Energy Quantization from Reissner-Nordström Anti-de Sitter Black Hole

M. Jakir Hossain and M. Ilias Hossain

Department of Mathematics, Rajshahi University, Rajshahi - 6205, Bangladesh

DOI: https://doi.org/10.51584/IJRIAS.2023.8802

Received: 01 August 2023; Accepted: 06 August 2023; Published: 03 September 2023

Abstract: In this work we address the study of Bekenstein-Hawking entropy of Reissner-Nordström anti-de Sitter (RNdS) black hole using energy quantization method like Bohr's atomic model. We have quantize the energy of the test particle orbiting around RNAdS black hole and the change of entropy between two nearby states approaches to zero for large quantum numbers.

Keywords: Energy Quantization, RNAdS black hole, Circular orbit.

I. Introduction

Hawking radiation is black-body radiation that is expected to be released by black holes as a result of quantum effects near the event horizon. Hawking radiation is named after Stephen Hawking, who provided a theoretical argument for its existence in 1974, [1] and after Jacob Bekenstein in 1972, who forecasted that black holes should have a finite, non-zero temperature and entropy [2]. Hawking's visited Moscow in 1973 where the Soviet physicists Yakov Zeldovich and Alexei Starobinsky showed him that, consistent with the quantum mechanical uncertainty principle, rotating black holes should create and emit particles [3]. Hawking radiation reduces the mass and energy of black holes and as such it is also known as black hole evaporation. Owing to this, black holes that lose more mass than they gain through other means are expected to reduce in size and ultimately disappear. Micro black holes are predicted to be larger net emitters of radiation than larger black holes and should get smaller and disappear faster. In September 2010, a signal that is closely related to black hole Hawking radiation was claimed to have been observed in a laboratory experiment involving optical light pulses. However, the results remain unverified and debatable [4, 5]. Other projects have been launched to look for this radiation within the framework of analog gravity. In June 2008, NASA launched the Fermi space telescope, which is searching for the terminal gamma-ray flashes expected from evaporating primordial black holes. In the event that speculative large extra dimension theories are correct, CERN's Large Hadron Collider may be able to create micro black holes and observe their evaporation [6, 7, 8, 9]. In Refes. [10, 11] many research works on the thermal radiation of black holes have been made by Kraus and Wilezek [12, 13]; Parikh and Wilezek [14]; Hemming and Keski-Vakkuri [15]. It seems that an initially pure quantum state, by collapsing to a black hole and then evaporating completely. When the black hole has evaporated down to the Planck size, quantum fluctuations dominate and the semi classical calculations would no longer be valid, as spacetime is subject to violent quantum fluctuations on this scale.

A conjectured de Sitter / conformal field theory (CFT) correspondence [16] defined in a manner analogous to the very successful AdS/CFT correspondence [17, 18] has been proposed recently that there is dual between a quantum gravity on a de Sitter space and a Euclidean conformal field on a boundary of the de Sitter space. Many works on Hawking radiation from massive uncharged particle tunneling [19, 20] and charged particle tunneling [19, 20, 21] from black hole was first proposed by Zhang and Zhao. Accomplishment this work, a few researches have been carried out as charged particle tunneling [22, 23]; Ali [24]; Wu and Jiang [23, 25] and for rotating black hole [26] and using this method Hawking radiation of Kerr-NUT black hole [27] and the charged black hole with a global monopole [28] have been developed but the quantization of black hole by the method of energy quantization developed not yet. The quantization of black hole or its gravity is very limited and therefore there is ample scope for further study. Recently we have investigated the quantization of black hole [29, 30] using the energy quantization method. We will use the same method to investigate the quantization of black hole for RNAdS black hole.

The rest of the composition is planned as follows: Firstly in section 2 we have discussed Reissner-Nordström anti-de Sitter(RNAdS) Spacetime and secondly in section 3 we have calculated the Lagrangian and canonical momentum of a test particle of the RNAdS metric according to our previous work [31]. In Section 4, we have designed the effective possibility of radial motion using the Lagrangian and canonical momentum. Here we have also quantized the angular momentum of the test particle moving around the black hole while in section 5 and in section 6 we have established by quantizing the energy of the RNAdS black hole gravity. Finally in section 7, we present our concluding remarks.

