A Bayesian Perspective on Error-Driven Cobb-Douglas Models: Revisiting Intrinsic Linearity

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ABSTRACT

Non-linear models in the classical approach have been used by different researchers in the estimation of production function (case study: Cobb-Douglas), which is intrinsically linear. Bayesian approach to non-linear models had gained much ground but little attention had been paid towards using independent Normal-Gamma prior. This study aimed at investigating the behaviour of the parameters in the Cobb-Douglas Model using Bayesian approach. Metropolis–Within-Gibbs Algorithm (Posterior simulator) with non-informative prior was adopted to generate posterior estimates in this work. The results obtained from the simulation study showed that as sample size increases, the true values are significantly close to the posterior estimates. Also, the estimated standard deviations and the numerical standard errors decreased consistently when production function is “less one”, “greater than one” and “equal to one”. As the prior changed in all measures of return to scale, the considered criteria remained unchanged using the Geweke’s convergence diagnostics tool to show the good performance of non-informative prior of the error precision.

It is recommended that the researchers should prioritize appropriate priors, sufficient sample sizes, effective simulations, and convergence validation to ensure credible and reliable Bayesian estimates when dealing with Non-linear models.

Keyword: Cobb-Douglas, Normal-Gamma, Non-informative prior, Numerical Standard Error, Metropolis-Within-Gibbs Algorithm.

INTRODUCTION

Bayesian approaches offer a thorough comprehensive framework for problems involving dynamic state estimation. This method involves developing the probability density function (PDF) of the state based on all the information that is currently available. The required PDF remains gaussian at every iteration of the filter in intrinsically linear estimation (transformation of non-linear model). The choice of Bayesian approach in non-linear production function is fast gaining attention in health, agriculture, industrial and many more sectors. Using a broad Metropolis-Hastings implementation, Edward and David (1999) found that a Bayesian framework for estimating nonlinear flood event models offered more modeling flexibility and was considerably simpler to design than a Gibbs sampler.

Cobb Douglas production function proposed in 1928 as a fairly universal law of production for different industries in the world chosen in Solow (1956) was important contribution when he specified growth theory. Berndt (1976) concluded that the Cobb-Douglas aggregate production function cannot be rejected. The validity of his findings has been questioned by many because Berndt assumed that every technological advancement had no impact on labour. A bias towards one is caused in estimates of elasticity by labor-augmenting technological change. Although the capital-labour ratio is increasing, the capital share of costs remains constant at the aggregate level, leading regressions to conclude that the production function is Cobb Douglas. However, any elasticity is in line with a steady aggregate capital share of cost, assuming that labour productivity increases at the same rate as the capital-labor ratio.

Numerous economic models assert that fluctuations in the business cycle stem from changes in productivity. In light of this, Abiola (2010) examined productivity in the banking sector by analyzing two important production functions well-established in economic research. He found that the results of the estimates of the substitution parameters a and b (representing the substitution parameters for capital and labour, respectively), obtained using the Ordinary Least Squares (OLS) method, confirmed the a priori prediction that the values of both a and b are positive and less than one. Adetunji et.al (2012) examined the performance of the banking sector and capital-labour substitution in Nigeria from 1960 to 2008. Additionally, they researched the Nigerian economy, which is likely to display constant returns to scale between 1990 and 2009. The average Capital/Labor ratio increased by 1%, while labour productivity saw a 1% rise. When compared with the findings of Liedholm (1964) and Osagie and Odaro (1975), the models produced acceptable outcomes in terms of goodness of fit. This indicated that the Cobb-Douglas Production Function (CDPF) offers a superior explanation of the overall production process in Nigeria's banking sector during the period under study.

Musa et al (2020) explained how to estimate the parameters of the Cobb-Douglas production function using the Ordinary Least Squares (OLS) method for a specific economy or industry. It also demonstrates how hypothesis tests can be conducted, such as determining whether the sum of labour and capital inputs elasticities is greater than one. It is suggested that cross-sectional data should be used for estimation, as OLS is not suitable for time series or longitudinal data according to available literature. Knoblach Michael (2020) et.al employed a meta-regression analysis in their study by estimating a long-term meta-elasticity for the overall economy by providing the first analysis of meta-regression for the US economy. The findings reveal that the variations in previous estimates are primarily influenced by modelling choices related to technological dynamics and the hypothesis of a Cobb-Douglas production function is consistently rejected throughout the study. Most estimates for specific industries do not significantly differ from the estimate for the overall economy.

