Implicit Seven Step Simpson's Hybrid Block Second-derivative Method with One Off-Step Point for Solving Second Order Ordinary Differential Equations

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***Abstract*: This paper is concerned with the construction of continuous Seven-Step implicit hybrid block Simpson's Second derivative method for solving initial value problems of second order ordinary differential equations were derived through interpolation and collocation method using maple software. Power series approximation method was used to generate the unknown parameters in the corrector. These Continuous formulations were evaluated at some desired points to give the discrete schemes which constitute the hybrid block method. The constructed block method is consistent, zero-stable and A(α)-Stable. Numerical results obtained using the new block method show that it superior on some system of initial value problems. The study revealed that our new method performed better.**

I. INTRODUCTION

A Considerable Literature exists for the hybrid linear multi-step method for the solution of ordinary differential equation of the form

$$y''(x) = f(x, y, y') \qquad (1.1)$$

$$y(a) = \eta_0 \qquad y'(a) = \eta_1$$

where y satisfies a given set of initial conditions, Ibijola et al (2011), We assume that the function f also satisfies the Lipschitz condition which guarantees existence, uniqueness and continuous differentiable solution, (Jain et al 2014).

For the discrete solution of (1.1) linear multi-step methods has been studied by Lambert (1973). One important advantage of the continuous over the discrete approach is the ability to provide discrete schemes for simultaneous integration. These discrete schemes can as well be reformulated as general linear methods by Butcher (1993). Block method for solving ODEs were first proposed by Milne (1953). The block method are self-starting and can directly be applied to both initial and boundary value problems, Skwame et al (2018) and Donald et al (2009). The block methods show that the proposed block hybrid methods are zero-stable, consistent and A-stable.

In this paper we present Implicit hybrid block Simpson's Second derivatives method with one off-grid point using Onumanyi et al (1994) approach, the derived schemes will be applied in block form in order to achieve its order, error constant and the region of absolute stability.

II. DERIVATION OF THE BLOCK METHOD

Consider an approximate solution to (1.1) in power series of the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \qquad (2.1)$$

Where r and s are the numbers of collocation and interpolation points respectively, a_j's are parameters to be determined. We consider a sequence of points $\{x_n\}$ in the interval $I = [a, b]$ defined by $a = x_0 < x_1 < \ldots < x_\eta = b$, such that $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, \ldots, N-1$.

We now consider the derivation of the multistep collocation method for constant step size h defined for the step $[x_n, x_{n+1}]$ by

$$y(x) = \sum_{j=0}^{s} \alpha_j(x) y_{n+j} + h\sum_{j=0}^{r} \beta_j(x) f_{n+j} + h^2 \sum_{j=0}^{r} \gamma_j(x) g_{n+j} \qquad (2.2)$$

Such that it satisfies the conditions

$$\bar{y}(x_{n+j}) = y_{n+j}, \quad j \in (0, 1, 2, \ldots, s) \qquad (2.3)$$

$$\bar{y}'(\bar{x}_{n+j}) = f_{n+j}, \quad j \in (0, 1, 2, \ldots, r) \qquad (2.4)$$

Where s denotes the number of interpolation points x_{n+j}, $j = 0, 1, 2, \ldots s$ and r

Denotes the number of collocation points $\bar{x}_j$, $f(x_n \ldots, x_{n+1})$, $j = 0, 1, 2, \ldots r$.

The points x_j are chosen from the step x_{n+j} as well as one off grid point from (2.2) the coefficient polynomials are of the form

$$\alpha_j(x) = \sum_{j=0}^{s+r-1} \alpha_{j,j+1} x^j,\ j \in (o,1,2,\ldots s+r-1) \qquad (2.5)$$

$$h\beta_j(x) = h\sum_{j=0}^{s+r-1} \beta_{j,j+1} x^j,\ j \in (o,1,2,\ldots s+r-1) \qquad (2.6)$$

$$h^2\gamma_j(x) = h\sum_{j=0}^{s+r-1} \gamma_{j,j+1} x^j,\ j \in (o,1,2,\ldots s+r-1) \qquad (2.7)$$

To determine $\alpha_j(x), \beta_j(x)$ *and* $\gamma_j(x)$, Sirisena (1997) arrived at a matrix of the form

DC = I (2.8)