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

II. Reissner-Nordström anti-de Sitter Spacetime

We have used the canonical formulation [32, 33, 34, 35] to quantize the RNAdS black hole gravity for this section. With the negative cosmological constant ($\Lambda = -3/\ell^2$), the metric of the Reissner-Nordström anti-de Sitter black hole is given by [25, 36]

$$\mathrm{ds}^2 = -\left(1-\frac{2M}{r}+\frac{r^2}{\ell^2}+\frac{Q^2}{r^2}\right)c^2dt^2+\left(1-\frac{2M}{r}+\frac{r^2}{\ell^2}+\frac{Q^2}{r^2}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\ d\varphi^2), \tag{1}$$

where M being the mass, ℓ is the cosmological radius , Q the total charge with respect to the static de Sitter space are defined such that $-\infty \le t \le \infty, r \ge 0, 0 \le \theta \le 2\pi$. At large r, the metric (1) tends to the ds space limit. It is seen that the explicit ds case is obtained by setting $M = 0$ while the explicit Reissner-Nordström case is obtained by taking the limit $\ell \to \infty$. For the simplicity we can rewrite the metric given in Eq. (1) of the following form

$$\mathrm{ds}^2 = -\left(1-\frac{2M}{r}\left(1-\frac{r^3}{2\mathrm{M}\ell^2}-\frac{Q^2}{2\mathrm{M}r^2}\right)\right)c^2dt^2+\left(1-\frac{2M}{r}\left(1-\frac{r^3}{2\mathrm{M}\ell^2}-\frac{Q^2}{2\mathrm{M}r^2}\right)\right)^{-1}dr^2$$
$$+r^2(d\theta^2+\sin^2\theta\ d\varphi^2). \tag{2}$$

The characteristic polynomial of RNAdS spacetime can be written as $r^4 - \ell^2 r^2 + 2M\ell^2 r - \ell^2 Q^2 = 0$. Solving this equation and using $= \sqrt{1-\frac{4Q^2}{\ell^2}}$, since $\ell^2 >> Q^2$, therefore $\alpha \approx 1$, thus the position of the black hole horizon [25] can be obtained as

$$r_h \approx \left(1-\frac{4M^2}{\ell^2}+\cdots\right)M\left(1+\sqrt{1-\frac{Q^2}{M^2}}\right). \tag{3}$$

For minisuper space $Q^2/M^2 \approx 0$ and neglecting higher power of $\frac{4M^2}{\ell^2}$, the first approximated position of horizon can be simplified from Eq. (3) of the following form

$$r_0 \approx 2M\left(1-\frac{4M^2}{\ell^2}+\cdots\right) = 2M\left(1-\frac{4M^2}{\ell^2}\right). \tag{4}$$

Using Eq. (4), into Eq. (2), the metric equation can be taken of the following form

$$\mathrm{ds}^2 = -\left(1-\frac{2M}{r}\left(1-\frac{4M^2\left(1-\frac{4M^2}{\ell^2}\right)^3}{\ell^2}-\frac{Q^2}{4M^2\left(1-\frac{4M^2}{\ell^2}\right)^3}\right)\right)c^2dt^2$$
$$+\left(1-\frac{2M}{r}\left(1-\frac{4M^2\left(1-\frac{4M^2}{\ell^2}\right)^3}{\ell^2}-\frac{Q^2}{4M^2\left(1-\frac{4M^2}{\ell^2}\right)^3}\right)\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\ d\varphi^2), \tag{5}$$

If we let $\chi = 1-\frac{4M^2}{\ell^2}$, then the above equation becomes as

$$\mathrm{ds}^2 = -\left(1-\frac{2M}{r}\left(1-\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)c^2dt^2+\left(1-\frac{2M}{r}\left(1-\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}dr^2$$
$$+r^2(d\theta^2+\sin^2\theta\ d\varphi^2), \tag{6}$$

