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On New Runge-Kutta Second and Third Orders for Solving First Order ODE

Modiu A. MOHAMMED[1]; Johnson F. BAIYERI[2]; Olayinka M. AYENI [3]; Ismaila S. AMUSA [4] and Temitope O. OGUNBAYO [5]

[12345]*Department of Mathematics, Yaba College of Technology, P.M.B. 2011, Yaba, Lagos, Nigeria*

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Abstract:- Runge-Kutta methods are iterative methods for the approximation of solutions of ODE's that were developed around 1900 by the German Mathematicians C. Runge (1856–1927) and M.W. Kutta (1867–1944). Runge-Kutta methods provide a popular way to solve the initial value problem for a system of ordinary differential equations and many Mathematicians have developed these methods in different ways. In this research work, we gave the overview of Runge-Kutta second and third orders in a simplified way and obtained new Runge-Kutta methods for these orders; our new schemes are better than the previous results obtained on the method.

I. Introduction

Runge-Kutta methods are part of the methods used in solving first order initial value problem of the form:

$$\frac{dy}{dx} = f(x, y) \;\; subject\ to \quad y(x_0) = y_0 \tag{1.1.1}$$

The General form of the Runge-Kutta method of "r" order is given as

$$y_{n+1} = y_n + h(c_1k_1 + c_2k_2 + c_3k_3 + \ldots + c_rk_r) = y_n + h\sum_{s=1}^{r} c_sk_s$$

where $k_1 = f(x_n, y_n)$

$$k_2 = f(x_n + a_2h, \; y_n + hb_{21}k_1)$$

$$\vdots \qquad \qquad \vdots$$

$$k_r = f\left(x_n + a_rh, \; y_n + h\sum_{s=1}^{r-1} b_{rs}k_s\right) \qquad r \geq 2 \qquad a_r = \sum_{s=1}^{r-1} b_{rs}$$

II Literature Review

Various Mathematicians have obtained several results on Runge-Kutta methods, some of these results are as follows:

1. Heun's (1900) obtained the following results on the second and third order as follows

Second order method $y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$ (2.2.1)

where

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + h, \; y_n + hk_1)$$

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Third order method $y_{n+1} = y_n + \frac{h}{4}\left(k_1 + 3k_3\right)$ (2.2.2)

where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \tfrac{h}{3},\;\; y_n + \tfrac{h}{3}k_1\right)$$

$$k_3 = f\left(x_n + \tfrac{2}{3}h,\; y_n + \tfrac{2}{3}h\,k_2\right)$$

2. **Ralston's** (1905) also obtained another results but not as good as Heun's results

Second order method $y_{n+1} = y_n + h/3\left(k_1 + 2k_2\right)$ (2.2.3)

where

$$k_1 = f\left(x_n,\, y_n\right)$$

$$k_2 = f\left(x_n + 3/4\, h,\; y_n + 3/4\, hk_1\right)$$

3. **Kutta's**(1901) obtained the following results on the third and forth order as follows

Third order method

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 4k_2 + k_3\right) \quad (2.2.4)$$

where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \tfrac{h}{2},\;\; y_n + \tfrac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + h,\;\; y_n - h\,k_1 + 2h\,k_2\right)$$

Forth order method

$$y_{n+1} = y_n + \frac{h}{8}\left(k_1 + 3k_2 + 3k_3 + k_4\right) \quad (2.2.5)$$

Where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \frac{h}{3},\, y_n + \frac{h}{3}k_1\right)$$

$$k_3 = f\left(x_n + \frac{2h}{3},\, y_n - \frac{h}{3}k_1 + hk_2\right)$$

$$k_4 = f\left(x_n + h,\;\; y_n + hk_1 - hk_2 + h\,k_3\right)$$

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4. **Runge and Kutta** (1901) obtained a better result on the forth order as called the Classical Runge-Kutta Fourth Order Method as follows

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad (2.2.6)$$

Where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}k_2\right)$$

$$k_4 = f\left(x_n + h,\;\; y_n + h\,k_3\right)$$

In 1969, R. England [4] developed another fourth order Runge-Kutta method as follows

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 4k_3 + k_4\right) \qquad (2.2.7)$$

Where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{4}k_1 + \frac{h}{4}k_2\right)$$

$$k_4 = f\left(x_n + h,\;\; y_n - k_2 h + 2h\,k_3\right)$$

Delin Tan and Zheng Chen [1] developed a general formula for the fourth order Runge-Kutta method as follows

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + \left(4 - t\right)k_2 + tk_3 + k_4\right) \qquad (2.2.8)$$

Where

$$k_1 = f(x_n, y_n) = f_n$$

$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + \frac{h}{2}, y_n + \left(\frac{1}{2} - \frac{1}{t}\right)hk_1 + \frac{1}{t}hk_2\right)$$

$$k_4 = f\left(x_n + h, y_n + \left(1 - \frac{t}{2}\right)k_2 h + \frac{1}{3}hk_3\right)$$

III. New Results

In this section, we obtained new Runge-Kutta second and third orders that are better than the above listed ones.