Where I is an identity matrix of dimension (s + r) x (s + r) while D and C are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & . & . & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{r+s-1} \\ 0 & 1 & 2x_n & 3x_n^2 & . & . & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 & . \\ . & . & . & . & . & . & 6x_{n+1}^5 & 7x_{n+1}^6 & 8x_{n+1}^7 & 9x_{n+1}^8 & . \\ . & . & . & . & . & . & 6x_{n+2}^5 & 7x_{n+2}^6 & 8x_{n+2}^7 & 9x_{n+2}^8 & . \\ . & . & . & . & . & . & 6x_{n+3}^5 & 7x_{n+3}^6 & 8x_{n+3}^7 & 9x_{n+3}^8 & . \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & 7x_{n+4}^6 & 8x_{n+4}^7 & . & . \\ 0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & 7x_{n+5}^6 & 8x_{n+5}^7 & . & jx_{n+5}^{r+s-1} \\ 0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & 7x_{n+6}^6 & 8x_{n+6}^7 & . & jx_{n+k6}^{r+s-1} \\ 0 & 1 & 2x_{n+7} & 3x_{n+7}^2 & 4x_{n+7}^3 & 5x_{n+7}^4 & 6x_{n+7}^5 & 7x_{n+7}^6 & 8x_{n+7}^7 & . & jx_{n+7}^{r+s-1} \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 2 & . & . & . & . & . & . & . & j(j-1)x_{n+k}^{r+s-2} \end{bmatrix} \qquad (2.9)$$

And

$$C = \begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{s-1,1} & h\beta_{0,1} & . & h\beta_{r-1,1} & h^2\gamma_{0,1} & \cdots & \cdots & h^2\gamma_{r-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{s-1,2} & h\beta_{0,2} & . & h\beta_{r-1,2} & h^2\gamma_{0,2} & \cdots & \cdots & h^2\gamma_{r-1,2} \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,s+r} & \alpha_{1,s+r} & \cdots & \alpha_{s-r,s+r} & h\beta_{0,s+r} & . & h\beta_{r-1,s+r} & h^2\gamma_{0,s+r} & . & . & h^2\gamma_{r-1,s+r} \end{bmatrix} \qquad (2.10)$$

Where $\overline{x_0} = x_n,\ \overline{x_1} = x_{n+1}$

From (2.8), we have that $C = D^{-1}$, where the columns of C give the Continuous coefficients $\alpha_j(x),\ \beta_j(x)$ *and* $\gamma_j(x)$ of the continuous schemes.

The parameters required for equation (2.9) are k = 7, s = 1 and r = k + 2. We propose to use the following off grid.

$$\bar{x}_0 = x_n,\ \bar{x}_1 = x_{n+1}, \bar{x}_2 = x_{n+2}, \bar{x}_3 = x_{n+3},\ \bar{x}_4 = x_{n+4},\ \bar{x}_5 = x_{n+5},$$
$$\bar{x}_6 = x_{n+6},\ \bar{x}_{\frac{13}{2}} = x_{n+\frac{13}{2}},\ \bar{x}_7 = x_{n+7}$$