Now let us consider a test particle of mass m orbiting along the circular geodesics in the equatorial plane $\theta = 2\pi$ around RNAdS black hole. Thus the above Eq. (7) as described in [37, 38] can becomes in the following form

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$\mathrm{ds}^2=-\left(1-\frac{2M}{r}\left(1-\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)c^2dt^2+\left(1-\frac{2M}{r}\left(1-\frac{4M^2\xi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}dr^2+r^2d\theta^2. \quad (7)$$

III. Lagrangian and Canonical Momentum

Now we assume that the black hole mass is larger than the Planck mass so that the Compton radius $r_c=\hbar/mc\ll r_h$, where r_h is the radius of the RNAdS black hole. In this circumstances the quantum fluctuations of the black hole disregards [39]. In terms of the metric components g_{ij}, the Lagrangian of the test particle can be written as $\mathcal{L}=g_{ij}\dot{x}^i\dot{x}^j$

$$=-\frac{m}{2}\left[\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)c^2dt^2+\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}dr^2\right]$$

$$+\frac{r^2m}{2}(d\theta^2+\sin^2\theta\ d\varphi^2). \quad (8)$$

For static and spherically charged RNAdS black hole, there exist two constants of motion for the test particles, associated with two Killing vectors in terms of energy E, angular momentum L and the canonical momenta components p_r, p_θ can be written in the following form

$$E=\frac{\partial\mathcal{L}}{\partial\dot{t}}=mc^2\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)\dot{t};\ \ L=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}=mr^2\sin^2\theta\,\dot{\varphi}. \quad (9)$$

$$p_r=\frac{\partial\mathcal{L}}{\partial\dot{r}}=mc^2\left(1-\frac{2M}{r}\left(1+\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}\dot{r};\quad p_\theta=\frac{\partial\mathcal{L}}{\partial\dot{\theta}}=mr^2\dot{\theta}. \quad (10)$$

IV. Radial Motion and Effective Potential for RNAdS Line Element

The radial motion of a geodesic can be written as

$$g^{00}p_0^2+g^{rr}p_r^2+g^{\phi\phi}p_\phi^2+g^{\theta\theta}p_\theta^2+m^2c^2=0, \quad (11)$$

the four vectors $(p_0=E/c,p)$ can be expressed in the magnitude of the energy-momentum. Inserting Eqs. (9)-(10) and Eq. (7) into Eq. (11), we have

$$\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{E^2}{c^2}+m^2\dot{r}^2\right)+\frac{L^2}{r^2\sin^2\theta}+\frac{m\dot{\theta}^2}{r^2}+m^2c^2=0. \quad (12)$$

Using $\dot{\theta}^2=0$ and $\sin^2\theta=1$, the above equation becomes as

$$\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{E^2}{c^2}+m^2\dot{r}^2\right)+\frac{L^2}{r^2}+m^2c^2=0. \quad (13)$$

In Refs. [31, 40], the energy and momentum of the test particle per unit rest mass $\tilde{E}=\frac{E}{m}$, and $\tilde{L}=\frac{L}{m}$ are respectively and setting these are into Eq. (13), we have

$$\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)^{-1}\left(-\frac{\tilde{E}^2}{c^4}+\frac{\dot{r}^2}{c^2}\right)+\frac{\tilde{L}^2}{c^2r^2}+1=0. \quad (14)$$

Simplifying the above equation, we get the radial motion of the test particle in the following form

$$\frac{\dot{r}^2}{2}=\frac{\tilde{E}^2}{2c^2}-\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right). \quad (15)$$

For the time like particle orbit the velocity is expressed in terms of two parameters energy and angular momentum. Thus we have the effective potential V_{eff} for the radial motion from Eq. (15) of the following form

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$V_{eff}=\frac{1}{2}\left(\frac{\tilde{L}^2}{r^2}+c^2\right)\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right). \quad (16)$$

