On New Probabilistic Hermite Polynomials

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Abstract: - In the theory of differential equation and probability, Probabilistic Hermite polynomials $H_r(x)\ \{r=0,1,2,...,n\}$ are the polynomials obtained from derivatives of the standard normal probability density function (pdf) of the form $\alpha(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$. These polynomials played an important role in the Gram-Charlier series expansion of type *A* and the Edgeworth's form of the type *A* series (see [18]).

In this paper, we obtained new Probabilistic Hermite polynomials by considering a standard normal distribution with probability density function (pdf) given as $\beta(x)=\frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}x^2}$. The generating function, recurrence relations and orthogonality properties are studied. Finally, a differential equation governing these polynomials was presented which enables us to obtain the expression of the polynomial in a closed form.

Keywords: Generating Function, recurrence relation, differential equation, Power series, Orthogonality.

I. Introduction and Preliminary

Special functions and polynomials are solutions of special differential equations; they appear in mathematics, statistics, Lie group theory, and number theory. Probabilistic Hermite polynomials $H_r(x)$ are also special polynomials that occur in the theory of advanced statistics and are given by the series expansion

$$H_r(x)=x^r-\frac{r^{[2]}}{2.1!}x^{r-2}+\frac{r^{[4]}}{2^2.2!}x^{r-4}-\frac{r^{[6]}}{2^3.3!}x^{r-6}+... \tag{1.1}$$

The polynomials are obtained by the series of differentiations of the standard normal distribution function of the form

$$\alpha(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2} \tag{1.2}$$

The generating function of the polynomial is given as

$$\sum_{r=0}^{\infty}\frac{H_r(x)}{n!}t^r=e^{\left(tx-\frac{1}{2}t^2\right)} \tag{1.3}$$

and the polynomials satisfy the second order differential equation

$$y''-xy'+ry=0 \tag{1.4}$$

[18] gave the following results on the Probabilistic Hermite polynomials

$$H_0(x)=1H_1(x)=x$$
$$H_2(x)=x^2-2H_3(x)=x^3-3x$$
$$H_4(x)=x^4-6x^2+3H_5(x)=x^5-10x^3+15x$$
$$H_6(x)=x^6-15x^4+45x^2-15H_7(x)=x^7-21x^5+105x^3-105x$$
$$H_8(x)=x^8-28x^6+210x^4-420x^2+105$$

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CF. Charlier (1931) proved that the equation in x, $H_r(x) = 0$, has r real roots, each not greater in absolute value than $\sqrt{\tfrac{1}{C}\, r(r-1)}$ [18].

J.P. Gram (1879) considered a series of the form

$$f(x) = \sum_{n=0}^{\infty} c_r H_r(x) \alpha(x) \tag{1.5}$$

where $c_r = \int_{-\infty}^{\infty} f(x) H_r(x) dx$

This series (1.5) is called Gram-Charlier series of type A, although it appeared in the work of P. L. Chebyshev and L. H. F. Oppermann before that of J.P. Gram in 1879, Thiele (1903) and CF Charlier (1931) [18].

II. Main Results

In this section, the results on generating function, recurrence relations for the new Probabilistic Hermite polynomials ($H^*_n(x)$) and their proofs will be presented. The special differential equation governing these polynomials will also be obtained.

2.0.1 Generating Function

As for other special differential equations, generating functions are used to generate the polynomials in series form and are also used to deduce the recurrence relations. We now start by considering the standard normal distribution of the form

$$\beta(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}x^2} \tag{2.1}$$

On differentiating (2.1) severally, we have

$$\beta'(x) = \frac{-1}{2\sqrt{\pi}} \frac{x}{2} e^{-\frac{1}{4}x^2}, \quad \beta''(x) = \frac{(-1)^2}{2\sqrt{\pi}} \frac{(x^2-2)}{2^2} e^{-\frac{1}{4}x^2}, \quad \beta'''(x) = \frac{(-1)^3}{2\sqrt{\pi}} \frac{(x^3-6x)}{2^3} e^{-\frac{1}{4}x^2}$$

$$\vdots$$

$$D^n \beta(x) = \frac{(-1)^n}{2\sqrt{\pi}} \frac{A_n(x)}{2^n} e^{-\frac{1}{4}x^2} \quad or \quad D^n \beta(x) = \frac{(-1)^n A_n(x) \beta(x)}{2^n} \tag{1}$$

where $H^*_n(x)$ represents the Probabilistic Hermite polynomials. Evidently $H^*_n(x)$ is a polynomial of degree n in x and the coefficient of x^n is one.

