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Some Inclusion Results of Operators Associated with a Generalization of the Mittag-Leffler Function

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Abstract: In this study, we consider one of the generalizations of the well-known Mittag-Leffler function, namely $E^{\theta}_{\alpha,\beta}(z)$. We normalize the latter by multiplication with the factor $z\Gamma(\beta)$ to generate a power series that belongs to the well-known class of analytic functions A , in the unit disk D. Consequently, and using spiral-like functions, we investigate some inclusion results.

Keywords: Convolution, Inclusion results, Integral Transform, Spiral-like functions, Mittag-Leffler function

Mathematics Subject Classification: 30C45, 30C50

I. Preliminaries

One of the functions that characterize exponential behavior was developed by a Swedish mathematician and is known as "The Mittag-Leffler Function" (see [1]). The Mittag-Leffler function (M-L) has become more significant due to its widespread applicability in many scientific and technical domains. In some branches of the physical and applied sciences, including probability and statistical distribution theory, fluid mechanics, biological issues, electrical networks, and others, the (M-L) function has been used recently. Leѵ́y flights, random walks, and—most importantly—generalization of kinetic equations are examples of integro-differential equations in which this function naturally occurs [2, 3]. The (M-L) function has been studied extensively in the literature for its normalization, generalization, characteristics, applications, and extension. One can check out [4, 5] and [6] for further information. The study of fractional generalization of kinetic equations, random walks, Leѵ́y flights, super-diffusive transport, complex systems, and delayed fractional reaction diffusion all involve fractional-order differential and integral equations; the solutions invariably contain (M-L) function (see [7–10]). Recently, the one-parameter (M-L) function has also been suggested as a solution for mathematical models in biology and tourism (see [11,12]).

Initially, the one-parameter (M-L) function $E_{\alpha}(z)$ for $\alpha \in C,\ with\ Re\ (\alpha > 0)$ (see [13] and [14]) is defined as:

$$E_{\alpha}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, z \in C,$$

then, the extension of (M-L) function in two-parameters was studied by Wiman [15]. For all $\alpha, \beta \in C,\ with\ Re\ (\alpha, \beta > 0)$, the two parameters function $E_{\alpha,\beta}(z)$ is defined as:

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, z \in C.$$

Many studies have provided some generalizations of the (M-L) function (see [16-19]). The main focus of this study is the form given by Prabhakar [20]:

$$E^{\theta}_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{(\theta)_n}{\Gamma(\alpha n + \beta)} . \frac{z^n}{n!}, \quad z, \beta, \theta \in C\ ;\ Re\ \alpha > 0.$$

Note that $(\theta)_v$ denotes the familiar Pochhammer symbol which is defined as:

$$(\theta)_v := \frac{\Gamma(\theta + v)}{\Gamma(\theta)} = \{1, \qquad if\ v = 0, \theta \in C\backslash\{0\}\ \theta(\theta+1)\ldots(\theta+n-1), \qquad if\ v = n \in N, \theta \in C,$$

$$(1)_n = n!, \qquad n \in N_0, N_0 = N \cup \{0\}, \qquad N = \{1,2,3,\ldots\},$$

and$\left(q \in N, j = 1,2,3, \ldots q;\ Re\{\theta_j, \beta_j\} > 0, and\ \ Re\ \alpha_j > max\{0, Re\ k_j - 1;\ Re\ k_j\};\ Re\ k_j > 0\right)$.

Let A be the class of analytic functions in the open unit disk $D:=\{z\in C:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$ with the form

