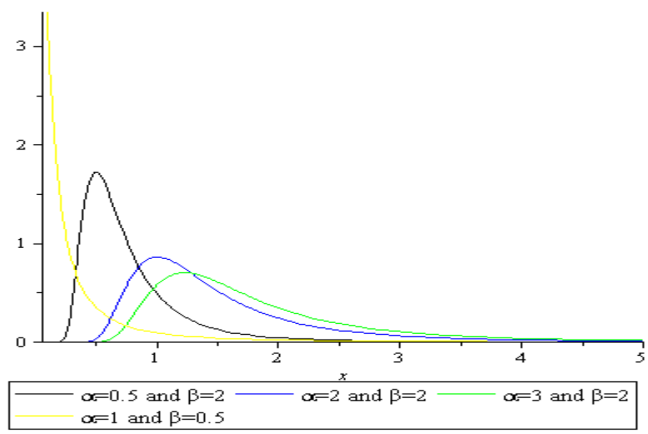

Figure 1: Plot of probability density function of WIW distribution

The plot in Figure 1 show that the shape of the WIW distribution is unimodal (inverted bathtub) and decreasing shapes depending on the value of the shape parameter.

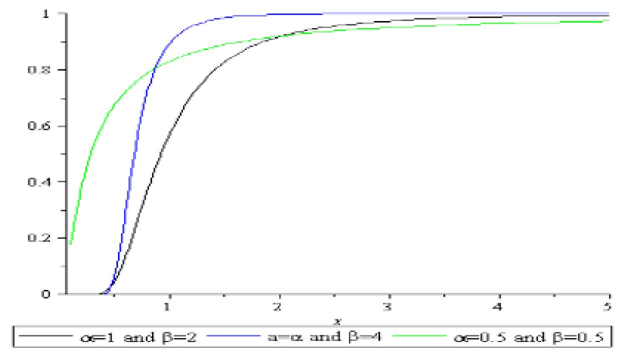

Figure 3: Plot of cumulative density function of WIW distribution

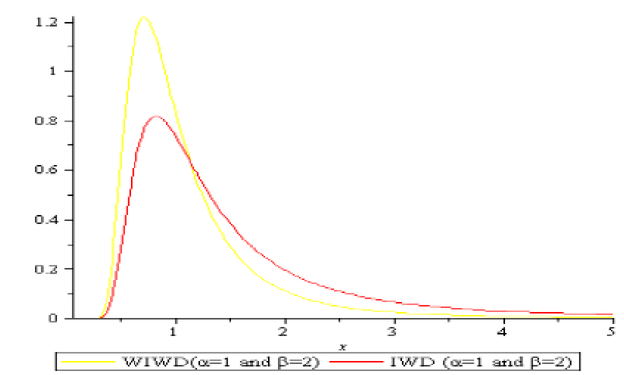

Figure 2: Plot of probability density function of WIW and IW distributions

2.1 Reliability Analysis

Survival Function: The Survival function is given by:

$$S(x) = 1 - F(x)$$

$$S(x) = 1 - \left[\frac{\Gamma\left(1+\frac{1}{\beta}, \alpha x^{-\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)} \right] \quad 10$$

Hazard function: The Hazard function is also given by:

$$h(x) = \frac{\beta \alpha^{\left(1+\frac{1}{\beta}\right)} x^{-(\beta+2)} e^{-\alpha x^{-\beta}}}{\Gamma\left(1+\frac{1}{\beta}\right)\left[1 - \frac{\Gamma\left(1+\frac{1}{\beta}, \alpha x^{-\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right]} \quad 11$$

$$E(X^r) = \frac{\alpha^{\frac{r}{\beta}}}{\Gamma\left(1+\frac{1}{\beta}\right)} \int_0^\infty t^{\left(\frac{1-r}{\beta}\right)} e^{-t} dt$$