Generally, the Runge-Kutta of "r" order is given as

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$$y_{n+1} = y_n + h\left(c_1k_1 + c_2k_2 + c_3k_3 + \ldots + c_rk_r\right) = y_n + h\sum_{s=1}^{r} c_s k_s \tag{3.0.1}$$

where

$$\left.\begin{aligned} k_1 &= f\left(x_n,\, y_n\right) \\ k_2 &= f\left(x_n + a_2h,\, y_n + hb_{12}k_1\right) \\ &\vdots \qquad\qquad \vdots \\ k_r &= f\left(x_n + a_rh,\, y_n + h\sum_{s=1}^{r-1} b_{rs}k_s\right) \qquad r \geq 2 \\ a_r &= \sum_{s=1}^{r-1} b_{rs} \end{aligned}\right\} \tag{3.0.2}$$

To carry out the analysis of the proofs, the following concepts are important

3.0.1 Taylor's Series of One Variable

The Taylor series of the function $y\left(x+h\right)$ is given as

$$y\left(x+h\right) = y\left(x\right) + hy'\left(x\right) + \frac{h^2}{2!}y''\left(x\right) + \frac{h^3}{3!}y'''\left(x\right) + \ldots + \frac{h^n}{n!}y^{(n)}\left(x\right)$$

$$\Rightarrow y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots + \frac{h^n}{n!}y_n^{(n)} \tag{3.0.3}$$

Since the Runge-Kutta intends to solve an initial/boundary value problem of the form

$$\frac{dy}{dx} = f\left(x,\, y\right) \quad \text{given that} \quad y\left(x_n\right) = y_n \tag{3.0.4}$$

Therefore the last expression of (3.0.3) becomes

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2!}f_n' + \frac{h^3}{3!}f_n'' + \ldots + \frac{h^n}{n!}f_n^{(n-1)} \tag{3.0.5}$$

We shall using (3.0.5) in the analysis of our proof as a Taylor's series.

3.0.2 Taylor's Series of Two Variables

The Taylor's series of function two variables $f\left(x+a, y+b\right)$ at $x = x_n$ and $y = y_n$, becomes

$$f\left(x_n + a,\, y_n + b\right) = f_n + \left(a\frac{\partial f_n}{\partial x} + b\frac{\partial f_n}{\partial y}\right) + \frac{1}{2!}\left(a^2\frac{\partial^2 f_n}{\partial x} + 2ab\frac{\partial^2 f_n}{\partial x \partial y} + b^2\frac{\partial^2 f_n}{\partial y^2}\right) \tag{3.0.6}$$

$$+\frac{1}{3!}\left(a^3\frac{\partial^3 f_n}{\partial x^3} + 3a^2b\frac{\partial^3 f_n}{\partial x^2 \partial y} + 3ab^2\frac{\partial^3 f_n}{\partial x \partial y^2} + b^3\frac{\partial^3 f_n}{\partial y^3}\right) + \ldots$$

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Total Differential

Suppose f is a function of x and y, then the total differential of $f$is given as

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad \Rightarrow \frac{df}{dx} = f'(x,\ y) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{3.0.7}$$

Now at $x = x_n,\ y = y_n$ and $\frac{dy}{dx} = f\left(x_n,\ y_n\right)$, (3.0.7) becomes

$$f'\left(x_n,\ y_n\right) = f_n' = \frac{\partial f_n}{\partial x} + f_n\frac{\partial f_n}{\partial y} = f_x + ff_y = A \tag{3.0.8}$$

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)\frac{dy}{dx}$$

$$f''(x,\ y) = \frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(A)$$

$$\therefore \frac{d^2 f}{dx^2} = f''(x,\ y) = B + Af_y \tag{3.0.9}$$

$$where\ \ B = f_{xx} + 2ff_{xy} + f^2 f_{yy}$$

3.1 Derivation of Second order Runge-Kutta Method

The second order Runge-Kutta method is of the form

$$y_{n+1} = y_n + h\left(c_1 k_1 + c_2 k_2\right) \tag{3.1.1}$$

where

$$k_1 = f\left(x_n,\ y_n\right)$$

$$k_2 = f\left(x_n + a_2 h,\ y_n + hb_{12}k_1\right)$$

Our aim here is to find the exact values of the constants $c_1,\ c_2,\ a_2$ and b_{12}

Now, by equation (3.0.5), we have, by truncating the order higher that h^3 that $y_{n+1} = y_n + hf_n + \frac{h^2}{2!}f_n' + O\left(h^3\right)$

(3.1.2)

where $O\left(h^3\right)$ is error in the method. Substituting equation (3.0.8) into (3.1.1), we have

$$\Rightarrow y_{n+1} = y_n + hf_n + \frac{h^2}{2!}\frac{\partial f_n}{\partial x} + \frac{h^2}{2!}f_n\frac{\partial f_n}{\partial y} + O\left(h^3\right) \tag{3.1.3}$$