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n, z\in D. \tag{1.1}$$

By observing that the above generalized (M-L) function $E_{\alpha,\beta}^{\theta}$ does not belong to family A, we follow the same method of Bansal and Prajapat [21] to obtain the following *normalization* of the generalized (M-L) function as follows:

$$E_{\alpha,\beta}^{\theta}(z):=z\Gamma(\beta)E_{\alpha,\beta}^{\theta}(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)(\theta)_n}{n!\,\Gamma(\alpha(n-1)+\beta)}z^n, \tag{1.2}$$

that holds for parameters $\alpha,\beta\in C$ with $Re\,\alpha>0, Re\,\beta>0$ and $z\in C$. In this study, we discuss the special case when α and β are real-valued parameters. For functions f and g in A, f of the form (1.1) and $g(z)=z+\sum_{n=2}^{\infty}b_n z^n$, , $z\in D$, the Hadamard product is defined by

$$(f*g)(z):=z+\sum_{n=2}^{\infty} a_n b_n z^n, z\in D.$$

Let S be the subclass of A whose members are univalent in D. Robertson [22] studied two well-known subclasses of S, namely, the classes of starlike and convex functions. Function $f\in A$ given by (1.1) is said to be starlike of order $\gamma, 0\le\gamma<1$, if and only if $Re\left(\frac{zf'(z)}{f(z)}\right)>\gamma, z\in D$,and the function class is denoted as $S^*(\gamma)$. We also write $S^*(0)=:S^*$, where S^* denotes the classof functions $f\in A$ such that $f(D)$ is starlike domain with respect to the origin. Function $f\in A$ is said to be convex of order $\gamma, 0\le\gamma<1$, if and only if $Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\gamma, z\in D$ and the class is denoted as $K(\gamma)$. Furthermore, $K:=K(0)$ represents the well-known standard class of convex functions. By Alexander's duality relation (see [23]), we know that $f\in K\Leftrightarrow zf'(z)\in S^*$. Function $f\in A$ is said to be spiral-like if

$$Re\left(e^{-i\xi}\frac{zf'(z)}{f(z)}\right)>0, z\in D,$$

for some $\xi\in C$ with $|\xi|<\frac{\pi}{2}$, these classes were introduced in [24].

In this study, we consider the subclasses of spiral-like functions $S(\xi,\gamma,\rho)$and$K(\xi,\gamma,\rho)$,that were introduced and studied by Murugusundramoorthy [25,26] and the class $R^{\tau}(\vartheta,\delta)$ that was introduced by Swaminathan [27].

Definition 1.1. *For* $0\le\rho<1, 0\le\gamma<1$ *and* $|\xi|<\frac{\pi}{2}$, *define the class* $S(\xi,\gamma,\rho)$ *by*

$$S(\xi,\gamma,\rho):=\left\{f\in A: Re\left(e^{i\xi}\frac{zf'(z)}{(1-\rho)f(z)+\rho zf'(z)}\right)>\gamma\cos\xi, z\in D\right\}$$

Definition 1.2.*For* $0\le\rho<1, 0\le\gamma<1$ *and* $|\xi|<\frac{\pi}{2}$, *define the class* $K(\xi,\gamma,\rho)$ *by*

$$K(\xi,\gamma,\rho):=\left\{f\in A: Re\left(e^{i\xi}\frac{zf''(z)+f'(z)}{f'(z)+\rho zf''(z)}\right)>\gamma\cos\xi, z\in D\right\}.$$

Next, Murugusundramoorthy [25,26], provided sufficient conditions for function f to be in the above classes.

Lemma 1.1. *Function* f *given by (1.1) is a member of* $S(\xi,\gamma,\rho)$ *if*

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]|a_n|\le 1-\gamma$$

where $|\xi|<\frac{\pi}{2}, 0\le\rho<1, 0\le\gamma<1$.

Lemma 1.2. *Function f given by (1.1) is a member of $K(\xi,\gamma,\rho)$ if*

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]|a_n|\le 1-\gamma,$$

where $|\xi|<\frac{\pi}{2}, 0\le\rho<1, 0\le\gamma<1$.

