# Mathematical Model for the Determination of Optimal Salary in Defined Benefit Pension Plan with Early Retirement, An Application of Smooth-Pasting Condition

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**Abstract:** The paper seeks to apply the mathematical model formulated by Adaji, et al (2015) to determine the optimal financial value of the retirement benefit $S_f$ by applying Smooth Pasting condition. We examined the smooth pasting condition (tangency condition) along the optimal point at the boundary for an American put by considering the gradient, $\frac{\partial V}{\partial S}$more closely and found that as long as $V(S)$ coincides with the straight line, $K - S$ gradient equals$-1$. We used Microsoft excel spread sheet facility to perform the individual computation for the last 25 years of service for University Senior Lecturers and the last 10 years of service for University Professors. The optimal salary $S_f$ was determined by drawing a vertical line from the tangent perpendicularly to the horizontal axis (salary axis). We recommend the applictaion of this model to individuals as well as cooperate organisations so as maximise the expected retirement benefit. Early retirement alternate should be encouraged so that the teeming population of unemployed youths can take up the vacancies created as a result of early retirement.

**Keyword**:Smooth Pasting, Optimal Stopping, Value Function and Gain Function

## I. Introduction

There comes a time in our lives when we as employees consider starting a pension plan - either on the advice of a friend, a relative, on our own volition or by law. The choice of plan may depend on various factors, such as the age and salary of the individual, number of years of expected employment, as well as options to retire early or late.

Calvo-Garridoand Vazquez (2012) presented a partial differential equation (PDE) model governing the value of a defined pension plan without the option for early retirement. They say that it is important to develop mathematical models to compute the value of this liability in order to estimate the financial situation of the institution or company that has the obligation with the member of pension plan.

Calvo-Garridoand Vazquez (2012) proposed future work concerning the possibility of early retirement. According to them, linear complementarity formulations of the resulting variational inequality can be analyzed to obtain the existence of solutions, and suitable numerical methods are required to obtain the pension plan value.

American options are financial derivatives that can be exercised at any time before maturity, in contrast to European options which have fixed maturities. The prices of American options are evaluated as an optimization problem, in which one has to find the optimal time to exercise in order to maximize the claim option payoff.The smooth pasting property (condition), states that the value function must be continuously differentiable everywhere, and yields conditions, which uniquely determine the optimal stopping region. Art of smooth pasting includes a heuristic justification for the differentiability of value functions at optimal stopping thresholds. In pure stopping problems, “smoothness” requires (and means) that the value functionsis once differentiable, and is known as the smooth pasting condition.

## II. Literature Review

Based on the work of Hu et al. (2018) with an additional stochastic payoff function, Sun (2021) characterized the value function for the continuous problem via the theory of quadratic reflected backward stochastic differential equations (BSDEs for short) with unbounded terminal condition. In regard to the discrete problem, Sun (2021)gets the discretization form composed of piecewise quadratic BSDEs recursively under Markovian framework and the assumption of bounded obstacle, and provides some useful a priori estimates about the solutions with the help of an auxiliary forward-backward SDE system and Malliavin calculus. Finally, she obtains the uniform convergence and relevant rate from discretely to continuously quadratic reflected BSDE, which arise from corresponding optimal investment stopping problem through above characterization.

Optimal stopping problems are linked to free boundary problems. This connection was discovered by McKeen (1965) and it was formulated as a free boundary problem that can be solved, an extra condition is needed. The principle of smooth pasting provides this condition. It was first adopted by Oksendai (2000) and was studied in greater depth by Shreve (2000). Smooth pasting (also called high-contact condition) is a kind of boundary condition used to model the American option (Seyde, 2021). It tells that the American option value is maximized by an exercise strategy that makes the option value and option delta continuous (Wilmot, 2021).

Cox and Hoeggerl (2013) consider the pricing of American put options in a model-independent setting that is, they do not assume that asset prices behave according to a given model, but aim to draw conclusions that hold in any model. They incorporate market information by assuming that the prices of European options are known. Strulovici and Szydlowski (2012) prove that under standard Lipschitz and growth conditions, the value function of all optimal control problems for one-dimensional diffusions is twice continuously differentiable, as long as the control space is compact and the volatility is uniformly bounded below, away from zero. Under similar conditions, the value function of any optimal stopping problem is continuously differentiable. They also provide sufficient conditions for the existence of an optimal control, which is also shown to be Markov. These conditions are based on the theory of monotone comparative statics. It was Bensoussa (1984), and later Karatzas (1988), that first used no-arbitrage methods to show that the price of the American put is the solution to an optimal stopping problem. Adaji, et al (2015) formulated a mathematical model to determine the optimal financial value of the retirement benefit $S_f$ by applying Smooth Pasting condition.This work followed that of McKeen (1965) who was the first to derive a free boundary problem for the 'discounted' American call with gain function$\Phi(S) = e^{-r\tau}(S-K)^+$

### III. Assumptions of the Model

a) The model satisfies smooth pasting condition

b) A member of the plan would retire when he/she maximizes the benefits of retirement among all possible dates (stopping times) to retire.

c) Optimal stopping problem with a value function $V(S_t) = \sup_{\tau \le T} E_S e^{-\tau r} V_\tau(K - S_t)$ satisfies geometric Brownian motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$

d) The infinitesimal generator of the (strong) Markov process S is given by$\mathbb{L}_S V = rS\frac{\partial}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2}$.

e) Standard Markovian arguments suggest that $V$ from assumption (g) solves the following free boundary problem of parabolic type$\mathbb{L}_S V = rV$

### 3.1 Parameters and Variables of the Model

The following parameters (functions) and variables are used in this research work:

$V = V(S, t) = V(S_t)$is the financial value of retirement benefit in the time interval $0 < t \le T$;

$S = S_t =$ salary at time, $t$;

$S_f =$ Optimal salary;

$K$= strike salary;

$S - K =$ Payoff for the call option (American call option for a fixed $K$ and any given salary, $S$) representing an employer's option

$K - S =$ Payoff for the put option (American put option for a fixed $K$ and any given salary, $S$) representing an employee's option