Maxwell and Essi (2019) emphasized on utilizing Monte Carlo Methods to estimate the unknown parameters of Cobb Douglas production function, the ordinary least square (OLS) method was employed. The data generating process was conducted through the application of Monte Carlo simulation techniques. It was discovered that the value of the mean square error (MSE) changes depending on the sum of the powers of the input variable.

In situations of uncertainty, Bayesian statistics offers a rationalist perspective on personality beliefs, aiming to explain how an individual should act to avoid specific types of undesirable behavioural inconsistencies. Zellner (1971) wrote a paper on Bayesian econometrics, which became widely known and has since been applied in various contexts (Koop, 1994). A key concept of Bayesian statistics is that probability should be used to describe any uncertainty we may have on a phenomenon.

However, Zhang and Wang (2019) propose a Bayesian approach for estimating the parameters of a Cobb-Douglas production function with endogenous technological change. The proposed approach uses a Markov chain Monte Carlo (MCMC) algorithm to sample from the posterior distribution of the parameters. The results show that the proposed approach is able to provide more accurate and robust estimates of the parameters than the traditional frequentist approaches. Iyaniwura et. al (2019) estimated the parameters of the Cobb-Douglas production function with additive and multiplicative errors using the Gauss-Newton and Bayesian approaches and was reported that the results of the gauss newton method was better in estimating the Cobb-Douglas with additive errors than multiplicative errors Cobb Douglas function while the estimates of Bayesian approached generally performed better.

Adesina (2019), proposes Bayesian estimators for estimating the parameters of nonlinear production functions, addressing the shortcomings of the classical approach. It compares the performance of Bayesian and classical approaches using a Monte Carlo study. The Bayesian estimators outperform the classical approach for the production functions considered, making them more suitable for handling nonlinear production functions, regardless of error specification. Braganca and Stern (2001) Showed how the full Bayesian Significance test can be used as a model selection criterion where modern numerical optimization and integration techniques were used to test if one of the parameters is null and such a test is used for a model selection procedure.

Iyaniwura et.al (2018) developed a way to estimate parameters of intrinsically nonlinear models using the Gauss-Newton Method via the Kmenta approximation and the Metropolis-within Gibbs Algorithm. The Bayesian approach was preferred due to minimal Numerical Standard Error and posterior estimates converging to the true values. David et.al (2000) proposed and implemented a coherent statistical framework for fusing theoretical and empirical models of macroeconomic activity using the Bayesian framework. In their approach, they illustrated a neoclassical business-cycle model to study out-of-sample forecasting of output and investment. The forecasts generated in this way can be compared to those from a Bayesian vector autoregression and found their result to be impressive, showing that the theory-based measurement had some advantages.

Edward and David (1999) presented a Bayesian procedure for parameter estimation in nonlinear flood event models. They also develop a pooling diagnostic using Bayes factors to identify when it is reasonable to pool model parameters across storm events. Markov chain Monte Carlo methods based on the Metropolis-Hastings algorithm was employed for parameter estimation. They found that pooling is not justified for the model and data at hand. This paper is aimed at investigating the sensitivity of the measure of returns to scale parameter in Cobb-Douglas production function.

Other aspects of this paper can be categorized into four sections; Section 2 reveals the Methodology using a Normal-Gamma prior, while a detailed Empirical Illustration and discussion of results is carried out in Section 3 and Section 4 gives the Conclusion

METHODOLOGY

The Nonlinear Regression Model Using an Independent Normal-Gamma Prior

The nonlinear regression model considered in this work will be of the form

$$y_i = \beta_0 X_1^{\beta_1} X_2^{\beta_2} e^{u_i} \qquad (2.1)$$

The above expression can be transformed into a linear form by applying the natural logarithm to both sides of the equation. This can then be simplified as:

$$Y_i{}' = \beta'_0 + \beta_1 X'_1 + \beta'_2 X'_2 + \varepsilon'_i$$

Where: $ln(y_i) = Y_i{}'; ln(\beta_0) = \beta_0'; ln(X_{i1}) = X_1';\ ln(X_{i2}) = X_2';\ u_i = \varepsilon_i'$