The matrix (2.9) becomes

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 & 9x_n^8 \\ 0 & 1 & 2x_n+2h & 3(x_n+h)^2 & 4(x_n+h)^3 & 5(x_n+h)^4 & 6(x_n+h)^5 & 7(x_n+h)^6 & 8(x_n+h)^7 & 9(x_n+h)^8 \\ 0 & 1 & 2x_n+4h & 3(x_n+2h)^2 & 4(x_n+2h)^3 & 5(x_n+2h)^4 & 6(x_n+2h)^5 & 7(x_n+2h)^6 & 8(x_n+2h)^7 & 9(x_n+2h)^8 \\ 0 & 1 & 2x_n+6h & 3(x_n+3h)^2 & 4(x_n+3h)^3 & 5(x_n+3h)^4 & 6(x_n+3h)^5 & 7(x_n+3h)^6 & 8(x_n+3h)^7 & 9(x_n+3h)^8 \\ 0 & 1 & 2x_n+8h & 3(x_n+4h)^2 & 4(x_n+4h)^3 & 5(x_n+4h)^4 & 6(x_n+4h)^5 & 7(x_n+4h)^6 & 8(x_n+4h)^7 & 9(x_n+4h)^8 \\ 0 & 1 & 2x_n+10h & 3(x_n+5h)^2 & 4(x_n+5h)^3 & 5(x_n+5h)^4 & 6(x_n+5h)^5 & 7(x_n+5h)^6 & 8(x_n+5h)^7 & 9(x_n+5h)^8 \\ 0 & 1 & 2x_n+12h & 3(x_n+6h)^2 & 4(x_n+6h)^3 & 5(x_n+6h)^4 & 6(x_n+6h)^5 & 7(x_n+6h)^6 & 8(x_n+6h)^7 & 9(x_n+6h)^8 \\ 0 & 1 & 2x_n+13h & 3(x_n+\frac{13}{2}h)^2 & 4(x_n+\frac{13}{2}h)^3 & 5(x_n+\frac{13}{2}h)^4 & 6(x_n+\frac{13}{2}h)^5 & 7(x_n+\frac{13}{2}h)^6 & 8(x_n+\frac{13}{2}h)^7 & 9(x_n+\frac{13}{2}h)^8 \\ 0 & 0 & 2 & 6x_n+42h & 12(x_n+7h)^2 & 20(x_n+7h)^3 & 30(x_n+7h)^4 & 42(x_n+7h)^5 & 56(x_n+7h)^6 & 72(x_n+7h)^7 \end{bmatrix} \qquad (2.11)$$

The inverse of (2.11) is obtained by the use of Maple software to give the values for $\alpha's$ *and* $\beta's$ respectively.Therefore, the discrete hybrid block method is

$$y_{n+1} = y_n + \frac{4444408363}{15166569600}hf_n + \frac{4368780757}{3499977600}hf_{n+1} - \frac{3866321}{2880640}hf_{n+2} + \frac{50424103}{29166480}hf_{n+3} - \frac{5983867297}{3499977600}hf_{n+4} + \frac{499949323}{388886400}hf_{n+5} - \frac{1086080999}{1166659200}hf_{n+6} + \frac{16868608}{39496275}hf_{n+\frac{13}{2}} - \frac{250999}{8333280}h^2g_{n+7}$$

$$y_{n+2} = y_n + \frac{134639923}{473955300}hf_n + \frac{488344268}{300779325}hf_{n+1} - \frac{3151471}{7291620}hf_{n+2} + \frac{2197676}{1822905}hf_{n+3} - \frac{139681417}{109374300}hf_{n+4} + \frac{999784}{1012725}hf_{n+5} - \frac{26451689}{36458100}hf_{n+6} + \frac{62255104}{186196725}hf_{n+\frac{13}{2}} - \frac{6184}{260415}h^2g_{n+7}$$

$$y_{n+3} = y_n + \frac{53534779}{187241600}hf_n + \frac{253146717}{158435200}hf_{n+1} - \frac{35247}{2880640}hf_{n+2} + \frac{701719}{360080}hf_{n+3} - \frac{22147227}{14403200}hf_{n+4} + \frac{16377777}{14403200}hf_{n+5} - \frac{11813687}{14403200}hf_{n+6} + \frac{6051072}{16091075}hf_{n+\frac{13}{2}} - \frac{2727}{102880}h^2g_{n+7}$$

$$y_{n+4} = y_n + \frac{33788054}{118488825}hf_n + \frac{483213736}{300779325}hf_{n+1} - \frac{2824}{67515}hf_{n+2} + \frac{4486432}{1822905}hf_{n+3} - \frac{24624086}{27343575}hf_{n+4} + \frac{3002984}{3038175}hf_{n+5} - \frac{6762832}{9114525}hf_{n+6} + \frac{149061632}{434459025}hf_{n+\frac{13}{2}} - \frac{6368}{260415}h^2g_{n+7}$$

$$y_{n+5} = y_n + \frac{173304035}{606662784}hf_n + \frac{224157725}{139999104}hf_{n+1} - \frac{662125}{46666368}hf_{n+2} + \frac{13778875}{5833296}hf_{n+3} - \frac{44686025}{139999104}hf_{n+4} + \frac{8067905}{5185152}hf_{n+5} - \frac{40088575}{46666368}hf_{n+6} + \frac{262400}{677079}hf_{n+\frac{13}{2}} - \frac{44875}{1666656}h^2g_{n+7}$$