Definition 1.3. *Function $f\in A$ is said to be in class $R^{\tau}(\vartheta,\delta)$, where $\tau\in C\setminus\{0\}$, $0<\vartheta\le 1$, and $\delta<1$, if it satisfies the inequality*

$$\left|\frac{(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}{2\tau(1-\delta)+(1-\vartheta)\frac{f(z)}{z}+\vartheta f'(z)-1}\right|<1, z\in D.$$

Lemma 1.3. *If $f\in R^{\tau}(\vartheta,\delta)$ is of the form* (1.1), *then*

$$|a_n|\le\frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)}, n\in N\setminus\{1\} \tag{1.3}$$

The bound given in (1.3) is sharp for

$$f(z)=\frac{1}{\vartheta z^{1-\frac{1}{\vartheta}}}\int_0^z t^{1-\frac{1}{\vartheta}}\left[1+\frac{2(1-\delta)\tau t^{n-1}}{1-2^{n-1}}\right]dt.$$

Next, we obtain sufficient conditions for the function $E_{\alpha,\beta}^{\theta}(z)$ to be in the classes $S(\xi,\gamma,\rho)$ and $K(\xi,\gamma,\rho)$ respectively.

II. Inclusion Results for The Normalized (M-L)

We follow the same approach of authors in [28-30], who studied the two-parameter (M-L): $E_{\alpha,\beta}(z)$. Prior to proving our main results, we compute the following:

$$E_{\alpha,\beta}^{\theta}(1)-1=\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \tag{2.1}$$

$$\left(E_{\alpha,\beta}^{\theta}\right)'(1)-1=\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{(n-1)!\,\Gamma(\alpha(n-1)+\beta)}, \tag{2.2}$$

$$\left(E_{\alpha,\beta}^{\theta}\right)''(1)=\sum_{n=2}^{\infty}\frac{(\theta)_n\Gamma(\beta)}{(n-2)!\,\Gamma(\alpha(n-1)+\beta)}. \tag{2.3}$$

Theorem 2.1. *If*

$$[(1-\rho)\sec\xi+\rho(1-\gamma)]\left(E_{\alpha,\beta}^{\theta}\right)'(1)+(1-\rho)(1-\gamma-\sec\xi)E_{\alpha,\beta}^{\theta}(1\le 2(1-\gamma), \tag{2.4}$$

then$E_{\alpha,\beta}^{\theta}\in S(\xi,\gamma,\rho)$.where$E_{\alpha,\beta}^{\theta}$ is defined by (1.2)

Proof. Because $E_{\alpha,\beta}^{\theta}$ are defined by (1.2), according to Lemma 1.1 it is sufficient to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)se\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma. \tag{2.5}$$

Because the left-hand side of inequality (2.5) could be written as

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$$J_1(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}\ [(1-\rho)sec\,\xi(n-1)+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}$$
$$=[(1-\rho)sec\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\ \frac{(\theta)_n\Gamma(\beta)}{(n-1)!\,\Gamma(\alpha(n-1)+\beta)}+(1-\rho)(1-\gamma$$
$$-sec\,\xi)\sum_{n=2}^{\infty}\ \frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)},$$

therefore, by using (2.1) and (2.2), we get

$$J_1(\xi,\gamma,\rho)=[(1-\rho)sec\,\xi+\rho(1-\gamma)]\left[\left(E^{\theta}_{\alpha,\beta}\right)'(1)-1\right]+(1-\rho)(1-\gamma-sec\,\xi)\left[E^{\theta}_{\alpha,\beta}(1)-1\right]$$
$$=[(1-\rho)sec\,\xi+\rho(1-\gamma)]\left(E^{\theta}_{\alpha,\beta}\right)'(1)+(1-\rho)(1-\gamma-sec\,\xi)E^{\theta}_{\alpha,\beta}(1)\ -(1-\gamma)$$

Thus, from assumption (2.4), it follows that $J_1(\xi,\gamma,\rho)\le 1-\gamma$, that is, (2.5) holds;, therefore, $E^{\theta}_{\alpha,\beta}\in S(\xi,\gamma,\rho)$. □

Theorem 2.2.*If*

$$[(1-\rho)sec\,\xi+\rho(1-\gamma)]\left(E^{\theta}_{\alpha,\beta}\right)''(1)+(1-\gamma)\left(E^{\theta}_{\alpha,\beta}\right)'(1)\le 2(1-\gamma), \tag{2.6}$$

then$E^{\theta}_{\alpha,\beta}\in K(\xi,\gamma,\rho)$. where$E^{\theta}_{\alpha,\beta}$ is defined by (1.2)