The radial acceleration of the test particle can be obtained by taking the derivative of Eq. (16) with respect to the proper time and then equating to zero, we obtain

$$\frac{c^2}{r^4}M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)\times\left(r^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)}r+\frac{3\tilde{L}^2}{c^2}\right). \quad (17)$$

Now our prime task in this section is to find the radius of the circular orbit namely R. For this purpose, we can be written the above equation of the following form

$$\left(R^2-\frac{\tilde{L}^2}{c^2M\left(1-\frac{2M}{r}\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)\right)}R+\frac{3\tilde{L}^2}{c^2}\right)=0. \quad (18)$$

Solve the above equation for the radius of the circular orbit R, we get the two roots of the form

$$R_{\pm}=\left(\frac{\tilde{L}^2}{2c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right)\pm\left[\left(\frac{\tilde{L}^2}{2c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right)^2-\frac{3\tilde{L}^2}{c^2}\right]^{\frac{1}{2}}. \quad (19)$$

Simplifying the Eq. (19) we can get

$$R_{\pm}=\left(\frac{\tilde{L}^2}{2c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right)\times\left(1\pm\left(1-\frac{12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}^2}\right)^{\frac{1}{2}}\right). \quad (20)$$

We observe that the roots $R_{\pm}$ of the above equation is real only when $\tilde{L}^2\geq 12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2$ and for the smallest stable orbit the square root on the right hand side of Eq. (20) dissolves. Therefore, we must have

$$\tilde{L}^2=12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2. \quad (21)$$

For large and largest stable circular orbits the conditions $\tilde{L}^2\geq 12c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2$ and $\tilde{L}^2>>12c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2$ are holds respectively.

V. Quantization of Angular Momentum of the Test Particle

According to Wilson [41] and Sommerfeld [42] idea, angular momentum can be quantized as a periodic function of time and with the help of quantize energy for the reason that it is closely related to the quantize angular momentum of the orbiting test particle. In order to quantize angular momentum J_ϕ with the help of canonical momentum L conjugate to the angular variable of the form

$$J_\phi=\int_0^{2\pi}Ld\emptyset=n\,h. \quad (22)$$

Since L is a constant of motion, thus $\tilde{L}=\frac{L}{m}$ gives the quantization condition for the angular momentum of the following form

$$L=m\tilde{L}=n\hbar,\text{so that } \tilde{L}_0=n_0\hbar/m\,. \quad (23)$$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

With the help of compton radius $r_c = \hbar/mc$ and equation Eq. (23), the Eq. (21) can be written as

$$\frac{n_0^2\hbar^2}{m^2} = 12c^2M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2$$

$$\Rightarrow \frac{n_0^2\hbar^2}{m^2c^2} = 12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2$$

$$\therefore n_0^2r_c^2 = 12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2. \quad (24)$$

By using Eq. (24) into Eq. (20), we get the radius of the different stable circular orbit of the particle corresponds to n_0 of the form

$$R_+ = \left(\frac{n_0^2r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}\right) \times \left(1 + \left(1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{n_0^2r_c^2}\right)^{\frac{1}{2}}\right). \quad (25)$$

Inserting Eq. (24) into Eq. (25) we get the approximate radius of the first circular orbit R_0 of the form

$$R_0 \approx \left(\frac{n_0^2r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}\right). \quad (26)$$

In the limiting case when $Q \to 0$ and $\ell \to \infty$ the above equation reduces to $R_0 = n_0^2\frac{r_c^2}{r_s}$ which agrees with the result given in Ref. [40], where $r_s = 2M$ represents the Schwarzschild radius. The position of the next higher circular orbit R_1 can be obtained from Eq. (25) by replacing n_0 with $n_1 = n_0 + 1$ of the form

$$R_1 = \left(\frac{(n_0+1)^2r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}\right) \times \left(1 + \left(1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{(n_0+1)^2r_c^2}\right)^{\frac{1}{2}}\right). \quad (27)$$