$(S-K)^+ = \max_S(S-K, 0)$assumed to occur at the optimal boundary (call option)

$(K-S)^+ = \min_S(K-S, 0)$assumed to occur at the optimal boundary (put option)

$V(S_f)$= Optimal financial value of retirement benefit (is also the same as the optimal retirement benefits) with respect to salary

$t =$ Time (in year) spent with the pension plan

$r$ =the salary growth rate or Accrual rate

$\mu = (r - \delta)$ is the expected return of the salary (asset)

$\delta$ =annual dividend yield $\delta \geq 0$ of the asset (salary) (when $\delta = 0$, then $\mu = r$)

$\sigma =$ The volatility of the salary (also the standard deviation)

$T$ =Worker's expected retirement time (maturity or expiry time) in years

$\tau$= stopping time

$\tau_f$= optimal stopping time

$\boldsymbol{C}, \mathcal{D} =$ continuation set and stopping set respectively

$W_t$ =geometric Brownian process, (Disturbance factor)

$k_1$ , $k_2$= arbitrary constants,

$w_+$,$w_-$= respective positive and negative roots of an auxiliary equation

$\mathbb{R}^d = d -$ dimensionalEuclidean space

$\mathbb{L}_S$ = infinitesimal operator of S

$V_\tau$ = value function at stopping time, $\tau$,

$\Phi_\tau$ = gain function at stopping time, $\tau$,

$E_S$ =expectation with respect to $S$

$(\Omega, \mathcal{F}, \mathcal{P})$ =probability space

$\mathcal{F}_t$ =filtration

**Remark 1**: **The Shift Operator**

The shift operator is useful in defining the (strong) Markov property.

**The Measure $P_S$**

Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion under the measure $P$. Thus each $W_t$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $W_0 = 0$ under $P$. Now define $S_t = S + W_t$, for all $0 \leq t < \infty$. Then $S_t$ is a random variable on the same probability space. Moreover, we see that $S_0 = S$ under $P$.

**Definition 1.**

Let $\Omega$ be some space of functions from $[0, \infty]$into $\mathbb{R}$. The shift operator $\theta_t : \Omega \to \Omega$ defined by $\big(\theta_t(\omega)\big)(s) = \omega(t + s)$

for $\omega \epsilon \Omega$ (Typically, we regard $\omega \epsilon \Omega$ as a sample path of some stochastic process.) Suppose that$S = (S_t)_{t \geq 0}$ is a stochastic process on the probability space $\Omega, \mathcal{F}, \mathcal{P}$ the following useful results are given without proofs.

**Definition 2.**

If a process $S = (S_t)_{t \geq 0}$is equipped with the filtration$((\mathcal{F}_t)_{t \geq 0}$, with $\mathcal{F} = \sigma(\cup_{t \geq 0} \ \mathcal{F}_t)$, then $S$ has the (strong) Markov property if any of the following three equivalent conditions hold

$$\mathbb{E}_S(f(X_{\tau+h})/\mathcal{F}_t) = \mathbb{E}_S(f(S_{\tau+h})|S_\tau) \qquad (1)$$

$$\mathbb{E}_S(S_{\tau+h})|\mathcal{F}_\tau) = \mathbb{E}_{S_\tau}(f(S_h)) \qquad (2)$$

$$\mathbb{E}_S(Y \circ \theta_\tau|\mathcal{F}_\tau) = \mathbb{E}_{S_\tau}(Y) \qquad (3)$$

for all $S$, all stopping times $\tau$, all $h > 0$, any bounded Borel-measurable function $f$, and any (bounded) $\mathcal{F}$-measurable random variable $Y$.

**Remark 1:The (Strong) Markov Property**

The future behaviour of a Markov process is not dependent on its past, but only on its current value.

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**Definition 3**

Let $(S_t)_{t\geq 0}$ be an Ito diffusion with stochastic differential equation given by $dS_t = \mu S_t dt + \sigma S_t dW_t,$ (4)

Then the infinitesimal generator is given by $\mathbb{L}_S = \sum_{i=1}^{n} \mu_i(S)\frac{\partial}{\partial S_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma\sigma_{i,j}^T(S)\frac{\partial^2}{\partial S_i \partial S_j}$ (5)

It is important to note that for $n = 1$ the above infinitesimal generator of the strong Markov process $S$ becomes

$$\mathbb{L}_S = \mu(S)\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(S)\frac{\partial^2}{\partial S^2} \quad (6)$$

which may be written as $\mathbb{L}_S = \mu S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}.$ (7)

**Remark 2: The Infinitesimal Generator**

The infinitesimal generator enables us to associate a second order partial differential operator with a stochastic process.

**Definition 4.**

A function $V$ is called *superharmonic*if$\mathbb{E}_S(V(S_\tau)) \leq V(S)$ (8)

for all $S$, and for every stopping time $\tau$. If the process $S$ has the (strong) Markov property and is adequately regular, this is equivalent to saying that the process $(V(S_t))_{t\geq 0}$ is a supermartingaleunder $P_S$, for each S.

**Remark 3**:

Dynkin'superharmonic characterisation of the value function for Markov processes is captured in the four statements of the following theorem.

**Theorem 1.**

*Let*$V(S) = \min_{\tau\in\mathcal{T}} \mathbb{E}_x(\Phi(S_\tau)$ (9)

where$S = (S_t)_{t\geq 0}$ is a (strong) Markov process, started at $S$ under $P_S$, and $T$ is a set of stopping times. Suppose that the stopping time $\tau_f \in \mathcal{T}$is optimal, that is

$$V(S) = \mathbb{E}_S(\Phi\left(S_{\tau_f}\right)).$$

Then (under certain regularity conditions):

i. *The value function*$V$ *is the smallest superharmonic function which dominates the gain function* $\Phi$.

ii. $\tau_{\mathcal{D}} \leq \tau_f P_{S-a.s.,}$ where the *stopping time* $\tau_{\mathcal{D}}$ *is defined by*$\tau_{\mathcal{D}} = inf\{t \geq 0 | S_t \in \mathcal{D}\}$ (10)

*and where* $\boldsymbol{C} = \{S \,|V(S) > \Phi(S)\}$*and* $\mathcal{D} = \boldsymbol{C}^c = \{S \,|V(S) = \Phi(S)\}$

iii. *The stopping time*$\tau_{\mathcal{D}}$*defined in (10) is optimal.*

**IV. Proposition 1: Optimum time for Exercising**

There exists $S_f$ such that early exercise is worthwhile for $S \leq S_f$, but not for $S > S_f$.