Now $k_2 = f\left(x_n + a_2 h,\ y_n + hb_{12}k_1\right)$, expanding in Taylor series using (3.0.6), we have

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$$k_2 = f\left(x_n + a_2 h,\ y_n + h b_{12} k_1\right) = f_n + \left(a_2 h \frac{\partial f_n}{\partial x} + h b_{12} k_1 \frac{\partial f_n}{\partial y}\right) + O\left(h^2\right)$$

Hence Runge-Kutta (3.1.1) becomes

$$y_{n+1} = y_n + h c_1 k_1 + h c_2 k_2 = y_n + \left(c_1 + c_2\right) h f_n + a_2 c_2 \left(h^2 \frac{\partial f_n}{\partial x}\right) + c_2 b_{12} \left(h^2 f_n \frac{\partial f_n}{\partial y}\right) + O\left(h^3\right) \qquad (3.1.4)$$

Now comparing (3.1.3) and (3.1.4), we obtain the following

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ a_2 c_2 &= \frac{1}{2} \\ c_2 b_{21} &= \frac{1}{2} \end{aligned}\right\} \qquad (3.1.5)$$

Here we have three equations and four unknown variables; this means that we must assign a value to one of these unknowns; various Mathematicians have given different values to the constant c_2 which yield different results. We consider the following cases for the assignment of value to the constant c_2.

Case 1: Heun's Method

Let $c_2 = 1/2$, then solving the system (3.1.5), we get $c_1 = 1/2,\ a_2 = 1$ and $b_{12} = 1$

Hence the Runge-Kutta (3.1.1) becomes

$$y_{n+1} = y_n + h/2\left(k_1 + k_2\right) \qquad (3.1.6)$$

Where

$$k_1 = f\left(x_n,\ y_n\right)$$

$$k_2 = f\left(x_n + h,\ y_n + h k_1\right)$$

Case 2: Midpoint's Method

Let $c_2 = 1$, then solving the system (3.1.5), we get $c_1 = 0,\ a_2 = 1/2$ and $b_{12} = 1/2$

Hence the Runge-Kutta (3.1.1) becomes

$$y_{n+1} = y_n + h k_2 \qquad (3.1.7)$$

where

$$k_1 = f\left(x_n,\ y_n\right)$$

$$k_2 = f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} k_1\right)$$

Case 3: Ralston's Method

Let $c_2 = 2/3$, then solving the system (3.1.5), we get $c_1 = 1/3,\ a_2 = 3/4$ and $b_{12} = 3/4$

Hence the Runge-Kutta (3.1.1) becomes

$$y_{n+1} = y_n + h/3\left(k_1 + 2k_2\right) \qquad (3.1.8)$$

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where

$$k_1 = f\left(x_n,\, y_n\right)$$

$$k_2 = f\left(x_n + \tfrac{3}{4}h,\; y_n + \tfrac{3}{4}hk_1\right)$$

Case 4: New Result on second order RK (NRK2)

Let $c_2 = \tfrac{3}{5}$, then solving the system (3.1.5), we get $c_1 = \tfrac{2}{5},\; a_2 = \tfrac{5}{6}$ and $b_{12} = \tfrac{5}{6}$

Hence the Runge-Kutta (3.1.1) becomes

$$y_{n+1} = y_n + \tfrac{h}{5}\left(2k_1 + 3k_2\right) \tag{3.1.9}$$

where

$$k_1 = f\left(x_n,\, y_n\right)$$

$$k_2 = f\left(x_n + \tfrac{5}{6}h,\; y_n + \tfrac{5}{6}hk_1\right)$$

3.2 DERIVATION OF THIRD ORDER RUNGE-KUTTA METHOD

In view of (3.0.1) and (3.0.2), the general form of Runge-Kutta third order method is given as

$$y_{n+1} = y_n + h\left(c_1k_1 + c_2k_2 + c_3k_3\right) \quad \text{or} \quad y_{n+1} - y_n = h\left(c_1k_1 + c_2k_2 + c_3k_3\right) \tag{3.2.1}$$

where

$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + a_2h,\; y_n + ha_2k_1\right)$$

$$k_3 = f\left(x_n + a_3h,\; y_n + h\left(a_3 - b_{32}\right)k_1 + hb_{32}k_2\right)$$

The Taylor's series (3.0.5) can be written as follows

$$y_{n+1} - y_n = h\left(f_n + \frac{h}{2!}f_n' + \frac{h^2}{3!}f_n'' + O\left(h^3\right)\right) \tag{3.2.2}$$