The Likelihood Function of the Non-Linear Regression Model

The log-likelihood can be represented using the multivariate Normal density:

$$p(y_i'|\beta, g) = \sum_{i=1}^{N}\{\frac{g^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} exp[-\frac{g}{2}(y_i' - X'\beta)'(y_i' - X'\beta)]\} \qquad (2.2)$$

The Prior

The knowledge we possess concerning a particular study before observing the data; the independent prior is indicated as $P(\beta, g)$

In the case of independent random variables, it is true that,

$P(\beta, g) = P(\beta) * P(g)$ with $P(\beta)$ represents Normal while $P(g)$ signifies Gamma:

$P(\beta) = \frac{1}{(2\pi)^{\frac{N}{2}}} |\underline{V}|^{-\frac{1}{2}} exp\left[-\frac{1}{2}\left(\beta - \underline{\beta}\right)' \underline{V}^{-1}\left(\beta - \underline{\beta}\right)\right]$ and

$$P(g) = C_G{}^{-1} g^{\frac{\underline{v}-2}{2}} exp\left(\frac{-g\underline{v}}{2\underline{s}^{-2}}\right)$$

Where, C_G is the Gamma's integrating constant, It follows that: $E[\beta|y] = \underline{\beta}$ denote the prior mean of β and $Var(\beta|g) = \underline{V}$ is the prior covariance matrix of $\beta$with the mean of g, as $\underline{s}^{-2}$and $\underline{v}$ as the degree of freedom.

The Posterior

The posterior is directly related to the prior and the likelihood. which are both the information gained from examining the data and using some mathematical tools, it can be a conjugate or independent or not taking a familiar distribution form, usually denoted

$P(\beta, g|y_i')$.

Note: $P(\beta, g|y_i') \neq P(\beta|y_i', g) \cdot P(g|y_i', \beta)$

Then, the posterior:

$$P(\beta, g|y_i') = \frac{g^{\frac{N}{2}}}{(2\pi)^{\frac{N}{2}}} exp\left[-\frac{g}{2}(Y_i' - X_i'\beta)'(Y' - X'\beta)\right] . \frac{1}{(2\pi)^{\frac{N}{2}}} |V|^{-\frac{1}{2}} exp\left[-\frac{1}{2}\left(\beta - \underline{\beta}\right)' \underline{V}^{-1}\left(\beta - \underline{\beta}\right)\right] \cdot C_G{}^{-1} g^{\frac{\underline{v}-2}{2}} exp\left(\frac{-g}{2\underline{s}^{-2}}\right) P(\beta, g|y_i') \propto exp\left[-\frac{1}{2}\{g(Y_i' - X'\beta)'(Y_i' - X'\beta) + \left(\beta - \underline{\beta}\right)' \underline{V}^{-1}\left(\beta - \underline{\beta}\right)\}\right] \cdot g^{\frac{N+\underline{v}-2}{2}} exp\left(\frac{-g\underline{v}}{2\underline{s}^{-2}}\right) \quad (2.3)$$

As there is no well-known distributional form that this joint posterior density for cc and g has, it cannot be computed analytically and must instead be solved using the posterior simulation approach. By eliminating the terms that (2.3) doesn't involve β we obtain,

$$P(\beta|y_i', g) \propto exp\left[-\frac{1}{2}\{\left(\beta - \overline{\beta}\right)' \overline{V}^{-1}\left(\beta - \overline{\beta}\right)\}\right] \quad (2.4)$$

which indicates that $\beta|y_i', g \sim N(\overline{\beta}, \overline{V})$, is a **Multivariate Normal density**

Where, $\overline{V} = (\underline{V}^{-1} + gX''X')^{-1}$ and $\overline{\beta} = \overline{V}(gX''Y' + \underline{V}^{-1}\underline{\beta})$

Similarly, by discarding the terms that exclude g we obtain,

$$P(g|y^{**}, \beta) \propto g^{\frac{N+\underline{v}-2}{2}} exp[-\frac{g}{2}\{(Y_i' - X'\beta)'(Y_i' - X'\beta) + \underline{v}\underline{s}^2\}] \quad (2.5)$$