$$y_{n+6} = y_n + \frac{1668739}{5851300}hf_n + \frac{180708}{112525}hf_{n+1} - \frac{3303}{90020}hf_{n+2} + \frac{54668}{22505}hf_{n+3} - \frac{213747}{450100}hf_{n+4} + \frac{245808}{112525}hf_{n+5} - \frac{121697}{450100}hf_{n+6} + \frac{405504}{1462825}hf_{n+\frac{13}{2}} - \frac{72}{3215}h^2g_{n+7}$$

$$y_{n+\frac{13}{2}} = y_n + \frac{85212856469}{298664755200}hf_n + \frac{1437841741193}{895994265600}hf_{n+1} - \frac{1871156339}{59732951040}hf_{n+2} + \frac{18026277347}{7466618880}hf_{n+3} - \frac{398909199923}{895994265600}hf_{n+4} + \frac{70770768029}{33184972800}hf_{n+5} - \frac{14836707899}{298664755200}hf_{n+6} + \frac{8931169}{118229050}hf_{n+\frac{13}{2}} - \frac{52778531}{2133319680}h^2g_{n+7}$$

$$y_{n+7} = y_n + \frac{616787773}{2166652800}hf_n + \frac{8860060097}{5499964800}hf_{n+1} - \frac{72373}{1234560}hf_{n+2} + \frac{10365313}{4166640}hf_{n+3} - \frac{291676567}{499996800}hf_{n+4} + \frac{130179133}{55555200}hf_{n+5} - \frac{66275489}{166665600}hf_{n+6} + \frac{81498368}{62065575}hf_{n+\frac{13}{2}} - \frac{93737}{8333280}h^2g_{n+7} \qquad (2.12)$$

III. STABILITY OF THE BLOCK METHOD

From (2.12), we arrange the block as a matrix finite difference equation of the form

$$A^{(1)}y_{n+1}-A^{(0)}y_n=hB^{(1)}F_n+h^2C'G_{n+j}$$

Where

$$Y_{n+1}=\left[y_{n+1}\ y_{n+2}\ y_{n+3}\ y_{n+4}\ y_{n+5}\ y_{n+6}\ y_{n+\frac{13}{2}}\ y_{n+7}\right]^T$$

$$Y_n=\left[y_{n+7}\ y_{n+\frac{13}{2}}\ y_{n+6}\ y_{n+5}\ y_{n+4}\ y_{n+3}\ y_{n+2}\ y_{n+1}\ y_n\right]^T$$

$$F_n=\left[f_{n+1}\ f_{n+2}\ f_{n+3}\ f_{n+4}\ f_{n+5}\ f_{n+6}\ f_{n+\frac{13}{2}}\ f_{n+7}\right]^T$$

$$G_n=g_{n+7}$$

$$\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\\y_{n+\frac{13}{2}}\\y_{n+7}\end{pmatrix}=\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+7}\\y_{n+\frac{13}{2}}\\y_{n+6}\\y_{n+5}\\y_{n+4}\\y_{n+3}\\y_{n+2}\\y_n\end{pmatrix}$$