Proof. Using definition (1.2) of $E^{\theta}_{\alpha,\beta}$, in view of Lemma 1.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty}\ n[(1-\rho)(n-1)se\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma. \tag{2.7}$$

The left-hand side of inequality (2.7) could be written as

$$J_2(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}\ n[(1-\rho)(n-1)sec\,\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}$$
$$=[(1-\rho)sec\,\xi+\rho(1-\gamma)]\sum_{n=2}^{\infty}\ \frac{(\theta)_n\Gamma(\beta)}{(n-2)!\,\Gamma(\alpha(n-1)+\beta)}+(1-\gamma)\sum_{n=2}^{\infty}\ \frac{(\theta)_n\Gamma(\beta)}{(n-1)!\,\Gamma(\alpha(n-1)+\beta)},$$

and from (2.2) and (2.3) we get

$$J_2(\xi,\gamma,\rho)=[(1-\rho)\,sec\,sec\,\xi\ +\rho(1-\gamma)]\left(E^{\theta}_{\alpha,\beta}\right)''(1)+(1-\gamma)\left[\left(E^{\theta}_{\alpha,\beta}\right)'(1)-1\right].$$

Hence, assumption (2.6) implies that $J_2(\xi,\gamma,\rho)\le 1-\gamma$ that is (2.7) holds, and consequently, $E^{\theta}_{\alpha,\beta}\in K(\xi,\gamma,\rho)$. □

III. Inclusion Results for The Image ofa Linear Operator

First, we introduce the following linear operator $\Lambda^{\alpha}_{\beta}: A\to A$ by the means of Hadamard product

$$\Lambda^{\alpha}_{\beta}f(z):=f(z)*E^{\theta}_{\alpha,\beta}(z)=z+\sum_{n=2}^{\infty}\ \frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}a_nz^n, z\in D.$$

Next, we explore the sufficient conditions for the images of the linear operator Λ^{α}_{β} on functions of the class $R^{\tau}(\vartheta,\delta)$. Thus, we provide sufficient conditions such that these images are in the classes $S(\xi,\gamma,\rho)$ and $K(\xi,\gamma,\rho)$, respectively.

Theorem 3.1.*For $f(z)\ \in A$. If*

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$$\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\sec\xi+\rho(1-\gamma)]\left[E^{\theta}_{\alpha,\beta}(1)-1\right]+(1-\rho)(1-\gamma-\sec\sec\xi)\int_0^1\left(\frac{E^{\theta}_{\alpha,\beta}(t)}{t}-1\right)dt\le 1-\gamma \tag{3.1}$$

then

$$\Lambda^{\alpha}_{\beta}(R^{\tau}(\vartheta,\delta))\subset S(\xi,\gamma,\rho).$$

Proof. Let $f\in R^{\tau}(\vartheta,\delta)$ be of the form (1.1). To prove that $\Lambda^{\alpha}_{\beta}(f)\in S(\xi,\gamma,\rho)$, from Lemma 1.1 it is necessary to show that

$$\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}|a_n|\le 1-\gamma.$$

We denote the left-hand side of the above inequality by

$$J_3(\xi,\gamma,\rho):=\sum_{n=2}^{\infty}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}|a_n|.$$

Because $f\in R^{\tau}(\vartheta,\delta)$, by Lemma 1.3 we have

$$|a_n|\le\frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)},n\in N\setminus\{1\},$$

and using the inequality $1+\vartheta(n-1)\ge\vartheta n$ we obtain

$$\begin{aligned}J_3(\xi,\gamma,\rho)&\le\frac{2|\tau|(1-\delta)}{\vartheta}\{\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\times\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\}\\&=\frac{2|\tau|(1-\delta)}{\vartheta}\{\sum_{n=2}^{\infty}[(1-\rho)\sec\xi+\rho(1-\gamma)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}+(1-\rho)(1-\gamma\\&\quad-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\}\end{aligned}$$