Considering also

$$\beta(x-t) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}(x-t)^2} = \beta(x) e^{-\frac{1}{4}(t^2-2xt)} \tag{2}$$

By using Taylor's series of function of two variables, $\beta(x-t)$ can also be written as

$$\beta(x-t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n D^n \beta(x)}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \frac{(-1)^n H^*_n(x)\beta(x)}{2^n} = \sum_{n=0}^{\infty} \frac{t^n H^*_n(x)\beta(x)}{2^n n!} \tag{3}$$

Substituting (1) and (2) in (3) and simplifying, yields

$$\sum_{n=0}^{\infty} \frac{t^n H^*_n(x)}{2^n n!} = e^{-\frac{1}{4}(t^2-2xt)} \tag{2.2}$$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VII July 2023

(2.2) is called the generating function of the polynomial $H^*_n(x)$.

2.0.2 Recurrence Relations for $H^*_n(x)$

Differentiating (2.2) with respect to *x*, we obtain

$$\sum_{n=0}^{\infty}\frac{t^n H^{*\prime}_n(x)}{2^n n!}=\frac{t}{2}e^{-\frac{1}{4}(t^2-2xt)}=\sum_{n=0}^{\infty}\frac{t^{n+1}H^*_n(x)}{2^{n+1}n!}=\sum_{n=0}^{\infty}\frac{t^n nH^*_{n-1}(x)}{2^n(n-1)!}$$

Equating the coefficients of t^n from both sides yields

$$H^{*\prime}_n(x)=nH^*_{n-1}(x) \quad\text{and}\quad H^{*\prime\prime}_n(x)=n(n-1)H^*_{n-2}(x) \tag{2.3}$$

Also, differentiating (2.2) *w.r.t t* yields

$$\sum_{n=0}^{\infty}\frac{nt^{n-1}H^*_n(x)}{2^n n!}=\frac{1}{2}(x-t)e^{-\frac{1}{4}(t^2-2xt)}=\frac{1}{2}(x-t)\sum_{n=0}^{\infty}\frac{t^n H^*_n(x)}{2^n n!}=\sum_{n=0}^{\infty}\frac{xt^n H^*_n(x)}{2^{n+1}n!}-\sum_{n=0}^{\infty}\frac{t^{n+1}H^*_n(x)}{2^{n+1}n!}$$

i.e. $$\sum_{n=0}^{\infty}\frac{nt^{n-1}H^*_n(x)}{2^n n!}=\sum_{n=0}^{\infty}\frac{nxt^{n-1}H^*_{n-1}(x)}{2^n n!}-\sum_{n=0}^{\infty}\frac{2t^{n-1}n(n-1)H^*_{n-2}(x)}{2^n n!}$$

Equating the coefficients of t^{n-1} from both sides and simplifying yields

$$2(n-1)H^*_{n-2}(x)-xH^*_{n-1}(x)+H^*_n(x)=0 \tag{2.4}$$

Equations (2.3) and (2.4) are the **recurrence relations** of the polynomial $H^*_n(x)$.

Now, substituting (2.3) in (2.4), we get

$$2(n-1)\left[\frac{H^*_n(x)}{n(n-1)}\right]-x\left[\frac{H^*_n(x)}{n}\right]+H^*_n(x)=0$$

Simplifying yields

$$2H^{*\prime\prime}_n(x)-xH^{*\prime}_n(x)+nH^*_n(x)=0 \tag{4}$$

(4) shows that $A_n(x)$ satisfies the second order differential equation

$$2y''-xy'+ny=0 \tag{2.5}$$

(2.5) is the required **differential equation** governing the new probabilistic Hermite polynomials $H^*_n(x)$.

2.0.3 Power Series Solution of the Equation (2.5)

In this section the power series solution of the differential equation obtained in (2.5) will be provided which will enable us to represent the polynomial is closed-series form.