$$+h\begin{pmatrix}\frac{4368780757}{3499977600}&-\frac{3866321}{2880640}&\frac{50424103}{29166480}&-\frac{5983867297}{3499977600}&\frac{499949323}{388886400}&-\frac{1086080999}{1166659200}&\frac{16868608}{39496275}&-\frac{250999}{8333280}\\\frac{488344268}{300779325}&-\frac{2880640}{7291620}&\frac{2197676}{1822905}&-\frac{139681417}{109374300}&\frac{999784}{1012725}&-\frac{26451689}{36458100}&\frac{62255104}{186196725}&-\frac{6184}{260415}\\\frac{253146717}{158435200}&\frac{35247}{2880640}&\frac{701719}{360080}&-\frac{22147227}{14403200}&\frac{16377777}{14403200}&-\frac{11813687}{14403200}&\frac{6051072}{16091075}&-\frac{2727}{102880}\\\frac{483213736}{300779325}&-\frac{2824}{67515}&\frac{4486432}{1822905}&-\frac{24624086}{27343575}&\frac{3002984}{3038175}&\frac{6762832}{9114525}&\frac{149061632}{434459025}&-\frac{6368}{260415}\\\frac{224157725}{139999104}&-\frac{662125}{46666368}&\frac{13778875}{5833296}&-\frac{44686025}{139999104}&\frac{8067905}{5185152}&-\frac{40088575}{46666368}&\frac{262400}{677079}&-\frac{44875}{1666656}\\\frac{180708}{112525}&-\frac{3303}{90020}&\frac{54668}{22505}&-\frac{213747}{450100}&\frac{245808}{112525}&-\frac{121697}{450100}&\frac{405504}{1462825}&-\frac{72}{3215}\\\frac{1437841741193}{895994265600}&-\frac{1871156339}{59732951040}&\frac{18026277347}{7466618880}&-\frac{398909199923}{895994265600}&\frac{70770768029}{33184972800}&\frac{14836707899}{298664755200}&\frac{8931169}{18229050}&-\frac{52778531}{2133319680}\\\frac{8860060097}{5499964800}&-\frac{72373}{1234560}&\frac{10365313}{416640}&-\frac{291676567}{499996800}&\frac{130179133}{55555200}&-\frac{66275489}{166665600}&\frac{81498368}{62065575}&\frac{93737}{8333280}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+\frac{13}{2}}\\g_{n+7}\end{pmatrix}$$

$$+h\begin{pmatrix}0&0&0&0&0&0&0&\frac{4444408363}{15166569600}\\0&0&0&0&0&0&0&\frac{134639923}{473955300}\\0&0&0&0&0&0&0&\frac{53534779}{187241600}\\0&0&0&0&0&0&0&\frac{33788054}{118488825}\\0&0&0&0&0&0&0&\frac{173304035}{606662784}\\0&0&0&0&0&0&0&\frac{1668739}{5851300}\\0&0&0&0&0&0&0&\frac{85212856469}{298664755200}\\0&0&0&0&0&0&0&\frac{616787773}{2166652800}\end{pmatrix}\begin{pmatrix}g_{n+7}\\f_{n+\frac{13}{2}}\\f_{n+6}\\f_{n+5}\\f_{n+4}\\f_{n+3}\\f_{n+2}\\f_n\end{pmatrix}\qquad(3.1)$$

3.1 Zero Stability block Method

According to Fatunla (1994), a block method is said to be zero –stable if

$\lambda_{ij}=1,2,\ldots,k$ specified as

$$\rho(\lambda)=\left|\sum_{i=0}^{k}A^{(i)}\lambda^{k=l}\right|=0$$

Satisfies $\rho(\lambda)=\left|\lambda_j\right|\le 1,$ the multiplicity must not exceed two, the block

method (2.12) is expressed in matrix equation form to obtain

$$\rho(R)=\det\left[RA^0-A\right],\ where\ A^0=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}\ and\ A^1=\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}$$

$$\rho(R)=\det\left[R\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{pmatrix}\right]=\det\begin{pmatrix}R&0&0&0&0&0&0&-1\\0&R&0&0&0&0&0&-1\\0&0&R&0&0&0&0&-1\\0&0&0&R&0&0&0&-1\\0&0&0&0&R&0&0&-1\\0&0&0&0&0&R&0&-1\\0&0&0&0&0&0&R&-1\\0&0&0&0&0&0&0&R-1\end{pmatrix}=0$$

$$R_1=0,R_2=0,R_3=0,R_4=0,R_5=0,R_6=0,R_{\frac{13}{2}}=0,R_7=1,$$

Following Henrici (1962), the new block method (2.12) is zero stable and consistent since its order P=8 >1. The method is therefore convergent.

3.2 Convergence Analysis of order and error constant of the block method

Consistency

The block integrator (2.12) is consistent since it has order ρ=8≥1

Convergence

The block integrator is convergent by consequence of Dahlquist theorem below.

Theorem 3.1 The necessary and sufficient conditions that a continuous linear multistep

method to be convergent are that it must be consistent and zero-stable, Dahlquist, G. G (1956).

To determine the order and error constants of the block hybrid method (2.12), we consider the general form of the hybrid method

$$\sum_{j=0}^{k}\alpha_j y_{n+j}=h\sum_{j=0}^{k}\beta_j f_{n+j}+h\beta_v f_{n+v}+h^2\gamma_j g_{n+j}\quad(3.1)$$

Where, $v\in[0,1,\ldots,k]$ which adds further freedom to the block hybrid linear multi-step form.