**Proof**

Let $\pi = V + S$ be a portfolio. As soon as

$V = (K - S)^+$ the option can be exercised since the amount

$\pi = (K - S)^+ \quad + S = K$at interest rate, r.

For $V > K - S$it is not worthwhile since the value of portfolio is

$\pi = V + S > (K - S) + S \geq K$but after exercising it is equal to $K$

The value $S_f$depends on time, and it is termed the free boundary value. We have

$V(S,t) = (K-S)^+, \quad S \le S_f(t)$

$V(S,t) > \ (K-S) > S_f(t)$

Since the free-boundary value is not known, it must be determined with the option price.

For large S, the put option satisfies the Black – Scholes equation (Merton, 1973).

That is, $S >> K$,

$V(S,t) \underset{S\to\infty}{\longrightarrow} 0$and

$V\big(S_f(t),t\big) = K - S_f(t)$

Additional conditions are required as these are not sufficient.

These are

$S \to \frac{\partial V}{\partial S}(S,t)$is continuous at $S = S_f(t)$

Since for $\quad S < S_f(t)$

$\frac{\partial V}{\partial S}(S,t) = \frac{\partial}{\partial S}(K-S) = -1$also$\frac{\partial V}{\partial S}\big(S_f(t),t\big) == \frac{\partial}{\partial S_f}\big(K-S_f\big) - 1$

This is smooth pasting condition.

According to Merton (1973), American Put option can be determined by solving

$$S \le S_f(t): \qquad V(S,t) = (K-S)^+$$

$$S > S_f(t): \qquad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + (r-\delta)\frac{\partial V}{\partial S} - rV = 0$$

With the endpoint condition

$$V(S,T) = (K-S)$$

And the boundary conditions

$$\lim_{S\to\infty} V(S,t) = 0$$

$$V\big(S_f(t),t\big) = K - S_f(t),$$

$$\frac{\partial V}{\partial S}\big(S_f(t),t\big) = -1$$

Our formulation follows perpetual American put option. Perpetual American put option has value function that is a function of the stock price, $V = V(S)$ only and its optimal stopping boundary is a constant function.

So, the boundary conditions become

$$\lim_{S\to\infty} V(S) = 0$$

$$V\big(S_f\big) = K - S_f,$$

$$\frac{\partial V}{\partial S}\big(S_f\big) = -1 \qquad (11)$$

Our formulation follows Perpetual (infinite) American put option. Perpetual American put option has value function that is a function of the stock price, $V = V(S)$ only and its optimal stopping boundary is a constant function.

**4.1 Derivation of the Solution by the Markovian Method**

Adaji, et al (2015) formulated a mathematical model to determine the financial value of retirement benefit, $V(S)$ (free arbitrage price or the option value) for perpetual American put. The study considers one pricing formulation of American options, namely, the optimal stopping formulation as equivalence of a free-boundary problem. The optimal stopping problem on perpetual

American put was formulated and its solutions found. The solutions found were analysed systematically by applying matching value condition, smooth pasting condition, asset equilibrium condition and the boundary condition. They used the free-boundary approach to derive the solution. The model was based on final salary given by

$$V(S) = \begin{cases} S^{\frac{-2r}{\sigma^2}}\left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} & \text{for } S \geq S_f \\ K - S & \text{for } S < S_f \end{cases} \quad (12)$$

where $K - S$ is the payoff in the case of American put option at any given value of salary, S.

The equation (12) is used to determine the financial value of retirement benefit, $V(S)$ (free arbitrage price or the option value) for perpetual American put. It can only be determined if its optimal value, $S_f$ is known. $V(S_f)$is the value of the pension benefit that is optimal to retire. Our task here (amongst other things) is to use (12) to determine the financial value of the retirement benefit $S_f$ by applying Smooth Pasting condition amongst other financial formulations.

We would like to examine the smooth pasting condition (tangency condition) along the optimal point at boundary for an American put. At $S = S_f$, the value of the optimal point of American put is $K - S_f$. This is termed as the value matching condition:

$$V(S_f)=K - S_f. \quad (13)$$

Suppose $S_f$is a known continuous function, the pricing model becomes a boundary value problem with a time dependent boundary. However, in the American put option model, $S_f$ is not known in advance. Rather, it must be determined as part of the solution.

To be able to calculate the unknown boundary$S_f$, we need the smooth pasting condition, and therefore we consider the gradient, $\frac{\partial V}{\partial S}$more closely. From our assumption that the model satisfies smooth pasting condition, it then implies that as long as $V(S)$ coincides with the straight line, $K - S$ with gradient equals $-1$ , and at the contact point, we will draw a vertical line from the tangent perpendicular to the horizontal axis to obtain $S_f$.

**V. Results**

The results obtained from computer programme of equations (12) in Chapter 3 are presented in Tables 1, 2, …, 6 and the corresponding graphs obtained from these tables are presented in Figures 1, 2, ..., 6 to illustrate the performance of the model. For this purpose, the following model parameters are presented.

We show how the values are simulated using Excel package. The data we use in the computation are from Appendix B1.

$$V(S) = \begin{cases} S^{\frac{-2r}{\sigma^2}}\left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} & \text{for } S \geq S_f \\ K - S & \text{for } S < S_f \end{cases}$$

We use Microsoft excel spread sheet facility to perform the individual computation for 25 years for University senior lecturers and 10 years for University Professors as shown in Tables 1 to 3.

This computation is applicable to various cadres of staff and institutions.

Substituting these parameters into the above programme, we have $= POWER(Ai, -D1) * (1/D1) * POWER(D2/(1 + D1), (D1 + 1))$

where $i = 1, 2, 3, \ldots, 25$.