Now, substituting (3.0.8) and (3.0.9) into (3.2.2) and simplifying gives

$$y_{n+1} - y_n = h\left(f_n + hA\left(\frac{1}{2}\right) + h^2Af_y\left(\frac{1}{6}\right) + h^2B\left(\frac{1}{6}\right) + O\left(h^3\right)\right) \tag{3.2.3}$$

We now expand k_2 and k_3 as a Taylor's of two variables using the result (3.0.6)

$$k_2 = f\left(x_n + a_2h,\; y_n + a_2hk_1\right) = f\left(x_n + a_2h,\; y_n + a_2hf_n\right)$$

$$= f_n + a_2h\left(f_x + ff_y\right) + \frac{a_2^2h^2}{2!}\left(f_{xx} + f_nf_{xy} + f^2f_{yy}\right) + O\left(h^3\right)$$

Using (3.0.8) and (3.0.9), we have that

$$k_2 = f_n + a_2hA + \frac{a_2^2h^2B}{2} + O\left(h^3\right) \tag{3.2.4}$$

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$$k_3 = f\left(x_n + a_3 h,\ y_n + h\left(a_3 - b_{32}\right)k_1 + hb_{32}k_2\right) = f\left(x_n + a_3 h,\ y_n + \left(ha_3 f - hb_{32} f\right) + hb_{32}k_2\right)$$

$$k_3 = f_n + a_3 h f_x + \left(ha_3 f - hb_{32} f\right) f_y + hb_{32}k_2 f_y + \frac{a_3^2 h^2}{2!} f_{xx} + \frac{2a_3 h\left[\left(ha_3 f - hb_{32} f\right) + hb_{32}k_2\right]}{2!} f_{xy}$$

$$+ \frac{\left[\left(ha_3 f - hb_{32} f\right) + hb_{32}k_2\right]^2}{2!} f_{yy} + O\left(h^3\right)$$

Any term involving h^3 will be neglected; hence we have by simplifying and rearranging that

$$k_3 = f_n + ha_3\left(f_x + ff_y\right) + a_2 b_{32} h^2 A f_y + \frac{a_3^2 h^2}{2!}\left(f_{xx} + 2ff_{xy} + f^2 f_{yy}\right) + O\left(h^3\right)$$

$$\therefore k_3 = f_n + a_3 hA + a_2 b_{32} h^2 A f_y + \frac{a_3^2 h^2}{2!} B + O\left(h^3\right) \qquad (3.2.5)$$

Substituting (3.2.4) and (3.2.5) into (3.2.1), we obtain

$$y_{n+1} - y_n = h\left\{c_1 f + c_2\left(f_n + a_2 hA + \frac{a_2^2 h^2 B}{2}\right) + c_3\left(f_n + a_3 hA + a_2 b_{32} h^2 A f_y + \frac{a_3^2 h^2}{2} B\right)\right\} + O\left(h^3\right)$$

$$= h\left\{\left(c_1 + c_2 + c_3\right) f_n + \left(c_2 a_2 + c_3 a_3\right) hA + \frac{\left(c_2 a_2^2 + c_3 a_3^2\right)}{2} h^2 B + \left(c_3 a_2 b_{32}\right) h^2 A f_y + O\left(h^3\right)\right\}$$

Now comparing the coefficients of the result to the Taylor's series (3.2.3), we obtain the following

$$c_1 + c_2 + c_3 = 1; \quad \left(c_2 a_2 + c_3 a_3\right) = \frac{1}{2}; \quad \frac{\left(c_2 a_2^2 + c_3 a_3^2\right)}{2} = \frac{1}{6} \quad \text{and} \quad \left(c_3 a_2 b_{32}\right) = \frac{1}{6}.$$

Rearranging and simplifying yields the following systems

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_2 a_2 + c_3 a_3 &= \frac{1}{2} \\ c_2 a_2^2 + c_3 a_3^2 &= \frac{1}{3} \\ c_3 a_2 b_{32} &= \frac{1}{6} \end{aligned}\right\} \qquad (3.2.6)$$

This leads to system of equations; there are four equations with six unknown variables, this means that we have to assign values for two of them (usually c_3 and c_2). Different Mathematicians have assigned different values to these constants (c_3 and c_2), so we consider the following cases

Case 1: Heun's Third Order Method (HRK3)

Let $c_3 = 3/4,\ c_2 = 0\ \therefore c_1 = 1/4$, solving the system (3.2.6) with these values, we have the remaining values as $a_2 = 1/3,\ a_3 = 2/3\ \therefore b_{32} = 2/3$. Hence the Runge-Kutta third order (3.2.1) becomes

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$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_3) \tag{3.2.7}$$

where

$$k_1 = f(x_n, y_n) = f_n$$

$$k_2 = f\left(x_n + \tfrac{h}{3},\ y_n + \tfrac{h}{3}k_1\right)$$

$$k_3 = f\left(x_n + \tfrac{2}{3}h,\ y_n + \tfrac{2}{3}hk_2\right)$$

Case 2: Kutta's Third Order Method (KRK3)