From the above inequality, using (2.1), we get

$$J_3(\xi,\gamma,\rho)\le\frac{2|\tau|(1-\delta)}{\vartheta}[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E^{\theta}_{\alpha,\beta}(1)-1\right]+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E^{\theta}_{\alpha,\beta}(t)}{t}-1\right)dt,$$

hence, the assumption (3.1) implies then $J_3(\xi,\gamma,\rho)\le 1-\gamma$, that is $\Lambda^{\alpha}_{\beta}(f)\in S(\xi,\gamma,\rho)$. □

Using Lemma 1.2 and following the same procedure as in the proof of Theorem 2.2, we obtain the following result:.

Theorem 3.2. *For* $f(z)\in A$. *If*

$$\frac{2|\tau|(1-\delta)}{\vartheta}\{[(1-\rho)\sec\sec\xi+\rho(1-\gamma)]\left(E^{\theta}_{\alpha,\beta}\right)'(1)+(1-\rho)(1-\gamma-\sec\sec\xi)E^{\theta}_{\alpha,\beta}(1)-(1-\gamma)\}\le 1-\gamma \tag{3.2}$$

then

$$\Lambda^{\alpha}_{\beta}(R^{\tau}(\vartheta,\delta))\subset K(\xi,\gamma,\rho)$$

Proof. Let $f\in R^{\tau}(\vartheta,\delta)$ be of the form (1.1). In view of Lemma 1.2, to prove that $\Lambda^{\alpha}_{\beta}(f)\in K(\xi,\gamma,\rho)$ we must show that

$$\sum_{n=2}^{\infty}n[(1-\rho)(n-1)\sec\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}|a_n|\le 1-\gamma. \tag{3.3}$$

Because $f\in R^{\tau}(\vartheta,\delta)$, then by Lemma 1.3 we have

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$$|a_n| \leq \frac{2|\tau|(1-\delta)}{1+\vartheta(n-1)}, n \in N \setminus \{1\},$$

and $1+\vartheta(n-1) \geq \vartheta n$. Denoting the left-hand side of the inequality (3.3) by

$$J_4(\xi,\gamma,\rho) := \sum_{n=2}^{\infty} n[(1-\rho)(n-1)se\xi + (1-\gamma)(1+n\rho-\rho)] \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} |a_n|.$$

We deduce that

$$J_4(\xi,\gamma,\rho) \leq \frac{2|\tau|(1-\delta)}{\vartheta} \sum_{n=2}^{\infty} [(1-\rho)sec\,\xi(n-1) + (1-\gamma)(1+n\rho-\rho)] \times \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}$$
$$= \frac{2|\tau|(1-\delta)}{\vartheta} \{[(1-\rho)sec\,\xi + \rho(1-\gamma)] \sum_{n=2}^{\infty} \frac{n(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} + (1-\rho)(1-\gamma - sec\,\xi) \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\}$$

Now, using (2.1) and (2.2), the above inequality yields

$$J_4(\xi,\gamma,\rho) \leq \frac{2|\tau|(1-\delta)}{\vartheta} \{[(1-\rho)sec\,\xi + \rho(1-\gamma)] \left[\left(E_{\alpha,\beta}^{\theta}\right)'(1) - 1\right] + (1-\rho)(1-\gamma - sec\,\xi)\left[E_{\alpha,\beta}^{\theta}(1) - 1\right]\}$$
$$= \frac{2|\tau|(1-\delta)}{\vartheta} \{[(1-\rho)sec\,\xi + \rho(1-\gamma)]\left(E_{\alpha,\beta}^{\theta}\right)'(1) + (1-\rho)(1-\gamma - sec\,\xi)E_{\alpha,\beta}^{\theta}(1) - (1-\gamma)\}.$$

Therefore, assumption (3.2) yields $J_4(\xi,\gamma,\rho) \leq 1-\gamma$, which implies inequality (3.3), that is $\Lambda_{\beta}^{\alpha}(f) \in K(\xi,\gamma,\rho)$. □