We compute financial values of retirement benefit for Senior lecturers in the last 25 years of service using simulation. For this category, we use

$D1 = \frac{2r}{\sigma^2} = \frac{2(2.5)}{1^2} = 5;$

$D2 = K = 6020349;$

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$S_0 = 3091505.$

We use the process to compute financial value of retirement benefit for Professors in the last 10 years of service. For this category, we use

$D1 = \frac{2r}{\sigma^2} = \frac{2(2.5)}{1^2} = 5;$

$D2 = K = 6020349;$

$S_0 = 4580349.$

Table 1:*Data of Financial Value, V(S) of Retirement* for University Senior Lecturers in the last 25 years of service when $K = 6020349$, $S_0 = 3091505,\ r = 2.5\sigma = 1$

| S | V=V(S) | V= (K-S) | X= | 0.5 |
|---|---|---|---|---|
| 3091505 | 158881.3024 | 2928844 | K= | 6020349 |
| 3216505 | 143892.1485 | 2803844 | | |
| 3341505 | 130810.4339 | 2678844 | | |
| 3466505 | 119335.0713 | 2553844 | | |
| 3591505 | 109221.0848 | 2428844 | | |
| 3716505 | 100267.6579 | 2303844 | | |
| 3841505 | 92309.02575 | 2178844 | | |
| 3966505 | 85207.46781 | 2053844 | | |
| 4091505 | 78847.86872 | 1928844 | | |
| 4216505 | 73133.46075 | 1803844 | | |
| 4341505 | 67982.46543 | 1678844 | | |
| 4466505 | 63325.42499 | 1553844 | | |
| 4591505 | 59103.06732 | 1428844 | | |
| 4716505 | 55264.58669 | 1303844 | | |
| 4841505 | 51766.25086 | 1178844 | | |
| 4966505 | 48570.26598 | 1053844 | | |
| 5091505 | 45643.84668 | 928844 | | |
| 5216505 | 42958.45013 | 803844 | | |
| 5341505 | 40489.14222 | 678844 | | |
| 5466505 | 38214.07047 | 553844 | | |
| 5591505 | 36114.02384 | 428844 | | |
| 5716505 | 34172.06353 | 303844 | | |
| 5841505 | 32373.21208 | 178844 | | |
| 5966505 | 30704.19051 | 53844 | | |
| 6091505 | 29153.19528 | -71156 | | |

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Table 2:Data of Financial Value, V(S) of Retirement *for University Senior Lecturers in the last 25 years of service when* $K = 5491505, \quad S_0 = 3091505,\ r = 2.5\sigma = 1$

| | S | V=V(S) | V= (K-S) | X= | 5 |
|---|---|---|---|---|---|
| 1 | 3091505 | 722.7744 | 2928844 | K= | 5491505, |
| 2 | 3216505 | 592.8316 | 2803844 | | |
| 3 | 3341505 | 489.9389 | 2678844 | | |
| 4 | 3466505 | 407.7494 | 2553844 | | |
| 5 | 3591505 | 341.5625 | 2428844 | | |
| 6 | 3716505 | 287.8584 | 2303844 | | |
| 7 | 3841505 | 243.9751 | 2178844 | | |
| 8 | 3966505 | 207.8799 | 2053844 | | |
| 9 | 4091505 | 178.0070 | 1928844 | | |
| 10 | 4216505 | 153.1403 | 1803844 | | |
| 11 | 4341505 | 132.3278 | 1678844 | | |
| 12 | 4466505 | 114.8189 | 1553844 | | |
| **13** | **4591505** | **100.0178** | **1428844** | | |
| 14 | 4716505 | 87.44823 | 1303844 | | |
| 15 | 4841505 | 76.72742 | 1178844 | | |
| 16 | 4966505 | 67.54576 | 1053844 | | |
| 17 | 5091505 | 59.65154 | 928844 | | |
| 18 | 5216505 | 52.83897 | 803844 | | |
| 19 | 5341505 | 46.93905 | 678844 | | |
| 20 | 5466505 | 41.81227 | 553844 | | |
| 21 | 5591505 | 37.34297 | 428844 | | |
| 22 | 5716505 | 33.43486 | 303844 | | |
| 23 | 5841505 | 30.00743 | 178844 | | |
| 24 | 5966505 | 26.99308 | 53844 | | |
| 25 | 6091505 | 24.33489 | -71156 | | |

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Table 3: *Data of Financial Value, V(S) of Retirement* for University Senior Lecturers in the last 25 years of service when $K = 5000000, S_0 = 3091505,\ r = 2.5\sigma = 1$

| S/NO | S | V=V(S) | V= (K-S) | X= 5 | |
|---|---|---|---|---|---|
| 1 | 3091505 | 117295.1 | 2428844 | K=5520349 | |
| 2 | 3216505 | 106229.3 | 2303844 | | |
| 3 | 3341505 | 96571.60 | 2178844 | | |
| 4 | 3466505 | 88099.84 | 2053844 | | |
| 5 | 3591505 | 80633.13 | 1928844 | | |
| 6 | 3716505 | 74023.21 | 1803844 | | |
| 7 | 3841505 | 68147.70 | 1678844 | | |
| 8 | 3966505 | 62904.93 | 1553844 | | |
| 9 | 4091505 | 58209.92 | 1428844 | | |
| 10 | 4216505 | 53991.22 | 1303844 | | |
| 11 | 4341505 | 50188.47 | 1178844 | | |
| 12 | 4466505 | 46750.38 | 1053844 | | |
| 13 | 4591505 | 43633.20 | 928844 | | |
| 14 | 4716505 | 40799.42 | 803844 | | |
| 15 | 4841505 | 38216.75 | 678844 | | |
| 16 | 4966505 | 35857.29 | 553844 | | |
| 17 | 5091505 | 33696.85 | 428844 | | |
| 18 | 5216505 | 31714.34 | 303844 | | |
| 19 | 5341505 | 29891.36 | 178844 | | |
| 20 | 5466505 | 28211.77 | 53844 | | |
| 21 | 5591505 | 26661.40 | -71156 | | |
| 22 | 5716505 | 25227.73 | -196156 | | |
| 23 | 5841505 | 23899.72 | -321156 | | |
| 24 | 5966505 | 22667.56 | -446156 | | |
| 25 | 6091505 | 21522.52 | -571156 | | |