This likewise suggests that $g|y', \beta \sim G(\overline{s}^{-2}, \overline{v})$, as **Gamma density**

Where, $\bar{v} = N + \underline{v}$ and $\overline{s}^2 = \frac{(Y_i' - X'\beta)'(Y_i' - X_i'\beta) + \underline{v}\underline{s}^2}{\overline{v}}$

The equations (2.4) and (2.5) does not directly correspond to the posterior of interest, since$P(\beta, g|y_i') \neq P(\beta|y_i', g) \bullet P(g|y_i', \beta)$. Therefore, the conditional posteriors do not provide a complete description of the posterior,$P(\beta, g|y_i')$. Nevertheless, the posterior simulator called the Metropolis-Within-Gibbs can now be applied.

Empirical Illustration and Discussion

The Monte Carlo simulation approach was applied to simulate the data used for analysis in this study. where, the explanatory variables are generated independently from a uniform [0,1] distribution,$X_{ij} \sim U[0,1]$, $i =$

0,1,2and $j = 1,2,\ldots,N$. Values fixed for the regression coefficients $\beta_i, i = 0,1,2$ $i.e$ $\beta_0 = 15, \beta_1 = 0.85, \beta_2 = 0.15$ (for the scenario considered $\beta_1 + \beta_2 = 1$,$\beta_1 + \beta_2 > 1$ and $\beta_1 + \beta_2 < 1$). Using four (4) different sample sizes N=30, N=50, N=100, and N=150 and a non-linear Cobb-Douglas model to obtain the response variable (data of interest) with an error component taken from a standard normal distribution, the posterior estimates were converged after 10,000 iterations and a burning of 1000.

CASE 1: When N =30, 50, 100 and 150, priors are: 0.02, 0.079 for scenario $\boldsymbol{\beta_1 + \beta_2 = 1}$

Table 3.1 showed the result obtained at various sample sizes 30, 50, 100 and 150 under the first scenario of the Cobb-Douglas multiplicative model where; $\boldsymbol{\beta_1 + \beta_2 = 1}$ with priors set at 0.02 for first prior variable and 0.079 for the second prior variable (note: the result for the Intercept is not shown in this study but the model included the Intercept in the process of analysis). True values set to be $\beta_1 = 0.85$ and $\beta_2 = 0.15$. The Metropolis-Within Gibbs algorithm is then used to carry out the analysis using a nonlinear regression model with independent normal-gamma prior and the result shown below; it is obvious that the true values are not too far from the posterior estimates, the standard deviation (S.D) decreased consistently as sample sizes increased in all level, the Numerical Standard Error (NSE) synonymous to the frequentist Standard Error showed a steady decrease as sample size increased, therefore; the aim of this work achieved that the result is accurate and finally, looking critically at the Geweke Convergence Diagnostic tool (GCD) values produced, it is shown that all produced values that are less than 1.96 in absolute value which buttressed the point that all the posterior estimates converged to the region of the true value.

Table 3.1: Posterior estimates for the first scenario; $\beta_1 + \beta_2 = 1$

True value	Sample size	Posterior	S.D	NSE	GCD
$\beta_1 = 0.85$	30	0.8314	0.1438	0.0015	0.6053
	50	0.8275	0.1138	0.0012	-0.4764
	100	0.7889	0.1108	0.0012	0.5378
	150	0.8206	0.0905	0.0010	-0.0651
$\beta_2 = 0.15$	30	0.1648	0.1038	0.0011	-0.6887
	50	0.1755	0.0871	0.0009	0.0823
	100	0.1696	0.0820	0.0009	1.7690
	150	0.1937	0.0742	0.0008	-0.9870

CASE 2: When N =30, 50, 100 and 150, priors are: 0.02, 0.079 for scenario $\boldsymbol{\beta_1 + \beta_2} > 1$