Let $c_3 = 1/6,\ c_2 = 2/3\ \therefore c_1 = 1/6$, solving the system (3.2.6) with these values, we have the remaining values as $a_2 = 1/2,\ a_3 = 1\ \therefore b_{32} = 2$. Hence the Runge-Kutta third order becomes

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_2 + k_3) \tag{3.2.8}$$

where

$$k_1 = f(x_n, y_n) = f_n$$

$$k_2 = f\left(x_n + \tfrac{h}{2},\ y_n + \tfrac{h}{2}k_1\right)$$

$$k_3 = f\left(x_n + h,\ y_n - hk_1 + 2hk_2\right)$$

Case 3: Our New Third Order Method (NRK3-1)

Let $c_3 = 1/4,\ c_2 = 3/4\ \therefore c_1 = 0$, solving the system (3.2.6) with these values, we have the remaining values as $a_2 = 1/3,\ a_3 = 1\ \therefore b_{32} = 2$. Hence the Runge-Kutta third order becomes

$$y_{n+1} = y_n + \frac{h}{4}(3k_2 + k_3) \tag{3.2.9}$$

where

$$k_1 = f(x_n, y_n) = f_n$$

$$k_2 = f\left(x_n + \tfrac{h}{3},\ y_n + \tfrac{h}{3}k_1\right)$$

$$k_3 = f\left(x_n + h,\ y_n - hk_1 + 2hk_2\right)$$

Case 4: Another New Third Order Method (NRK3-2)

Let $c_3 = 3/8,\ c_2 = 3/8\ \therefore c_1 = 1/4$, solving the system (3.2.6) with these values, we have the remaining values as $a_2 = 2/3,\ a_3 = 2/3\ \therefore b_{32} = 2/3$. Hence the Runge-Kutta third order becomes

$$y_{n+1} = y_n + \frac{h}{8}(2k_1 + 3k_2 + 3k_3) \tag{3.2.10}$$

where

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$$k_1 = f\left(x_n,\, y_n\right) = f_n$$

$$k_2 = f\left(x_n + \tfrac{2}{3}h,\;\; y_n + \tfrac{2}{3}hk_1\right)$$

$$k_3 = f\left(x_n + \tfrac{2}{3}h,\; y_n + \tfrac{2}{3}h\,k_2\right)$$

IV Problem Solving for the Comparison of Our Results and the Other Methods

In this section, we will consider some solved problems that will be used to compare the former Runge-Kutta methods and our new methods.

4.1 Problem Solving on Second Order Runge-Kutta

Solved Problem 1: Given the initial value problem $\frac{dy}{dx} = e^{-3x} - 2y$, $y(0) = 0$; find solution of $y(0.2)$ taking $h = 0.1$

Solution: Here $x_0 = 0,\;\; y = 0 \;\; and \;\; h = 0.1$

Case 1: Using Heun's Second order method in (3.1.6)

$$k_1 = f\left(x_0,\, y_0\right) = 1; \qquad k_2 = f\left(x_0 + h,\; y_0 + hk_1\right) = 0.540818$$

$$\therefore y_1(0.1) = y_0 + h/2\left(k_1 + k_2\right) = 0.077041$$

$$k_1 = f\left(x_1,\, y_1\right) = 0.586736; \qquad k_2 = f\left(x_0 + h,\; y_0 + hk_1\right) = 0.277382$$

$$\therefore y_2(0.2) = y_1 + h/2\left(k_1 + k_2\right) = 0.120247$$

Case 2: Midpoint Rule in (3.1.7)

$$k_1 = f\left(x_0,\, y_0\right) = 1 \qquad k_2 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}k_1\right) = 0.760708$$

$$\therefore y_1(0.1) = y_0 + hk_2 = 0.076071$$

$$k_1 = f\left(x_1,\, y_1\right) = 0.588676 \qquad k_2 = f\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2}k_1\right) = 0.426619$$

$$\therefore y_2 = y_1 + hk_2 = 0.118733$$

Case 3: Ralston's Method in (3.1.8)

$$k_1 = f\left(x_0,\, y_0\right) = 1 \qquad k_2 = f\left(x_n + \frac{3h}{4},\, y_n + \frac{3h}{4}k_1\right) = 0.648516$$

$$\therefore y_1(0.1) = y_0 + h/3\left(k_1 + 2k_2\right) = 0.076568$$

$$k_1 = f\left(x_1,\, y_1\right) = 0.587682 \qquad k_2 = f\left(x_1 + \frac{3h}{4},\, y_1 + \frac{3h}{4}k_1\right) = 0.350267$$

$$\therefore y_2(0.2) = y_1 + h/3\left(k_1 + 2k_2\right) = 0.119509$$

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Case 4: Our New Method in (3.1.9) **(NRK2)**

$$k_1 = f(x_0, y_0) = 1 \qquad k_2 = f\left(x_0 + \frac{5h}{6}, y_0 + \frac{5h}{6}k_1\right) = 0.612134$$

$$\therefore y_1(0.1) = y_0 + \frac{0.1}{5}(2k_1 + 3k_2) = 0.076728$$

$$k_1 = f(x_1, y_1) = 0.587362 \qquad k_2 = f\left(x_1 + \frac{5h}{6}, y_1 + \frac{5h}{6}k_1\right) = 0.325600$$

$$\therefore y_2(0.2) = y_1 + \frac{0.1}{5}(2k_1 + 3k_2) = 0.119759$$

Comment: The above solutions show that our result (3.1.9) is better than midpoint rule and the Ralston's method.