Now, since there is no solution using the ascending order power series solution method, then we solve it using descending order power series solution method.

Let $y=\sum_{r=0}^{\infty}a_r x^{c-r} \quad \therefore y'=\sum_{r=0}^{\infty}a_r(c-r)x^{c-r-1}$ and $\sum_{r=0}^{\infty}a_r(c-r)(c-r-1)x^{c-r-2}$

Plugging these into (2.5) and simplifying, we have

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$$2\sum_{r=0}^{\infty}(c-r)(c-r-1)a_r x^{c-r-2}+\sum_{r=0}^{\infty}a_r(n+r-c)x^{c-r}=0 \qquad (5)$$

Equating the coefficient of x^c in the second summation of (5) to zero gives

$$a_0(n-c)=0 \quad \therefore n=c \text{ since } a_0\neq 0.$$

Now, replacing r by $(r-2)$ in the first summation yields

$$2\sum_{r=0}^{\infty}(c-r+2)(c-r+1)a_r x^{c-r}+\sum_{r=0}^{\infty}a_r(n+r-c)x^{c-r}=0$$

Equating the coefficient of x^{c-r} and simplifying gives the recurrence relation as

$$a_r=\frac{-2(c-r+1)(c-r+2)}{n+r-c}a_{r-2}=\frac{-2(n-r+1)(n-r+2)}{r}a_{r-2} \quad where\ \ n=c$$

Putting r = 1, 3, 5…(2r – 1) in the recurrence relation, we will have that

$$a_1=a_3=...=a_{2r-1}=0$$

Now, putting r = 2, 4…(2r) we obtain $a_2=\dfrac{-2n(n-1)}{2}a_0,\qquad a_4=\dfrac{(-2)^2 n(n-1)(n-2)(n-3)}{2.4}a_0$

$$\therefore a_{2r}=\frac{(-1)^r 2^r n(n-1)...(n-2r+1)}{2.4.6...2r}a_0=\frac{(-1)^r 2^r n(n-1)...(n-2r+1)}{2^r r!}\frac{(n-2r)!}{(n-2r)!}a_0=\frac{(-1)^r n!a_0}{(n-2r)!r!}$$

Hence the general solution becomes

$$y=\sum_{r=0}^{\infty}a_{2r}x^{c-2r}=a_0\sum_{r=0}^{\infty}\frac{(-1)^r n!}{(n-2r)!\,r!}x^{n-2r} \qquad (6)$$

Let $a_0=1$ and that (6) can only exists (or be polynomial) if $n-2r\geq 0$ or $n\geq 2r$ $r\leq n/2$.

$$\therefore H^*_n(x)=\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r n!}{(n-2r)!\,r!}x^{n-2r} \qquad (2.6)$$

where $\left[\dfrac{n}{2}\right]=\begin{cases} n/2 & \text{if } n \text{ is } even \\ (n-1)/2 & \text{if } n \text{ is } odd \end{cases}$

(2.6) is the closed form representation of the polynomial $H^*_n(x)$.

A few of the polynomials are as follow;

$H^*_0(x)=1$ $\qquad$ $H^*_1(x)=x$

$H^*_2(x)=x^2-2$ $\qquad$ $H^*_3(x)=x^3-6x$

$H^*_4(x)=x^4-12x^2+12$ $\qquad$ $H^*_5(x)=x^5-20x^3+60x$

$H^*_6(x)=x^6-30x^4+180x^2-120$

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2.0.4 Orthogonality Properties of the polynomials

The orthogonality properties for $H^*_n(x)$ is given as

$$\int_{-\infty}^{\infty} \beta(x) H^*_m(x) H^*_n(x)\,dx = \begin{cases} 0 \quad if \quad m \neq n \\ 2^n n! \quad if \quad m = n \end{cases} \tag{2.7}$$

Proof: Ca

Since H^* ... ve that

$$2H^{*\prime\prime}_m(x) - xH^{*\prime}_m(x)(x) + mH^*_m(x) = 0 \qquad \text{(i)}$$

$$2H^{*\prime\prime}_n(x) - xH^{*\prime}_m(x)(x) + nH^*_n(x) = 0 \qquad \text{(ii)}$$