Table 3.2 showed the result obtained at various sample sizes 30, 50, 100 and 150 under the second scenario of the Cobb-Douglas multiplicative model where; $\boldsymbol{\beta_1 + \beta_2} > 1$ with the same set of priors at 0.02 for first prior variable and 0.079 for the second prior variable (note: the result for the Intercept is not shown in this study but the model included the Intercept in the process of analysis). The True values set to be $\beta_1 = 0.90$ and$\beta_2 = 0.20$. The Metropolis-Within Gibbs algorithm is then used to carry out the analysis using a nonlinear regression model with independent normal-gamma prior and the result shown below; it is obvious that the true values are not too far from the posterior estimates, the standard deviation (S.D) decreased consistently as sample sizes increased in all level, the Numerical Standard Error (NSE) showed a steady decrease; 0.0018, 0.014, 0.0013, 0.0010 for $\beta_1 = 0.90$ and 0.0010, 0.0009, 0.0009, 0.0008 for $\beta_2 = 0.20$ as sample size increased and finally, the Geweke Convergence Diagnostic tool (GCD) showed values that are less than 1.96 in absolute value which buttressed the point that all the posterior estimates converged to the region of the true value.

Table 3.2: Posterior estimates for the second scenario; $\beta_1 + \beta_2 > 1$

True value	Sample size	Posterior	S.D	NSE	GCD
$\beta_1 = 0.90$	30	0.9623	0.1683	0.0018	0.8325
	50	0.9396	0.1353	0.0014	-0.1138
	100	0.8130	0.1204	0.0013	-1.2330
	150	0.8618	0.0948	0.0010	-1.6140
$\beta_2 = 0.20$	30	0.2165	0.0992	0.0010	-0.5363
	50	0.2289	0.0886	0.0009	-0.3179
	100	0.2187	0.0844	0.0009	0.2169
	150	0.2453	0.0746	0.0008	-0.1371

CASE 3: When N =30, 50, 100 and 150, priors are: 0.02, 0.079 for scenario $\boldsymbol{\beta_1 + \beta_2} < 1$

Table 3.3 below; showed the result obtained at various sample sizes 30, 50, 100 and 150 under the third scenario of the Cobb-Douglas multiplicative model where; $\boldsymbol{\beta_1 + \beta_2} < 1$ with the same priors set at 0.02 for first prior variable and 0.079 for the second prior variable (note: the result for the Intercept is not shown in this study but the model included the Intercept in the process of analysis). True values set to be $\beta_1 = 0.65$ and $\beta_2 = 0.15$. The Metropolis-Within Gibbs algorithm is then used to carry out the analysis using a nonlinear regression model with independent normal-gamma prior and the result shown below; it is obvious that the true values are not too far from the posterior estimates, the standard deviation (S.D) decreased consistently as sample sizes increased, the Numerical Standard Error (NSE) showed that there is a steady decrease as sample size increased and finally, looking closely at the Geweke Convergence Diagnostic tool (GCD) values, it is observed that the values produced are less than 1.96 in absolute value, which signified that all the posterior estimates converged to the region of the true value.

Table 3.3: Posterior estimates for the third scenario; $\beta_1 + \beta_2 < 1$

True value	Posterior	S.D	NSE	GCD
$\beta_1 = 0.65$	0.6308	0.1466	0.0015	0.0352
	0.6294	0.1171	0.0012	0.2396
	0.5899	0.1088	0.0012	-0.0556
	0.6191	0.0955	0.0010	-0.4475
$\beta_2 = 0.15$	0.1669	0.1014	0.0011	-0.0509
	0.1761	0.0860	0.0009	-0.7938
	0.1727	0.0814	0.0009	-0.4955
	0.1980	0.0755	0.0008	0.1656