4.2 Problem Solving on Third Order Runge-Kutta

Solved problem 2: Solve the initial value problem $\frac{dy}{dx} = 3e^{3x} + 2y$ given that $y(0) = 1$. Hence solve for $0(0.1)0.3$

Solution: Here $f(x, y) = 3e^{3x} + 2y; \quad x_0 = 0, \; y_0 = 1 \quad h = 0.1$

Case 1: Using Heun's Third Scheme in (3.2.7)

$$k_1 = f(x_0, y_0) = 5, \qquad k_2 = f\left(x_0 + \frac{h}{3}, y_0 + \frac{h}{3}k_1\right) = 5.64885,$$

$$k_3 = f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_2\right) = 6.41739 \qquad \therefore y_1(0.1) = y_0 + \frac{0.1}{4}(k_1 + 3k_3) = 1.60630$$

$$k_1 = f(x_1, y_1) = 7.26218, \qquad k_2 = f\left(x_1 + \frac{h}{3}, y_1 + \frac{h}{3}k_1\right) = 8.17222,$$

$$k_3 = f\left(x_1 + \frac{2h}{3}, y_1 + \frac{2h}{3}k_2\right) = 9.24839 \qquad \therefore y_2(0.2) = y_1 + \frac{0.1}{4}(k_1 + 3k_3) = 2.48148$$

$$k_1 = f(x_2, y_2) = 10.42932, \qquad k_2 = f\left(x_2 + \frac{h}{3}, y_2 + \frac{h}{3}k_1\right) = 11.69951,$$

$$k_3 = f\left(x_2 + \frac{2h}{3}, y_2 + \frac{2h}{3}k_2\right) = 13.19952 \qquad \therefore y_3(0.23) = y_2 + \frac{0.1}{4}(k_1 + 3k_3) = 3.73218$$

Case 2: Using Kutta's Third Order Scheme in (3.2.8)

$$k_1 = f(x_0, y_0) = 5, \qquad k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1\right) = 5.9855,$$

$$k_3 = f(x_0 + h, y_0 - hk_1 + 2hk_2) = 7.44378 \qquad \therefore y_1(0.1) = y_0 + \frac{0.1}{6}(k_1 + 4k_2 + k_3) = 1.60643$$

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$$k_1 = f(x_1, y_1) = 7.26244, \qquad k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1\right) = 8.64404,$$

$$k_3 = f(x_1 + h, y_1 - hk_1 + 2hk_2) = 10.68434 \qquad \therefore y_2(0.2) = y_1 + \frac{0.1}{6}(k_1 + 4k_2 + k_3) = 2.48181$$

$$k_1 = f(x_2, y_2) = 10.42998, \qquad k_2 = f\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_1\right) = 12.35762,$$

$$k_3 = f(x_2 + h, y_2 - hk_1 + 2hk_2) = 15.19948 \qquad \therefore y_3(0.3) = y_2 + \frac{0.1}{6}(k_1 + 4k_2 + k_3) = 3.73281$$

Case 3: Using Our New Scheme in (3.2.9)

$$k_1 = f(x_0, y_0) = 5, \qquad k_2 = f\left(x_0 + \frac{h}{3}, y_0 + \frac{h}{3}k_1\right) = 5.64885,$$

$$k_3 = f(x_0 + h, y_0 - hk_1 + 2hk_2) = 7.30912 \qquad \therefore y_1(0.1) = y_0 + \frac{0.1}{4}(3k_2 + k_3) = 1.60640$$

$$k_1 = f(x_1, y_1) = 7.26238, \qquad k_2 = f\left(x_1 + \frac{h}{3}, y_1 + \frac{h}{3}k_1\right) = 8.17243,$$

$$k_3 = f(x_1 + h, y_1 - hk_1 + 2hk_2) = 10.49565 \qquad \therefore y_2(0.2) = y_1 + \frac{0.1}{4}(3k_2 + k_3) = 2.48175$$

$$k_1 = f(x_2, y_2) = 10.42986, \qquad k_2 = f\left(x_2 + \frac{h}{3}, y_2 + \frac{h}{3}k_1\right) = 11.70008,$$

$$k_3 = f(x_2 + h, y_2 - hk_1 + 2hk_2) = 14.93637 \qquad \therefore y_3(0.3) = y_2 + \frac{0.1}{4}(3k_2 + k_3) = 3.73267$$