Now $n_0 >> 1$ while $2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right) \gg r_c$ and therefore we can write

$$(n_0+1)^2 = n_0^2 + 2n_0 + 1 = n_0^2\left[1 + \frac{2}{n_0} + \frac{1}{n_0^2}\right] \approx n_0^2\left[1 + \frac{2}{n_0}\right]. \quad 28)$$

Using Eq. (28) into Eq. (27) we get

$$R_1 = \left(\frac{n_0^2\left(1 + \frac{2}{n_0}\right)r_c^2}{2M\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)}\right) \times \left(1 + \left(1 - \frac{12M^2\left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2}{n_0^2\left(1 + \frac{2}{n_0}\right)r_c^2}\right)^{\frac{1}{2}}\right). \quad (29)$$

By using the Eq. (24) into the right side of the above equation can be approximated of the form

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$1+\left(1-\frac{12M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{n_0^2\left(1+\frac{2}{n_0}\right)r_c^2}\right)^{\frac{1}{2}} \approx 1+\left(1-\frac{1}{\left(1+\frac{2}{n_0}\right)}\right)^{\frac{1}{2}} = 1+\left(1-\left(1+\frac{2}{n_0}\right)^{-1}\right)^{\frac{1}{2}} = 1+\left(1-\left(1-\frac{2}{n_0}+\cdots\right)\right)^{\frac{1}{2}}$$
$$\approx 1+\sqrt{\frac{2}{n_0}}. \qquad (30)$$

Simplifying Eq. (29) with the help of Eq. (30) as

$$R_1 = R_0\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right). \qquad (31)$$

Thus the next higher stage n_2^2 defined as

$$n_2^2 = (n_0+2)^2 = n_0^2+4n_0+4 = n_0^2\left[1+\frac{4}{n_0}+\frac{4}{n_0^2}\right] \approx n_0^2\left[1+\frac{4}{n_0}\right]. \qquad (32)$$

Progressing in this way, the radius of the next higher circular orbit R_2 of the test particle with the help of Eq. (32) can be obtained from Eq. (25) of the form

$$R_2 = \left(\frac{n_0^2\left(1+\frac{4}{n_0}\right)r_c^2}{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right)\times\left(1+\left(1-\frac{12M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{n_0^2\left(1+\frac{4}{n_0}\right)r_c^2}\right)^{\frac{1}{2}}\right), = R_0\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)$$
$$= R_1\frac{\left(1+\frac{4}{n_0}\right)\left(1+\sqrt{\frac{4}{n_0}}\right)}{\left(1+\frac{2}{n_0}\right)\left(1+\sqrt{\frac{2}{n_0}}\right)}. \qquad (33)$$

Proceeding in the similar manner the $(n+1)$th radius of the stable circular orbits of the particle can be written as

$$R_{n+1} = R_n\frac{\left(1+\frac{2n+2}{n_0}\right)\left(1+\sqrt{\frac{2n+2}{n_0}}\right)}{\left(1+\frac{2n}{n_0}\right)\left(1+\sqrt{\frac{2n}{n_0}}\right)}. \qquad (34)$$

Here we have $R_{n+1} = R_n$, when $n_0 \to \infty$. For large quantum number we can say that the two nearby states coincide for RNAdS black hole.

VI. Energy Quantization for RNAdS Black Hole

With the help of angular momentum we want to quantize the energy of the orbiting test particle in this section. For zero velocity at $r = R$ the Eq. (15) gives as

$$\tilde{E}^2 = c^2\left(\frac{\tilde{L}^2}{R^2}+c^2\right)\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right). \qquad (35)$$

But Eq. (17) gives at $r = R$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$\frac{\tilde{L}^2}{R^2} = \frac{c^2}{\left(\frac{R}{M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}-3\right)}. \tag{36}$$

Using Eq. (36) into Eq. (35) we get

$$\frac{E^2}{m^2} = \tilde{E}^2 = c^2\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)\frac{c^2}{\left(\frac{R}{M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}-3\right)},$$

$$\Rightarrow E^2 = m^2c^4\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)\frac{\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)}{\left(1-\frac{3M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)} \tag{37}$$