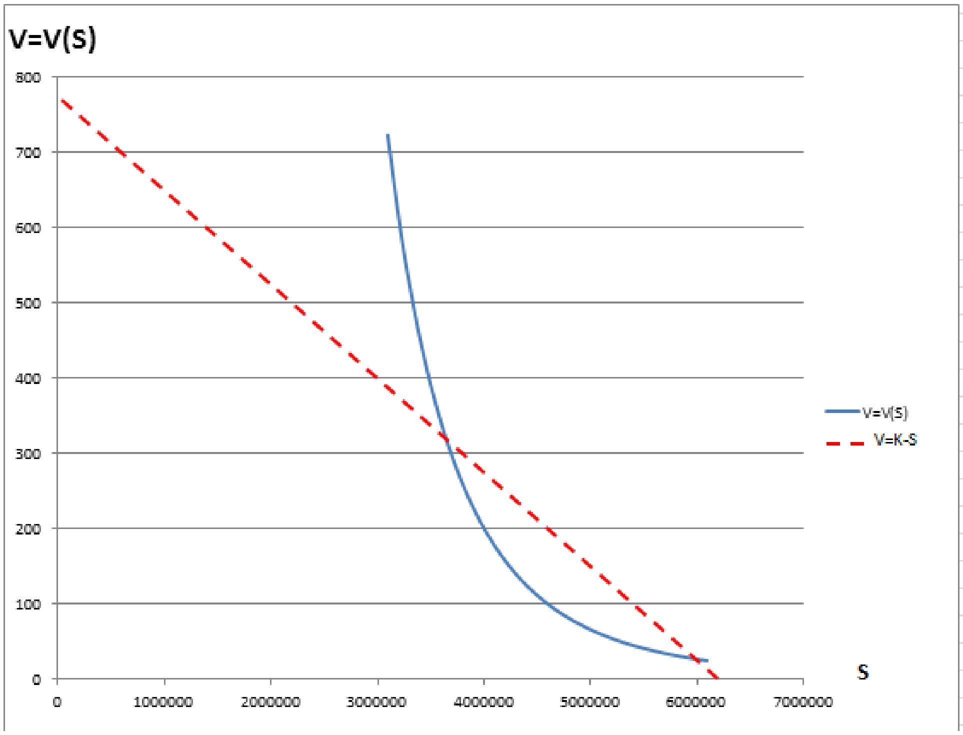


Figure 1:Graph of Financial Value, $V(S)$ of Retirement for University Senior Lecturers in the last 25 years of service when $K = 6020349, \quad S_0 = 3091505, \ r = 2.5, \sigma = 1$

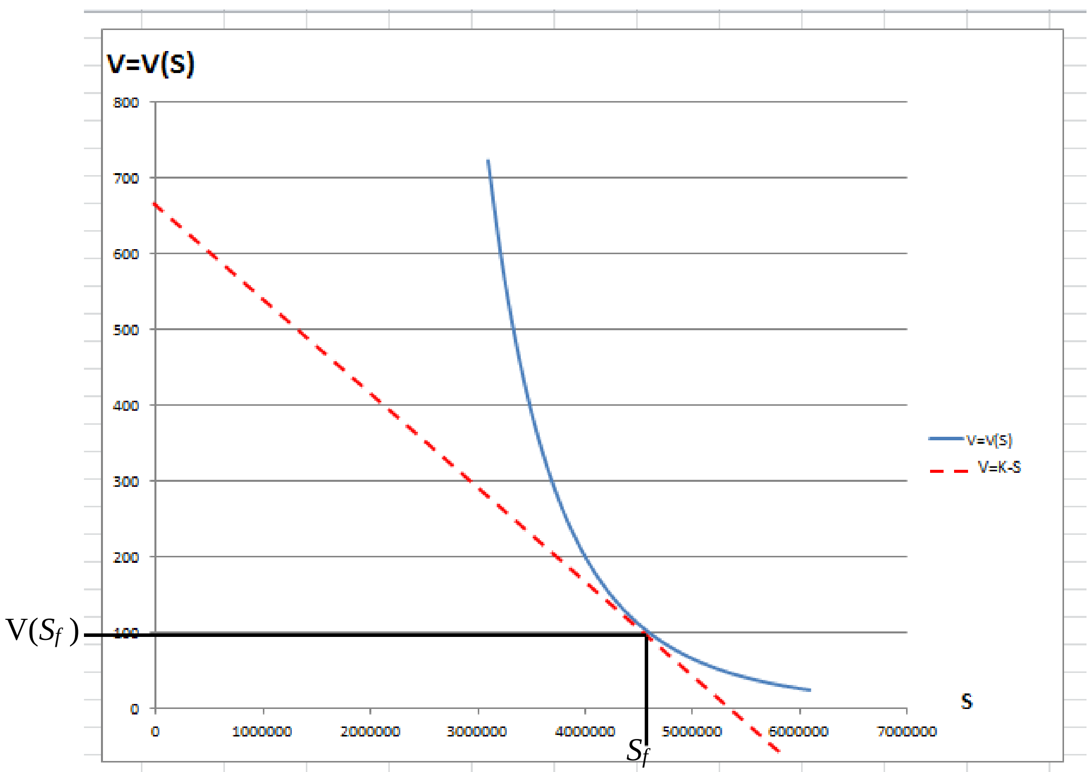


Figure 2:Graph of Financial Value, $V(S)$ of Retirement for University Senior Lecturers in the last 25 years of service when K=₦5491505,$S_0 = 3091505, \ r = 2.5, \sigma = 1$