Case 4: Using Our New Scheme in (3.2.10)

$$k_1 = f(x_0, y_0) = 5, \qquad k_2 = f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_1\right) = 5.33088,$$

$$k_3 = f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_2\right) = 6.50833 \qquad \therefore y_1(0.1) = y_0 + \frac{0.1}{8}(2k_1 + 3k_2 + 3k_3) = 1.60647$$

$$k_1 = f(x_1, y_1) = 7.26252, \qquad k_2 = f\left(x_1 + \frac{2h}{3}, y_1 + \frac{2h}{3}k_1\right) = 9.12744,$$

$$k_3 = f\left(x_1 + \frac{2h}{3}, y_1 + \frac{2h}{3}k_2\right) = 9.37610 \qquad \therefore y_2(0.2) = y_1 + \frac{0.1}{8}(2k_1 + 3k_2 + 3k_3) = 2.48192$$

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$$k_1 = f(x_2, y_2) = 10.4302, \quad k_2 = f\left(x_2 + \frac{2h}{3}, y_2 + \frac{2h}{3}k_1\right) = 13.02116,$$

$$k_3 = f\left(x_2 + \frac{2h}{3}, y_2 + \frac{2h}{3}k_2\right) = 13.37796 \qquad \therefore y_3(0.3) = y_2 + \frac{0.1}{8}(2k_1 + 3k_2 + 3k_3) = 3.73302$$

The actual solution to the initial value problem is given as $y(x) = 3e^{3x} - 2e^{2x}$

$$\therefore y_1(0.1) = 1.60677$$

$$y_2(0.2) = 2.48271$$

$$y_3(0.1) = 3.73457$$

We now make use of the absolute value error formula (to obtain the errors in the solutions for comparison) given as

$$|\in_t|\% = \frac{|Error|}{ActualValue} \times 100\%$$

Table 4.2.1: Comparison of the methods

$y_n(x_n)$	**HRK3** $\|\in_t\|$ **%**	**KRK3** $\|\in_t\|$ **%**	**(NRK3-1)** $\|\in_t\|$ **%**	**(NRK3-2)** $\|\in_t\|$ **%**	**EXACT Solution**
$y_0(0)$	1.00 (0.0%)	1.00 (0.0%)	1.00 (0.0%)	1.00 (0.0%)	1.0000
$y_1(0.1)$	1.60630 $\|\in_t\|$ = 0.0298	1.60643 $\|\in_t\|$ = 0.0212	1.60640 $\|\in_t\|$ = 0.0230	1.60643 $\|\in_t\|$ = 0.0187	1.60677
$y_2(0.2)$	2.48148 $\|\in_t\|$ = 0.0494	2.48181 $\|\in_t\|$ = 0.0363	2.48175 $\|\in_t\|$ = 0.0387	2.48192 $\|\in_t\|$ = 0.0318	2.48271
$y_3(0.3)$	3.73218 $\|\in_t\|$ = 0.0640	3.73281 $\|\in_t\|$ = 0.0471	3.73267 $\|\in_t\|$ = 0.0509	3.73392 $\|\in_t\|$ = 0.0415	3.73457

Solved Problem 3: Consider the initial value problem $\frac{dy}{dx} = x^2 + 2y$ given that $y(0) = 2$, obtain by Runge-Kutta third order methods the solution of $0(0.1)0.2$

Solution: Here $f(x, y) = x^2 + 2y; \quad x_0 = 0, \; y_0 = 2 \quad h = 0.1$

Case 1: Using Heun's Third Scheme

$$k_1 = f(x_0, y_0) = 4, \quad k_2 = f\left(x_0 + \frac{h}{3}, y_0 + \frac{h}{3}k_1\right) = 4.267778,$$

$$k_3 = f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}k_2\right) = 4.573482 \qquad \therefore y_1(0.1) = y_0 + \frac{0.1}{4}(k_1 + 3k_3) = 2.443011$$

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$$k_1 = f\left(x_1,\, y_1\right) = 4.896022, \qquad k_2 = f\left(x_1 + \frac{h}{3},\, y_1 + \frac{h}{3}k_1\right) = 5.230201,$$

$$k_3 = f\left(x_1 + \frac{2h}{3},\, y_1 + \frac{2h}{3}k_2\right) = 5.611160 \qquad \therefore y_2\left(0.2\right) = y_1 + \frac{0.1}{4}\left(k_1 + 3k_3\right) = 2.986249$$