The block hybrid method which are obtained with the help of maple software have the following order and error constants for each case

Table 1

$$\begin{bmatrix} Evaluating\ point & order & Error\ constant \\ y(x=x_{n+1}) & 8 & -\frac{29529781351}{47999692800} \\ y(x=x_{n+2}) & 8 & -\frac{90942677}{187498800} \\ y(x=x_{n+3}) & 8 & -\frac{35647647}{65843200} \\ y(x=x_{n+4}) & 8 & -\frac{23412151}{46874700} \\ y(x=x_{n+5}) & 8 & -\frac{1055899775}{1919987712} \\ y(x=x_{n+6}) & 8 & -\frac{117649}{257200} \\ y(x=x_{n+\frac{13}{2}}) & 8 & -\frac{6209341393619}{12287921356800} \\ y(x=x_{n+7}) & 8 & \frac{11028064313}{47999692800} \end{bmatrix}$$

The method k=7 second derivative is of order 8 and has error constants

$$C_9 = \left[-\frac{2952978135}{4799969280}, -\frac{90942677}{187498800}, -\frac{35647647}{65843200}, -\frac{23412151}{46874700}, -\frac{1055899775}{1919987712}, -\frac{117649}{257200}, -\frac{6209341393\ 19}{1228792136\ 800}, \frac{1102806433}{4799969280}\right]^T$$

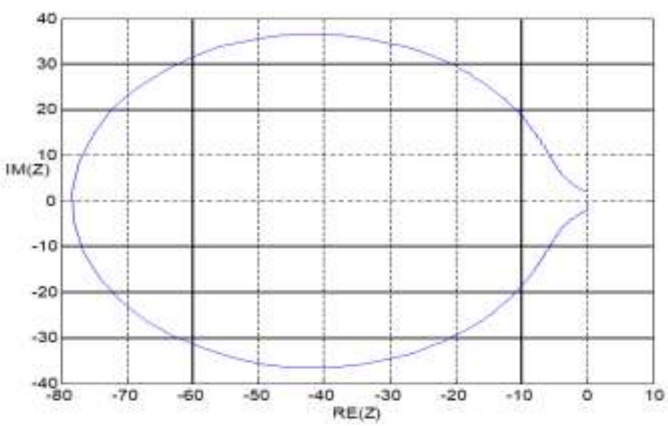
3.3 Region of Absolute Stability of the block method:

The absolute stability region of the block method is obtained using Chollom et al, (2007) and is as shown below:

$$\left(-\frac{1}{72}w^7+\frac{455}{23148}w^8\right)h^8+\left(-\frac{2024033}{5833296}w^8-\frac{6469801}{58332960}w^7\right)h^7+\left(\frac{2445199}{4861080}w^8+\frac{31748723}{29166480}w^7\right)h^6+$$
$$\left(\frac{147743467}{49999680}w^{78}-\frac{1424719}{925920}w^7\right)h^5+\left(-\frac{307967}{92592}w^7+\frac{16661233}{3124980}w^8\right)h^4+\left(\frac{11582545}{1666656}w^8-\frac{78383}{15432}w^7\right)h^3+$$
$$\left(-\frac{40663}{7716}w^7+\frac{108102631}{16666560}w^8\right)h^2+\left(-\frac{30425177}{8333280}w^8-\frac{38891}{11574}w^7\right)h\ -w^7+w^8$$

Using MATLAB software, the absolute stability region of the new method is plotted and shown in fig 1

Figure 1: Region of Absolute Stability of the HBSM (2.12)



IV. NUMERICAL EXPERIMENT

The newly constructed method in equation (2.12) is tested on stiff problems and the results are displayed in tables 2, 3 and 4below.