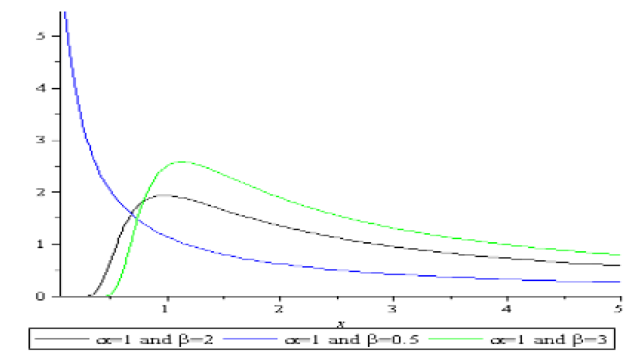

Figure 5: Hazard plot of the Weighted Inverse Weibull Distribution

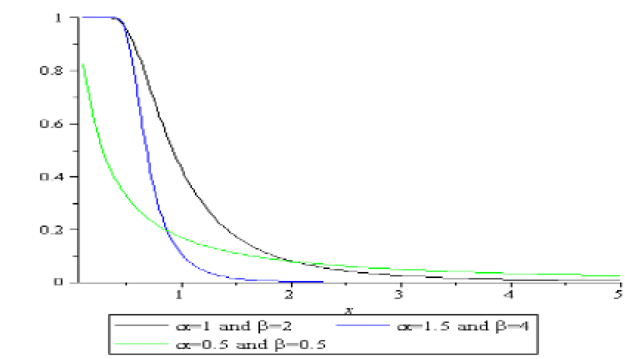

Figure 4: Survival plot of the Weighted Inverse Weibull Distribution

2.2 Moment of WIW Distribution

The moment of distribution is very important, it will help us to determine the mean, dispersion, coefficients of skewness and kurtosis. The kth moments of a non negative random variable X is defined as

$$E(X^r) = \int_0^\infty x^r f(x) dx$$

$$E(X^r) = \int_0^\infty x^r \frac{\beta \alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)} x^{-(\beta+2)} e^{-\alpha x^{-\beta}} dx \quad 12$$

Let $t = \alpha x^{-\beta}$ then $x = \alpha^{\frac{1}{\beta}} t^{-\frac{1}{\beta}}$ and $dx = -\alpha^{\frac{1}{\beta}} \beta^{-1} t^{-\frac{1}{\beta}-1} dt$

$$E(X^r) = \frac{\beta \alpha^{\left(1+\frac{1}{\beta}\right)}}{\Gamma\left(1+\frac{1}{\beta}\right)} \int_0^\infty \left(\alpha^{\frac{1}{\beta}} t^{-\frac{1}{\beta}}\right)^{(r-\beta-2)} e^{-t} (-1) \alpha^{\frac{1}{\beta}} \beta^{-1} t^{-\frac{1}{\beta}-1}$$

$$E\left(X^r\right)=\frac{\alpha^{\frac{r}{\beta}}\Gamma\left(\frac{1+\beta-r}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)} \quad 13$$

The Mean, Variance, Coefficient of Variation (CV),

Coefficient Skewness (CS) and Coefficient Kurtosis (CV) are

derived as follows:

$$\mu=\frac{\alpha^{\frac{1}{\beta}}}{\Gamma\left(1+\frac{1}{\beta}\right)} \quad 14$$

$$\sigma^2=E\left(X^2\right)-(\mu)^2$$

$$\sigma^2=\frac{\alpha^{\frac{2}{\beta}}\left[\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)-1\right]}{\Gamma^2\left(1+\frac{1}{\beta}\right)} \quad 15$$

$$CV=\frac{\sigma}{\mu}$$

$$CV=\left[\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)-1\right]^{1/2} \quad 16$$

$$CS=\frac{E\left(X^3\right)-3\mu\sigma^2-\mu^3}{\sigma^3}$$

$$CS=\frac{\Gamma^2\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-2}{\beta}\right)-3\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)+2}{\left[\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)-1\right]^{3/2}} \quad 17$$

$$CK=\frac{E\left(X^4\right)-4\mu E\left(X^3\right)+6\mu^2\sigma^2+3\mu^4}{\sigma^4}$$

$$CK=\frac{\Gamma^3\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-3}{\beta}\right)-4\Gamma^2\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-2}{\beta}\right)+6\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)-3}{\left[\Gamma\left(1+\frac{1}{\beta}\right)\Gamma\left(\frac{\beta-1}{\beta}\right)-1\right]^2} \quad 18$$

2.3 Moment Generating Function of WIW Distribution

Following (Cordeiro, 2011) the expression for moment generating function is given as

$$M_x(t)=\sum_{r=0}^{n}\left[\frac{t^r}{r!}\frac{\alpha^{\frac{r}{\beta}}\Gamma\left(\frac{1+\beta-r}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right]$$

The moment generating function is the expected value of exponential function of tX, i.e, the moment generating function of random variable X is given as:

$$M_x(t)=E\left(e^{tX}\right)$$

where $E\left(e^{tX}\right)=\int_0^\infty e^{tX}f(x)dx$

with the use of Taylor's series

$$M_x(t)=\int_0^\infty\left(1+\frac{tx}{1!}+\frac{t^2x^2}{2!}+\ldots+\frac{t^rx^r}{r!}\ldots\right)f(x)dx \quad 19$$

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}E\left(X^r\right)$$

and $E\left(X^r\right)$ is defined in 13 above, then

$$M_x(t)=\sum_{r=0}^{n}\left[\frac{t^r}{r!}\frac{\alpha^{\frac{r}{\beta}}\Gamma\left(\frac{1+\beta-r}{\beta}\right)}{\Gamma\left(1+\frac{1}{\beta}\right)}\right] \quad 20$$