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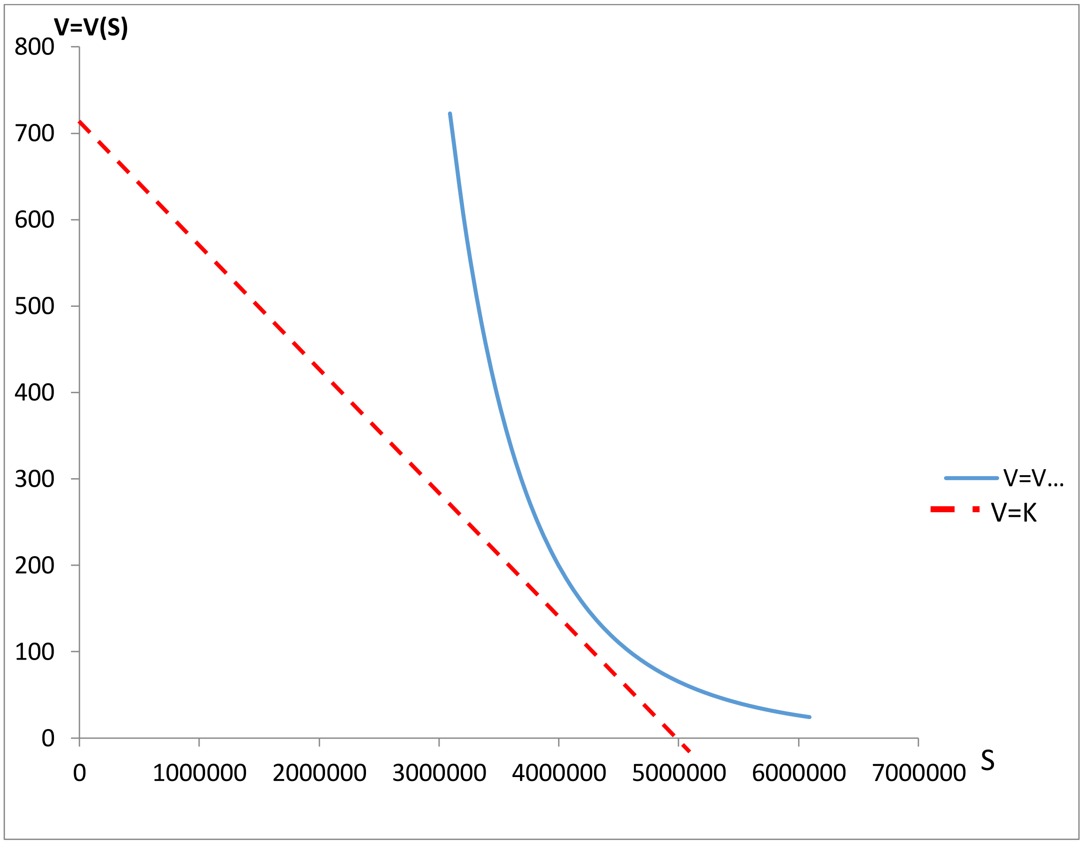


Figure 3:Graph of Financial Value, $V(S)$ of Retirement for University Senior Lecturers in the last 25 years of service when K= 500000,$S_0 = 3091505,\ r = 2.5, \sigma = 1$

**VI. Discussion**

In section 4, we validated the model by using empirical data from Consolidated University Salary Structure (CONUASS) to determine the optimal salary for early retirement for University Senior Lecturers in the last 25 years of service. Figures 1- 3 display the outputs of Tables 1-3 as we varied the values of K. We set $K = ₦6020349$ in Figure 1, K=₦5491505 in Figure 2, K=₦5000000 and in Figure 3, $K = ₦6020349$.

**Figure1**. The Figure displays the interaction of the salary curve and the straight line with negative slop. The line $(K - S)$ cuts the curve, $V = V(S)$ at two points. This means that we did not get the optimal point. This implies that the $K = 6020349$ is high and needed to be reduced.

**Figure 2:**It is a graph showing salary, $S$ on the horizontal axis and financial value of retirement benefit $V(S)$ of American put option on the vertical as for fixed strike salary, K=₦5491505 per annum and the payoff, $K - S$of American put option for University Senior lecturers in the last 25 years of service. $S_0 = 3091505$, $\frac{2r}{\sigma^2} = 5$. The value of the option meets the payoff function smoothly. The salary curve $V(S)$ touches tangentially the intrinsic line at the point representing the value of the optimal retirement salary, $S_f$. The optimal value of the financial retirement benefit is got by drawing a vertical line from the tangent point to the horizontal line. The vertical line intercepts the salary axis at $S_f$= 4591505 per annum). The corresponding vertical axis $V(S_f) = 100$.

**Figure 3:**This graph is showing salary, $S$ and financial value of retirement benefit$V(S)$ of American put option for fixed strike salary, K=₦5000000 per annum and the payoff, $K - S$of American put optionfor University Senior lecturers in the last 25 years of service. $S_0 = 3091505$, $\frac{2r}{\sigma^2} = 5$. The Figure 3 displays the interaction of the salary curve and the straight line with negative slop. The line $(K - S)$ and the curve, $V = V(S)$ do not have contact at all. This means that we did not get the optimal point. This implies that the $K = 5000000$ is low and needed to be increased.

The differences we observe in Figures 1 and 3, are direct consequences of varying $K - S$. We continue until we arrive at Figures 2 for the Senior Lecturers in the last 25 years of their service.

Figures 1 and 3 are not optimal while that of 2 is optimal for University Senior in the last 25 years of service.

## VII. Conclusion and Recommendations

### 7.1 Conclusion

In the examples, we used Microsoft excel spread sheet facility to perform the individual computation for 25 years for University senior lecturers and 10 years for University Professors as shown in Tables 1 to 3.

We assumed that the smooth pasting condition holds, this assumption was used to find the optimal stopping region, and verified that this solution equals the optimal value function. This is what is referred to as "Art of Smooth Pasting," which includes a heuristic justification for the differentiability of value functions at optimal stopping thresholds.