Case 2: Using Kutta's Third Order Scheme

$$k_1 = f\left(x_0,\, y_0\right) = 4, \qquad k_2 = f\left(x_0 + \frac{h}{2},\, y_0 + \frac{h}{2}k_1\right) = 4.4025,$$

$$k_3 = f\left(x_0 + h,\, y_0 - hk_1 + 2hk_2\right) = 4.971 \qquad \therefore y_1\left(0.1\right) = y_0 + \frac{0.1}{6}\left(k_1 + 4k_2 + k_3\right) = 2.443017$$

$$k_1 = f\left(x_1,\, y_1\right) = 4.896034, \qquad k_2 = f\left(x_1 + \frac{h}{2},\, y_1 + \frac{h}{2}k_1\right) = 5.398137,$$

$$k_3 = f\left(x_1 + h,\, y_1 - hk_1 + 2hk_2\right) = 6.106082 \qquad \therefore y_2\left(0.2\right) = y_1 + \frac{0.1}{6}\left(k_1 + 4k_2 + k_3\right) = 2.986261$$

Case 3: Using Our New Scheme

$$k_1 = f\left(x_0,\, y_0\right) = 4, \qquad k_2 = f\left(x_0 + \frac{h}{3},\, y_0 + \frac{h}{3}k_1\right) = 4.267778,$$

$$k_3 = f\left(x_0 + h,\, y_0 - hk_1 + 2hk_2\right) = 4.917111 \qquad \therefore y_1\left(0.1\right) = y_0 + \frac{0.1}{4}\left(3k_2 + k_3\right) = 2.443011$$

$$k_1 = f\left(x_1,\, y_1\right) = 4.896022, \qquad k_2 = f\left(x_1 + \frac{h}{3},\, y_1 + \frac{h}{3}k_1\right) = 5.230101,$$

$$k_3 = f\left(x_1 + h,\, y_1 - hk_1 + 2hk_2\right) = 6.038898 \qquad \therefore y_2\left(0.2\right) = y_1 + \frac{0.1}{4}\left(3k_2 + k_3\right) = 2.986249$$

Case 4: Using Our New Scheme

$$k_1 = f\left(x_0,\, y_0\right) = 4, \qquad k_2 = f\left(x_0 + \frac{2h}{3},\, y_0 + \frac{2h}{3}k_1\right) = 4.537778,$$

$$k_3 = f\left(x_0 + \frac{2h}{3},\, y_0 + \frac{2h}{3}k_2\right) = 4.609482 \qquad \therefore y_1\left(0.1\right) = y_0 + \frac{0.1}{8}\left(2k_1 + 3k_2 + 3k_3\right) = 2.443022$$

$$k_1 = f\left(x_1,\, y_1\right) = 4.896044, \qquad k_2 = f\left(x_1 + \frac{2h}{3},\, y_1 + \frac{2h}{3}k_1\right) = 5.566628,$$

$$k_3 = f\left(x_1 + \frac{2h}{3},\, y_1 + \frac{2h}{3}k_2\right) = 5.656039 \qquad \therefore y_2\left(0.2\right) = y_1 + \frac{0.1}{8}\left(2k_1 + 3k_2 + 3k_3\right) = 2.986273$$

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The actual solution to the initial value problem is given as $y(x)=\frac{9}{4}e^{3x}-\frac{x^2}{2}-\frac{x}{2}-\frac{1}{4}$

$$\therefore y_1(0.1)=2.443156 \quad \text{and} \quad y_2(0.2)=2.986606$$

Table 4.2.2: Comparison of the methods

$y_n(x_n)$	**HRK3** $\lvert\in_t\rvert$ **%**	**KRK3** $\lvert\in_t\rvert$ **%**	**(NRK3-1)** $\lvert\in_t\rvert$ **%**	**(NRK3-2)** $\lvert\in_t\rvert$ **%**	**EXACT Solution**
$y_0(0)$	1.00 (0.0%)	1.00 (0.0%)	1.00 (0.0%)	1.00 (0.0%)	1.0000
$y_1(0.1)$	2.443011 $\lvert\in_t\rvert$ = 5.935 x 10^{-3}	2.443017 $\lvert\in_t\rvert$ = 5.69 x 10^{-3}	2.443011 $\lvert\in_t\rvert$ = 5.94 x 10^{-3}	2.443022 $\lvert\in_t\rvert$ = 5.49 x 10^{-3}	2.443156
$y_2(0.2)$	2.986249 $\lvert\in_t\rvert$ = 0.0120	2.986261 $\lvert\in_t\rvert$ = 0.0116	2.986249 $\lvert\in_t\rvert$ = 0.0120	2.986273 $\lvert\in_t\rvert$ = 0.0112	2.986606

4.3 Discussion of Results

Discussion

1. Table 4.2.1 shows that our new scheme (3.2.9) has less error that Heun's scheme thereby making our scheme better than the Heun's. Also our new scheme (3.2.10) has less error than Heun's and Kutta's schemes, we conclude that our new scheme is better than all the Runge-kutta third order methods. To clearly be convinced about our new third order scheme, we consider another solved problem.
2. Table 4.2.2 also shows that our new scheme (3.2.10) has less error than Heun's and Kutta's schemes, we conclude that our new scheme is better than all the Runge-kutta third order method

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