2.4 Parameter Estimation of WIW distribution

The Estimation of Weighted Inverse Weibull distribution is obtained using the Method of Maximum Likelihood Estimation (MLE). The formula of MLE contains the unknown parameters of the distribution. The values of these parameters that maximize the sample likelihood are known as the ML estimates (Elgarhy, 2017)

Let $x_1, x_2, ..., x_n$ be a random sample of size "n" from Weighted Inverse Weibull distribution defined in equation (3) and (4), the Likelihood function $L(\tilde{x}/\alpha,\beta)$ is given by

$$L(\tilde{x}/\alpha,\beta)=\prod_{i=1}^{n} f(x_i/\alpha,\beta)$$

Let $l=\log L(\tilde{x}/\alpha,\beta)$

$$l=n\log\beta+n\left(1+\frac{1}{\beta}\right)\log(\alpha)-(\beta+2)\sum_{i=1}^{n}\log(x)-\sum_{i=1}^{n}\alpha x^{-\beta}-n\log\left(\Gamma\left(1+\frac{1}{\beta}\right)\right) \quad 21$$

Differentiating equation (21) with respect to α

$$\frac{dl(\beta)}{d\alpha}=\frac{n\left(1+\frac{1}{\beta}\right)}{\alpha}-\sum_{i=1}^{n}x^{-\beta} \quad 22$$

Differentiating equation (21) with respect to β

$$\frac{dl}{d\beta}=\frac{n}{\beta}-\frac{n\log(\alpha)}{\beta^2}-\sum\log(x)+\alpha\sum\left(x^{-\beta}\log(x)\right)-\frac{n\Psi\left(1+\frac{1}{\beta}\right)}{\beta^2} \quad 23$$

Setting equation (22) and (23) to zero and solving the resulting non-linear equations simultaneously will give the maximum likelihood estimates of parameters α and β .

III. APPLICATION TO DATA SETS

The application to real life data sets of the Weighted Inverse Weibull Distribution is provided. The performance of the WIW distribution was compared with that of existing Inverse Weibull distribution using log-likelihood and Akaike Information Criterion as selection criteria. The distribution that corresponds to the highest log-likelihood value and lowest AIC value is selected as the best for the data set used.

Data Set I: The first data set has been previously used by Lee and Wang (2003).The data represents the remission time (months) of a random sample of 128 bladder cancer patients.

Table 2: Summary of Remission time (Months) of Cancer Patient's

N	mean	Med.	Var.	Skewness	Kurtosis
128	9.365	6.395	110.433	3.286	18.483

3: Analysis of the performance of the competing distributions on Remission time (Months) of Cancer Patients

Models	Estimates	LL	AIC
WIWD	$\hat{\alpha}=6.20806(0.32313)$ $\hat{\beta}=0.44408(0.03869)$	-428.6478	861.2956
IWD	$\hat{\alpha}=2.43039(0.21867)$ $\hat{\beta}=0.75207(0.04243)$	-443.9773	891.9547

Data Set II for WIW & WR Distributions

The second data set has been previously considered by Alqallaf *et al*, (2015).

The data set represents waiting time before being served of 100 bank customers

Table 5: Summary of waiting time before being served of bank Customers

N	Mean	Med.	Var.	Skewness	Kurtosis
100	9.877	8.100	52.3741	1.4727	5.5403

Table 6: Analysis of the performance of the competing distributions

Models	Estimates	LL	AIC
WIWD	$\hat{\alpha}=8.92778\ (0.80856)$ $\hat{\beta}=0.76940(0.07370)$	-327.8677	659.7354
IWD	$\hat{\alpha}=6.53228(0.87686)$ $\hat{\beta}=1.16291(0.0799)$	-334.3810	672.7620

Remark: From Table 3, the WIW distribution has the highest log-likelihood value and the lowest AIC value, therefore, it can be concluded that it fits the data set better than the Inverse Weibull distribution.

IV. CONCLUSION

The Weighted Inverse Weibulll distribution has been successfully derived. The model has unimodal (inverted bathtub) and decreasing shapes (depending on the value of the parameters). Explicit expressions for its basic statistical properties such as reliability analysis, Moment and Moment Generating Function have been successfully derived. The Weighted Inverse Weibull distribution exhibits unimodal and decreasing failure rates, this implies that the distribution will be suitable to describe and model real life phenomena with unimodal or decreasing failure rates. In the real life application considered, the proposed Weighted Inverse Weibull distribution performs better than the existing Inverse

Weibull distribution; hence, it is a good and competitive distribution.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest

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