IV. Inclusion Results for The Alexander Integral Operator

Theorem 4.1. *For the Alexander Integral Operator* Ψ_{β}^{α} *be given by*

$$\Psi_{\beta}^{\alpha}(z) = \int_0^z \frac{E_{\alpha,\beta}^{\theta}(t)}{t} dt, z \in D \quad (4.1)$$

$$\Psi_{\beta}^{\alpha}(z) = \int_0^z \frac{E_{\alpha,\beta}^{\theta}(t)}{t} dt, z \in D.$$

If

$$[(1-\rho)sec\,\xi + \rho(1-\gamma)]\left(E_{\alpha,\beta}^{\theta}\right)'(1) + (1-\rho)(1-\gamma - sec\,\xi)E_{\alpha,\beta}^{\theta}(1) \leq 2(1-\gamma),$$

then $\Psi_{\beta}^{\alpha} \in K(\xi,\gamma,\rho)$.

Proof. Since

$$\Psi_{\beta}^{\alpha}(z) = z + \sum_{n=2}^{\infty} \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \cdot \frac{z^n}{n}, z \in D. \quad (4.2)$$

According to Lemma 1.2, it is sufficient to prove that

$$\sum_{n=2}^{\infty} n[(1-\rho)(n-1)sec\,\xi + (1-\gamma)(1+n\rho-\rho)] \frac{1}{n} \cdot \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \leq 1-\gamma$$

or, equivalently

$$\sum_{n=2}^{\infty} [(1-\rho)(n-1)sec\,\xi + (1-\gamma)(1+n\rho-\rho)] \frac{(\theta)_n \Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)} \leq 1-\gamma$$

The proof of Theorem 4.1 is parallel to that of Theorem 2.1. □

Theorem 4.2. *Let the function Ψ_β^α be given by (4.1). If*

$$[(1-\rho)\sec\sec\xi+\rho(1-\gamma)]\left(E_{\alpha,\beta}^{\theta}(1)-1\right)+(1-\rho)(1-\gamma-\sec\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt\le 1-\gamma, \tag{4.3}$$

then $\Psi_\beta^\alpha\in S(\xi,\gamma,\rho)$.

Proof. Because Ψ_β^α has the power series expansion (4.2), then by Lemma 1.1 it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}\le 1-\gamma.$$

The left-hand side of the above inequality could be rewritten as

$$J_5(\xi,\gamma,\rho)=\sum_{n=2}^{\infty}\frac{1}{n}[(1-\rho)(n-1)\sec\xi+(1-\gamma)(1+n\rho-\rho)]\times\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}$$
$$=\sum_{n=2}^{\infty}[(1-\rho)\sec\xi+\rho(1-\gamma)]\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}+(1-\rho)(1-\gamma-\sec\xi)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{(\theta)_n\Gamma(\beta)}{n!\,\Gamma(\alpha(n-1)+\beta)}$$

and using (2.1), we get

$$J_5(\xi,\gamma,\rho)\le[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}^{\theta}(1)-1\right]+(1-\rho)(1-\gamma-\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}^{\theta}(t)}{t}-1\right)dt.$$

Therefore, if the assumption (4.3) holds, then $J_5(\xi,\gamma,\rho)\le 1-\gamma$. Hence, $\Psi_\beta^\alpha\in S(\xi,\gamma,\rho)$. □

V. Conclusions

In this study, we normalized the generalized (M-L) to deduce the analytic form $E_{\alpha,\beta}^{\theta}$, that we investigated its inclusion results in the subclasses the classes $S(\xi,\gamma,\rho)$ and $K(\xi,\gamma,\rho)$. In addition, we discuss sufficient conditions for the linear operator $\Lambda_\beta^\alpha(f), f\in R^\tau(\vartheta,\delta)$ to be a member of the same subclasses, i.e. $S(\xi,\gamma,\rho)$ and $K(\xi,\gamma,\rho)$. Finally, the investigation has been extended to involve Alexander operator Ψ_β^α by the means of Hadamard product.

Conflict of Interest

The author has no conflicts of interest to declare.

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