Therefore, Eq. (37) becomes as

$$\Rightarrow E = mc^2\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)\left(1-\frac{3M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)^{-1/2} \tag{38}$$

With the help of mechanical stability condition $\tilde{L}^2 = MGR$ in Eq. (20), the second term in the parenthesis can be written as

$$\frac{12c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}^2} = 3c^2\frac{2GM}{c^2}\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{MGR} = \frac{12M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}, \tag{39}$$

which gives with the help of Eq. (21) for the circular orbits corresponding to $n_0 >> 1$

$$\frac{12M\left(1+\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R} << 1. \tag{40}$$

Neglecting higher terms and therefore, Eq. (38) can be written as

$$E = mc^2\left(1-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}\right)\left(1+\frac{3M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{2R}+\cdots\right),$$

$$= \left(1+\frac{3M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{2R}-\frac{2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{R}+\cdots\right),$$

$$\therefore E = mc^2\left(1-\frac{M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}{2R}\right). \tag{41}$$

Using Eq. (23) into Eq.(26) we have

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$2R \approx \left(\frac{n^2 r_c^2}{M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right) = \left(\frac{n^2\hbar^2}{m^2c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right)$$
$$= \left(\frac{\tilde{L}^2}{c^2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)}\right). \qquad (42)$$

Substituting the above equation into Eq. (41) we obtain

$$E \approx mc^2\left(1-\frac{c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}^2}\right). \qquad (43)$$

The quantized energy E_n for nth energy label can be written as

$$E_n \approx mc^2\left(1-\frac{c^2M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{\tilde{L}_n^2}\right). \qquad (44)$$

The above equation can be written with the help of Eq. (23) and $\hbar = r_c mc$ as

$$E_n \approx mc^2\left(1-\frac{M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{n^2r_c^2}\right). \qquad (45)$$

From the Eq. (45), we obtains the corresponding $(n+1)$th label energy of the form

$$E_{n+1} \approx mc^2\left(1-\frac{M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2}{r_c^2(n+1)^2}\right). \qquad (46)$$

Thus, the quantized energy difference between two nearby states of the form

$$\delta E = E_{n+1} - E_n$$
$$\approx \frac{mc^2}{r_c^2}\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]M^2\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right)^2. \qquad (47)$$

Neglecting the 4th and higher powers of (M/ℓ) of the RNAdS black hole radius given in Eq. (2) then we have

$$r_{RNAdS} \approx 2M\left(1-\frac{4M^2\chi^3}{\ell^2}-\frac{Q^2}{4M^2\chi}\right). \qquad (48)$$

When $\ell \to \infty$, we observe that the radius of the RNAdS black hole approaches to the radius of the RNdS black hole and which has been developed in [30]. For RNAdS black hole, the Eq. (47) can be written as

$$\delta E \approx \frac{mc^2 r_{RNAdS}^2}{4r_c^2}\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]^2. \qquad (49)$$

For large values of n, the first parenthesis of Eq. (49) can be replaced by $2/n^3$ so that

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

$$\delta E \approx \frac{m^3 c^4 r_{RNAdS}^2}{2\hbar^2 n^3}.$$

$$= \frac{c^4 m^3}{2\hbar^2 n^3} \times 4M^2 \left(1 - \frac{4M^2\chi^3}{\ell^2} - \frac{Q^2}{4M^2\chi}\right)^2 \quad . \tag{50}$$

We have $\delta E \to 0$ when $n \to \infty$ which indicates for a larger circular orbit the change of energy between two nearby states approaches to zero.