We demonstrated this by considering the behaviour of curve near $S_f(\tau)$. From our assumption that the model certifies smooth pasting condition, it then implies that as long as $V(S_t)$ coincides with the straight line, $K - S$ its gradient equals $-1$, and at the high contact point produces $S_f$(where $(K - S) = (K - S_f)$) we also have under the following two scenarios

$$\left.\frac{\partial V}{\partial S}\right|_{S=S_f(\tau)} < -1 \, , \left.\frac{\partial V}{\partial S}\right|_{S=S_f(\tau)} > -1$$

When $\left.\frac{\partial V}{\partial S}\right|_{S=S_f(\tau)} < -1$, the salary curve $V(S)$ at value of $S$ close to but greater than $S_f$ falls below the intrinsic value line (see the figure1).

When $\left.\frac{\partial V}{\partial S}\right|_{S=S_f(\tau)} > -1$, is above the intrinsic value line (see figure3). The value of the American put option at asset salary level close to $S_f(\tau)$ can be varied (increased by choosing a smaller value for $S_f(\tau)$). This explains why both cases do not correspond to the optimal exercise strategy.

In order to obtain optimality, we varied the value of $(K - S)$ which represents a straight line. We continued until the line touched the curve tangentially as seen in figure 2.

The (K-S) reveals the relation between the strike salary, $K$ (assumed to be greater than other salaries over the years). It has a negative slop which intercepts the salary axis (horizontal axis at $K$). The value of financial benefit (the vertical axis represents $V(S)$ and the horizontal axis represents the salary axis.

The model is people-friendly in application as individual employee who does not have a mathematical background can also use it with ease. The application can be extended to every employee's cadre. We also included the formula and the computation of gratuity in Tables 4 and 5.

### 7.2 Recommendations

We recommend the appliction of this model to individuals as well as cooperate organisations so as maximise the expected retirement benefit. This can only to achieved if we know the optmal point to retire.

We recommend that government at the federal, state and local level make public the salary growth rate as well as the strike salary of every cadre. Early retirement alternative should be encouraged so that the teeming population of unemployed youths can take up the vacancies created as a result of early retirement.

As was stated, this work is concerned with one dimension, however, it can be extended to higher dimensions. This is open to further research.

A computer programme could be developed to authomaticaly compute model equation (12) to determine the optimal salary instead of hearustic justification by the smooth pasting approach

**Contributions:** Investigation, Matthew Adaji; Writing – original draft, Matthew Adaji; Writing – review & editing, Matthew Adaji.

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## APPENDICES

### Appendix 1

Table 4:Computation of Expected Benefits under the Defined Benefit Pension Plan

| Annual Salary | Gratuity {S+S(N-5)(0.08)} for 4< N | Years of service (N) | Annual Pension ={S+S(N-10)(0.02)} for N >10 |
|---|---|---|---|
| 3091505 | | 1 | |
| 3216505 | | 2 | |
| 3341505 | | 3 | |
| 3466505 | | 4 | |
| 3591505 | 3878825.4 | 5 | |
| 3716505 | 4013825.4 | 6 | |
| 3841505 | 4456145.8 | 7 | |
| 3966505 | 5235786.6 | 8 | |
| 4091505 | 5400786.6 | 9 | |
| 4216505 | 5903107 | 10 | |
| 4341505 | 6772747.8 | 11 | 4216505 |
| 4466505 | 6967747.8 | 12 | 4428335.1 |
| 4591505 | 7530068.2 | 13 | 4645165.2 |
| 4716505 | 8112388.6 | 14 | 4866995.3 |
| 4841505 | 9102029.4 | 15 | 4866995.3 |
| 4966505 | 9337029.4 | 16 | 5093825.4 |
| 5091505 | 9979349.8 | 17 | 5325655.5 |
| 5216505 | 11058990.6 | 18 | 5562485.6 |
| 5341505 | 11323990.6 | 19 | 5804315.7 |
| 5466505 | 12026311 | 20 | 6051145.8 |
| 5591505 | 12748631.4 | 21 | 6302975.9 |
| 5716505 | 13948272.2 | 22 | 6559806 |
| 5841505 | 14253272.2 | 23 | 6821636.1 |
| 5966505 | 15035592.6 | 24 | 7088466.2 |
| 6091505 | 16325233.4 | 25 | 7360296.3 |
| 6341505 | 16995233.4 | 26 | 7637126.4 |
| 6466505 | 17847553.8 | 27 | 7918956.5 |
| 6591505 | 18719874.2 | 28 | 8370786.6 |
| 6716505 | 19612194.6 | 29 | 8665116.7 |
| 6841505 | 20524515 | 30 | 8964446.8 |
| 6966505 | 21456835.4 | 31 | 9268776.9 |
| 7091505 | 22409155.8 | 32 | 9578107 |
| 7216505 | 23381476.2 | 33 | 9892437.1 |
| 7341505 | 24373796.6 | 34 | 10211767.2 |
| 7466505 | 25386117 | 35 | 10536097.3 |

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**Appendix 2**

Table 5:Simulated values of data (from the formulae) for calculating Pension and Gratuity

| Year of service | Gratuity as %of terminal Salary including all Approved Allowances | Pension as % of terminal salary including all Approved Allowances |
|---|---|---|
| 5 | 100 | - |
| 6 | 108 | - |
| 7 | 116 | - |
| 8 | 124 | - |
| 9 | 132 | - |
| 10 | 100 | 30 |
| 11 | 108 | 32 |
| 12 | 116 | 34 |
| 13 | 124 | 36 |
| 14 | 132 | 38 |
| 15 | 140 | 40 |
| 16 | 148 | 42 |
| 17 | 156 | 44 |
| 18 | 164 | 46 |
| 19 | 172 | 48 |
| 20 | 180 | 50 |
| 21 | 188 | 52 |
| 22 | 196 | 54 |
| 23 | 204 | 56 |
| 24 | 212 | 58 |
| 25 | 220 | 60 |
| 26 | 228 | 62 |
| 27 | 236 | 68 |
| 28 | 244 | 64 |
| 29 | 252 | 66 |
| 30 | 260 | 70 |
| 31 | 268 | 72 |
| 32 | 276 | 74 |
| 33 | 284 | 76 |
| 34 | 292 | 78 |
| 35 | 300 | 80 |