Example 1:

$$y'1 = -8y_1 + 7y_2$$
$$y'2 = 42y_1 - 43y_2 \qquad where h = \frac{1}{10}, \qquad y_1(0)=1, \qquad y_2(0)=8$$

and Exact Solution

$$y_1(x) = 2e^{-x} - e^{-50x}, \qquad y_2(x) = 2e^{-x} + 6e^{-50x}$$

Example 2:

$$y'_1 = 998y_1 + 1998y_2$$
$$y'_2 = -999y_1 - 1999y_2 \qquad h = 0.1, \qquad y_1(0)=1 \qquad y_2(0)=1$$

Exact Solution

$$y_1(x) = 4e^{-x} - 3e^{-1000x} \qquad y_2(x) = -2e^{-x} + 3e^{-1000x}$$

Example 3:

$$y'_1 = 198y_1 + 199y_2$$
$$y'_2 = -398y_1 - 399y_2 \qquad h = 0.1, \qquad y_1(0)=1 \qquad y_2(0)=-1$$

Exact Solution

$$y_1(x) = e^{-x} \qquad y_2(x) = -e^{-x}$$

Table 2. Absolute stability Errors for Example 1

	HBSM with one off-grid point	
X	**Y_1**	**Y_2**
0.1	$8.978E^{-1}$	$1.791E^{0}$
0.2	$8.185E^{-1}$	$5.882E^{-1}$
0.3	$7.406E^{-1}$	$8.485E^{-1}$
0.4	$6.701E^{-1}$	$5.933E^{-1}$
0.5	$6.063E^{-1}$	$6.834E^{-1}$
0.6	$5.487E^{-1}$	$4.718E^{-1}$
0.7	$4.968E^{-1}$	$4.188E^{-2}$
0.8	$4.494E^{-1}$	$5.144E^{-1}$
0.9	$4.066E^{-1}$	$3.888E^{-1}$
1.0	$3.679E^{-1}$	$3.761E^{-1}$
1.1	$3.329E^{-1}$	$3.269E^{-1}$
1.2	$3.012E^{-1}$	$3.071E^{-1}$

Table 3. Absolute stability Errors for Example 2

	HBSM with one off-grid point	
X	**Y_1**	**Y_2**
0.1	$2.643E^{0}$	$1.141E^{0}$
0.2	$2.323E^{0}$	$1.211E^{0}$
0.3	$2.035E^{0}$	$9.950E^{-1}$
0.4	$1.776E^{0}$	$9.050E^{-1}$
0.5	$1.543E^{0}$	$7.556E^{-1}$
0.6	$1.334E^{0}$	$6.841E^{-1}$
0.7	$1.147E^{0}$	$1.117E^{1}$
0.8	$9.783E^{-1}$	$7.912E^{-1}$
0.9	$8.278E^{-1}$	$7.635E^{-1}$
1.0	$6.928E^{-1}$	$1.837E^{-1}$
1.1	$5.720E^{-1}$	$4.027E^{-1}$
1.2	$4.641E^{-1}$	$1.158E^{-1}$

Table 4. Absolute stability Errors for Example 3

HBSM with one off-grid point		
X	**Y_1**	**Y_2**
0.1	$1.004E^{0}$	$1.011E^{0}$
0.2	$7.956E^{-1}$	$7.933E^{-1}$
0.3	$7.499E^{-1}$	$7.511E^{-1}$
0.4	$6.647E^{-1}$	$6.639E^{-1}$
0.5	$6.113E^{-1}$	$6.122E^{-1}$
0.6	$5.431E^{-1}$	$5.420E^{-1}$
0.7	$5.034E^{-1}$	$5.045E^{-1}$
0.8	$4.896E^{-1}$	$5.094E^{-1}$
0.9	$4.025E^{-1}$	$4.001E^{-1}$
1.0	$3.688E^{-1}$	$3.694E^{-1}$
1.1	$3.325E^{-1}$	$3.322E^{-1}$
1.2	$3.014E^{-1}$	$3.015E^{-1}$

V. CONCLUSION

From the results of this finding the newly constructed hybrid block Simpson's method with one off-grid point was demonstrated on some stiff initial value problems (IVPs). The results are displayed on tables (2, 3 and 4), it can be seen that the HBSM performs efficient and converges very well on examples one and three and performs fairly on example two. Therefore, the newly constructed hybrid block Simpson's method is efficient, accurate and convergent on stiff problems.

In this paper, we have presented Implicit hybrid block Simpson's second derivative method for solving initial value problems of second order ordinary differential equations. The approximate solution adopted in this research produced a block method with A(α)-Stable stability region. This made it performed well on stiff problems. The block hybrid method proposed was found to be zero-stable, consistent and convergent. The new block hybrid method was also found to perform better.

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