VII. Concluding Remarks

In this paper, we have presented the change of entropy for two nearby circular orbit around Reissner-Nordström Anti-de Sitter Black Hole by Energy Quantization process. We have also found the different energy labels of RNAdS black hole in nature can be performed in the same way as that for the electron signal inside the atom like Bohr's quantum theory and leads to the results on quantization of black hole [43]. For further study, we may extend the quantum relation of Reissner-Nordström Anti-de Sitter black hole and their temperature can be evaluated in agreement with the Hawking temperature. The results we have obtained agree with the results of RN black hole [29] when cosmological constant goes to infinity. Again when $\ell \to \infty$ and $Q \to 0$ the RNdS black hole reduced to Schwarzschild black hole [40].

References

1. Rose. Charlie; A conversation with Dr. Stephen Hawking and Lucy Hawking. charlierose.com.
2. Levi Julian, Hana (3 September 2012). "'40 Years of Black Hole Thermodynamics in Jerusalem". Arutz Sheva. Retrieved 8 September 2012.
3. Hawking, S. W.: The Illustrated A Brief History of Time, Bantam Books, ISBN 0-553-38016-8, (1988).
4. Belgiorno, F., Cacciatori, S.L., Clerici, M., Gorini, V., Ortenzi, G., Rizzi, L., Rubino, E., Sala, V.G., Faccio, D.: Hawking radiation from ultrashort laser pulse filaments, DOI:10.1103/PhysRevLett.105.203901, (2010), arxiv.org/abs/1009.4634.
5. Lisa Grossman; Wired. Retrieved 30 April 2012.
6. Giddings, Steven B., Thomas Scott.: High Energy Colliders as Black Hole Factories: The End of Short Distance Physics Phys.Rev. D 65, 056010 (2002), arXiv:hep-ph/0106219.
7. Dimopoulos, S., Landsberg. G.: Black Holes at the LHC, doi:10.1103/Phys. Rev. Lett. 87, 161602 (2001), arXiv:hep-ph/0106295.
8. " The case for mini black holes". CERN courier. 12 Nov 2004.
9. Mark. Henderson; The Times (London), Retrieved May 4, 2010.
10. Hawking, S. W.: Black hole explosions, Nature (London) **248**, 30(1974).
11. Hawking, S. W.: Particle Creation by Black Holes, Commun. Math. Phys. **43**, 199-220(1975).
12. Kraus, P. and Wilczek, F.: Self-Interaction Correction to Black Hole Radiance, Nucl. Phys. B 433, 403 (1995), arXiv:gr-qc/9408003.
13. March-Russell, John. Sethi , Savdeep. and Wilczek, Frank.:Saltatory Relaxation of the Cosmological Constant, Nucl. Phys. B 437, 231 (1995), arXiv:hep-th/9411219.
14. Parikh, M. K. and Wilczek, F. : Hawking Radiation as Tunneling, Phys. Rev. Lett. 85, 5042 (2000), arXiv:hep-th/9907001.
15. Hemming, S. and Keski-Vakkuri, E. : Hawking Radiation from AdS Black Holes, Phys. Rev. D 64, 044006 (2001), arXiv:gr-qc/0005115.
16. Strominger, A.: The dS/CFT Correspondence, JHEP 10, 034(2001).
17. J.M. Maldacena, J. M. : Eternal Black Holes in AdS, Adv. Theor. Math. Phys. 2, 321 (1998).
18. Written, E. : Anti-de Sitter Space, Thermal Phase Transition, And Confinement In Gauge Theories, Adv. Theor. Math. Phys. 2, 253 (1998).
19. Zhang J. Y. and Zhao Z. : Hawking radiation of charged particles via tunnelling from the Reissner-Nordstr¨om black hole, JHEP **10**, 055 (2005).
20. Zhang J. Y., Zhao Z. : Massive particles' black hole tunneling and de Sitter tunneling, Nucl. Phys. B725, 173 (2005).
21. Zhang, J.Y. and Zhao Z. : Charged particles' tunnelling from the Kerr-Newman black hole, Phys. Lett. B **638**, 110 (2006), arXiv:gr-qc/0512153.
22. Wu, S. Q., Jiang, Q. Q.: Remarks on Hawking radiation as tunneling from the BTZ black holes, J. High Energy Phys. 03, 079 (2006a).
23. Wu, S. Q., Jiang, Q. Q.: Hawking Radiation of Charged Particles as Tunneling from Higher Dimensional Reissner-Nordstrom-de Sitter Black Holes, Jiang; (2006b), arXiv:hep-th/0603082.