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**Appendix 3**

Table 6:Recommended Consolidated University Salary Structure (CONUASS)

RECOMMENDED CONSOLIDATED UNIVERSITY ACADEMIC SALARY STRUCTURE (CONUASS) TABLE II

| CONUASS | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | N | N | N | N | N | N | N | N | N | N | N | N | N |
| | CONUASS | 694992 | 714843 | 734694 | 754545 | 774396 | 794247 | | | | | | | |
| 01 | CONPUAA | 439633 | 452465 | 465298 | 478131 | 490964 | 503797 | | | | | | | |
| | Rent | 128753 | 132947 | 137141 | 141335 | 145529 | 149723 | | | | | | | |
| | Total Emolument | 1263377 | 1300255 | 1337133 | 1374011 | 1410889 | 1447767 | | | | | | | |
| | CONUASS | 794260 | 817566 | 840872 | 864178 | 887484 | 910790 | 934096 | 957402 | | | | | |
| 02 | CONPUAA | 504947 | 520051 | 535154 | 550258 | 565362 | 580466 | 595570 | 610674 | | | | 0 | |
| | Rent | 151864 | 156859 | 161853 | 166848 | 171842 | 176837 | 181831 | 186826 | | | | | |
| | Total Emolument | 1451071 | 1494475 | 1537880 | 1581284 | 1624689 | 1668093 | 1711497 | 1754902 | | | | | |
| | CONUASS | 897501 | 922760 | 948019 | 973278 | 998537 | 1023796 | 1049055 | 1074314 | | | | | |
| 03 | CONPUAA | 573999 | 590411 | 606822 | 623234 | 639645 | 656056 | 672468 | 688879 | | | | | |
| | Rent | 178009 | 183500 | 188991 | 194482 | 199974 | 205465 | 210956 | 216447 | | | | | |
| | Total Emolument | 1649509 | 1696671 | 1743832 | 1790994 | 1838156 | 1885317 | 1932479 | 1979641 | | | | | |
| | CONUASS | 1122751 | 1162974 | 1203197 | 1243420 | 1283643 | 1323866 | 1364089 | 1404312 | 1444535 | | | | |
| 04 | CONPUAA | 723801 | 750074 | 776347 | 802621 | 828894 | 855167 | 881441 | 907714 | 933987 | | | | |
| | Rent | 233443 | 242449 | 251454 | 260460 | 269466 | 278471 | 287477 | 296483 | 305488 | | | | |
| | Total Emolument | 2079995 | 2155497 | 2230999 | 2306501 | 2382003 | 2457505 | 2533007 | 2608509 | 2684011 | | | | |
| | CONUASS | 1653415 | 1711322 | 1769229 | 1827136 | 1885043 | 1942950 | 2000857 | 2058764 | 2116671 | 2174578 | 2232485 | 2290392 | 2348299 |
| 05 | CONPUAA | 1075788 | 1115342 | 1154896 | 1194450 | 1234004 | 1273558 | 1313112 | 1352666 | 1392220 | 1431774 | 1471328 | 1510882 | 1550436 |
| | Rent | 362302 | 378508 | 394713 | 410919 | 427125 | 443331 | 459537 | 475742 | 491948 | 508154 | 524360 | 540566 | 556771 |
| | Total Emolument | 3091505 | 3205172 | 3318838 | 3432505 | 3546172 | 3659839 | 3773506 | 3887172 | 4000839 | 4114506 | 4228173 | 4341840 | 4455506 |
| | CONUASS | 2053124 | 2123232 | 2193340 | 2263448 | 2333556 | 2403664 | 2473772 | 2543880 | 2613988 | 2684096 | | | |
| 06 | CONPUAA | 1311273 | 1359083 | 1406893 | 1454703 | 1502513 | 1550323 | 1598133 | 1645943 | 1693753 | 1741563 | | | |
| | Rent | 403824 | 423298 | 442772 | 462246 | 481720 | 501194 | 520668 | 540142 | 559616 | 579091 | | | |
| | Total Emolument | 3768221 | 3905613 | 4043005 | 4180397 | 4317789 | 4455181 | 4592573 | 4729965 | 4867357 | 5004749 | | | |
| | CONUASS | 2485099 | 2565548 | 2645997 | 2726446 | 2806895 | 2887344 | 2967793 | 3048242 | 3128691 | 3209140 | | | |
| 07 | CONPUAA | 1644228 | 1701656 | 1759085 | 1816514 | 1873941 | 1931370 | 1988799 | 2046229 | 2103656 | 2161084 | | | |
| | Rent | 451022 | 473124 | 495226 | 517328 | 539429 | 561531 | 583633 | 605735 | 627837 | 649939 | | | |
| | Total Emolument | 4580349 | 4740328 | 4900308 | 5060287 | 5220265 | 5380245 | 5540225 | 5700206 | 5860184 | 6020163 | | | |

NOTE:
CONUASS: CONSOLIDATED UNIVERSITY ACADEMIC SALARY STRUCTURE
CONPUAA: CONSOLIDATED PECULIAR UNIVERSITY ACADEMIC ALLOWANCE
THE OPERATION OF THE ABOVE SALARY STRUCTURE IS RESTRICTED TO THE UNIVERSITIES

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**Appendix 4**

The computer programme is as follows: $V(S) = POWER(S, -2r/\sigma^2) * \left(1/\frac{2r}{\sigma^2}\right) * POWER\left(\frac{K}{\left(1+\frac{2r}{\sigma^2}\right)}, \left(\frac{2r}{\sigma^2}+1\right)\right)$

*We used the first 4 columns, A, B, C, D, E (to represent) as follows:*

$A = S$ = *salary (horizontal axis)*

$B = V(S)$ = financial values of retirement benefit *(vertical axis)*

$$C = K - S$$

$$D1 = \frac{2r}{\sigma^2}$$

$$D2 = K$$

*Substituting these parameters into the above programme, we have*

$$= POWER(Ai, -D1) * (1/D1) * POWER(D2/(1 + D1), (D1 + 1))$$

$where i = 1, 2, 3, \ldots, 25 \; for \; University \; Senior \; lecturers$ .