Graphical Illustrations

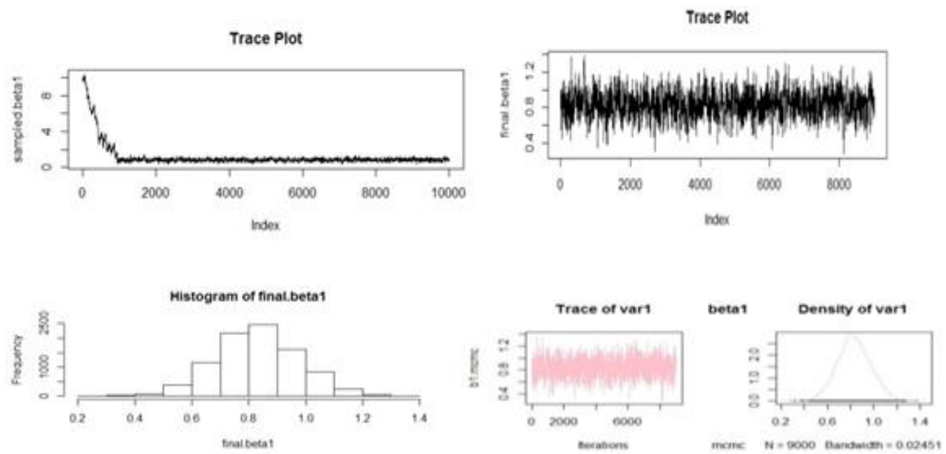

Figure 3.1: Summary of metropolis-within-Gibbs draw of β_1 at N=30

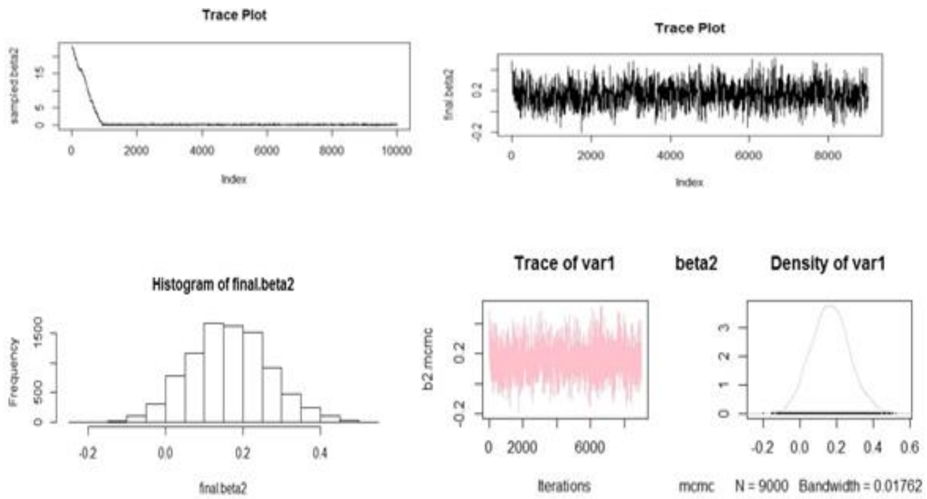

Figure 3.2: Summary of metropolis-within-Gibbs draw of β_2 at N=30

We use trace plots to examine the convergence of the posterior distribution. The trace plots display parameter estimations from the chains' successive iterations, showing rapid up-and-down variations without trends as the chains converge. For all the parameters represented by $\boldsymbol{\beta_1}$ ***and*** $\boldsymbol{\beta_2}$ when N=30, the chains converged, as shown by the trace plots and marginal density plots in Figures 3.1 and 3.2 respectively.