Multiplying (i) by $H^*_n(x)$ and (ii) by $H^*_m(x)$, subtracting the two new equations and simplifying yield

$$2\frac{dW}{dx} - xW = (n-m)H^*_m(x)H^*_n(x), \quad \text{where } W = H^{*\prime}_m(x)H^*_n(x) - H^{*\prime}_n(x)H^*_m(x).$$

Hence the general solution becomes

$$We^{-\int \frac{x}{2}dx} = \frac{1}{2}\int e^{-\int \frac{x}{2}dx}(n-m)H^*_m(x)H^*_n(x)\,dx$$

$$\therefore \frac{1}{\sqrt{\pi}}\left[We^{-\frac{1}{4}x^2}\right]_{-\infty}^{\infty} = \frac{n-m}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}x^2} H^*_m(x)H^*_n(x)\,dx$$

$$\Rightarrow 0 = (n-m)\int_{-\infty}^{\infty}\beta(x)H^*_m(x)H^*_n(x)\,dx$$

$$\text{Since } m \neq n \quad \therefore \int_{-\infty}^{\infty}\beta(x)H^*_m(x)H^*_n(x)\,dx = 0$$

Case 2: If $m = n$

Squaring both sides of the generating function (3.2), we obtain

$$\sum_{n=0}^{\infty}\frac{t^{2n}H^*_n(x)H^*_n(x)}{2^{2n}[n!]^2} = e^{-\frac{2}{4}(t^2-2xt)}$$

Multiplying both sides by $\beta(x) = \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}x^2}$ and integrating both sides in the interval $(-\infty, \infty)$ yields

$$\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}[n!]^2}\int_{-\infty}^{\infty}\beta(x)H^*_n(x)H^*_n(x)dx = \int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}x^2}e^{-\frac{2}{4}(t^2-2xt)}dx$$

$$= \frac{1}{2\sqrt{\pi}}e^{\frac{1}{2}t^2}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(x-2t)}dx = \frac{1}{2\sqrt{\pi}}e^{\frac{1}{2}t^2}\int_{-\infty}^{\infty}e^{-\frac{1}{4}(x-2t)}dx = \frac{1}{2\sqrt{\pi}}e^{\frac{1}{2}t^2}(2\sqrt{\pi})$$

$$= e^{\frac{1}{2}t^2} = \sum_{n=0}^{\infty}\frac{1}{n!}\left[\frac{t^2}{2}\right]^n = \sum_{n=0}^{\infty}\frac{t^{2n}}{2^n n!}$$

$$\therefore \sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}[n!]^2}\int_{-\infty}^{\infty}\beta(x)H^*_n(x)H^*_n(x)dx = \sum_{n=0}^{\infty}\frac{t^{2n}}{2^n n!} \tag{7}$$

Now, equating the coefficients of t^{2n} from both sides of (7) gives

$$\frac{1}{2^n n!}\int_{-\infty}^{\infty}\beta(x)H^*_n(x)H^*_n(x)dx=1 \qquad \therefore \int_{-\infty}^{\infty}\beta(x)H^*_n(x)H^*_n(x)dx=2^n n!.$$

This completes the proof.

III. Methodology

The methods employed in proving the results in this paper are the same methods adopted in the proof of other special differential equations. Taylor series of function of two variables, differentiations and manipulations are all employed to deduce the generating function and recurrence relations. Descending power series solution method was adopted to obtain the series solution of differential equation governing the new probabilistic Hermite polynomials $H^*_n(x)$. Finally, integrating factor method of solving first order differential equation was used to obtain the orthogonality properties of the polynomial

IV. Conclusion

The results obtained here are related to the well-known special differential equation called Probabilistics Hermite equation. It is of special interest to extend these results to application in engineering as other special differential equations; this will be presented in our further research work. We conclude that all the results obtained in this paper are entering into the literature for the first time.

V. Further Research

In our further research, we aim to consider the following

1. A series of the form $f(x)=\sum_{n=0}^{\infty}a_n\beta(x)H^*_n(x)$
2. Integral and confluent hypergeometry representations of the polynomial $H^*_n(x)$
3. Application of the polynomial $H^*_n(x)$.
4. Relationship to the Probabilistic Hermite Polynomials $H_r(x)$.

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