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS | Volume VIII Issue VIII August 2023

24. Ali, M.H.: Charged Particles' Tunneling from Hot-NUT-Kerr-Newman-Kasuya Spacetime, Int. J. Theor. Phys. 47, 2203 (2008).
25. Hossain, M. I. and Rahman, M. A.: Hawking Thermal and Non-thermal Radiation of Nordström anti-de Sitter Black Hole by Hamilton-Jacobi method, Astrophys Space Sci. 347, 91-97 (2013), DOI 10.1007/s10509-013-1505-3.
26. Chen D. Y., Zu X. T. and Yang S. H. : Massive particle Tunnels From The Taub-Nut Black Hole, Acta Physica Polonica B **39**, no.6, 1329 (2008).
27. Liu H. L., Liu Z. X., Hou J. S., Yang S. Z. : A New Method to Study the Hawking Radiation from the Kerr-NUT Black Hole, Int. J. Theor. Phys.47, 2960 (2008).
28. Liu, J.J., Chen, D.Y. and Yang, S.Z. : A new method to study the Hawking radiation of the charged black hole with a global monopole. Romanian J. Phys. 53, no. 5-6, 659–664, (2008).
29. Hossain, M.J., Rahman M. A. and Hossain M. I.: Bekenstein–Hawking entropy by energy quantization from Reissner–Nordström black hole, Int. J. Mod. Phys. D Vol. 25, No. 3 (2016) 1650034.
30. Hossain, M.J., Rahman M. A. and Hossain M. I.: Jahangirnagar University Physics Studies Vol. 23, ISSN 1999-6632, June 2020.
31. Rahman, M. A., Hossain, M.J. and Hossain M. I.: Bekenstein–Hawking Entropy by energy quantization from Schwarzschild-de Sitter black hole, Astroparticle Physics 71, 71-75 (2015).
32. Kastrup, H. A. and Thiemann, T.: Spherically symmetric gravity as a completely integrable system, Nucl. Phys. B 425, 665 (1994).
33. Kuchair, K. V.: Geometrodynamics of Schwarzschild black holes, Phys. Rev. D 50, 3961 (1994).
34. Cavagli'a , M., Alfaro, V. de, and Filippov, A. T. : Hamiltonian Formalism for Black Holes and Quantization, to appear in, Int. J. Mod. Phys. D (1995), DOI: 10.1142/S0218271895000442.
35. Cavagli'a , M., Alfaro, V. de, and Filippov, A. T. : Quantization of the Schwarzschild Black Hole, Int. J. Mod. Phys. D 5 P. 227-250(1996). [arXiv:gr-qc/9508062v1] .
36. Hossain, M. I. and Rahman, M. A.: Hawking radiation of Reissner-Nordstrom-de Sitter black hole by Hamilton-Jacobi method, arXiv:1309.0067v1 [gr-qc].
37. Misner, C. W., Thorne, K. S. and Wheeler, J. A. : Gravitation, San Francisco: Freeman (1973).
38. Wald, R. : General Relativity, University of Chicago Press, Chicago (1984).
39. Coleman, S., Preskill, J. and Wilczek, F.: Quantum Hair on Black Holes, Nucl. Phys. B 378, 175-246, (1992).
40. Simanek, E. : Energy quantization for matter orbiting black hole and Hawking radiation, arXiv:1209.3791v1 [gr-qc].
41. Wilson, W. : The quantum-theory of radiation and line spectra, Phil. Mag. 29, 795 (1915).
42. A. Sommerfeld; Ann. Physik 51, 1 (1916).
43. He, X. G.; Ma, B.Q.: Quantization of black holes. Mod. Phys. Lett. A, 26, 2299–2304 (2011).