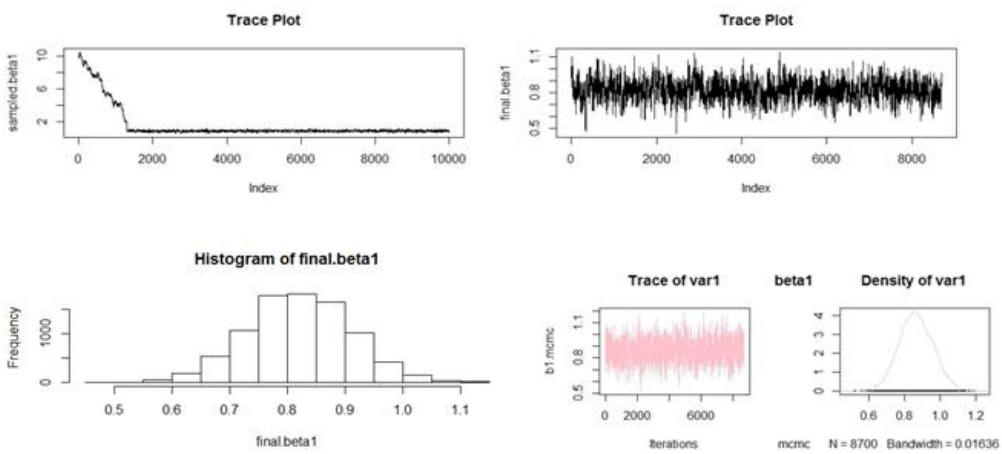

Figure 3.3: Summary of metropolis-within-Gibbs draw of β_1 at N=150

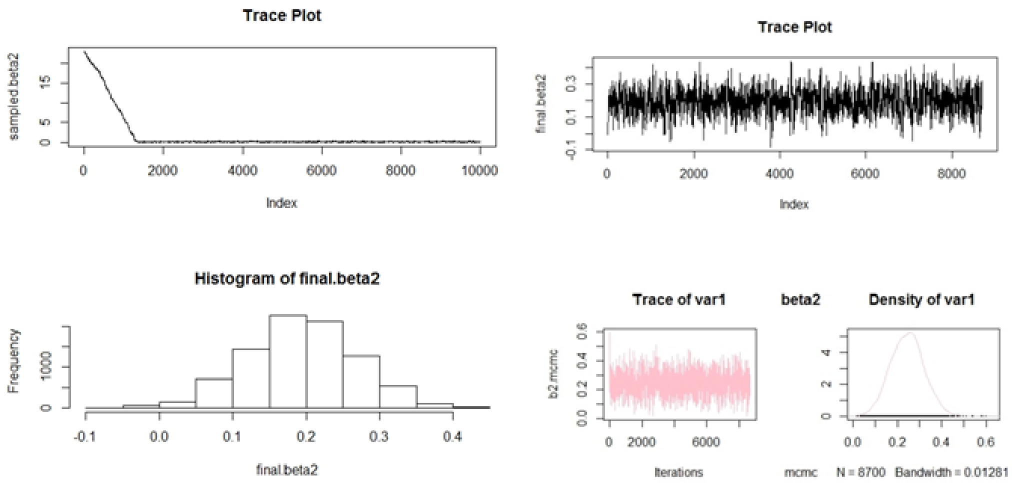

Figure 3.4: Summary of metropolis-within-Gibbs draw of β_2 at N=150

Figure 3.1 to 3.4 above showed a graphical illustration of the posterior estimates obtained from the empirical analysis, the trace plot on the left-hand side showed the pattern in which the estimates converged to the region of the posterior estimates using the Metropolis-within Gibbs algorithm, the trace plot on the right-hand side showed the outcome when a burn in process had taken place, this removed the effects of the initial values and gave accurate posterior estimates. The Histogram also showed the distributional form to follow a normal distribution of the data.

CONCLUSION

The nonlinear regression model i.e., the Cobb-Douglas production function was estimated with an independent normal-gamma prior which resulted in a truncated posterior and the methodologies stated to show how the estimates were derived using the posterior simulation technique called the Metropolis Within-Gibbs algorithm. Then, the estimates for the Cobb-Douglas production function with the simulated data in the three scenarios for below, above and unity as classified and explained in the Empirical Illustration and Discussion section showed that the posterior estimates are not far from the true values of the parameters set at N=30, 50, 100 and 150 for both $\beta_1$and $\beta_2$in Table 3.1, Table 3.2 and Table 3.3 respectively.

Also, the R-software was used to provide a useful framework for the Bayesian analysis of the Log- linear regression model where we carried out 10,000 iterations taking a 1000 burn-in–sample to remove the effect of the initial values. The remaining 9,000 were used for the parameter estimation. Convergence diagnostics tools to ascertain accuracy of the study of interest were considered. Graphical illustrations were also used to show the point of convergence to the region of the true value. The standard deviation (S.D) decreased consistently as sample sizes increased, the Numerical Standard Error (NSE) showed that there is steady decrease as sample size increased and finally, looking closely at the Geweke Convergence Diagnostic tool (GCD) values, it is observed that the values produced are less than 1.96 in absolute value for all scenarios, which signified that all the posterior estimates converged to the region of the true value.

Researchers are advised to prioritize the selection of appropriate priors, ensure sufficient sample sizes, utilize effective methods for posterior simulation, and validate convergence to achieve accurate parameter estimation in nonlinear regression models. Adhering to these practices will enhance the credibility and reliability of Bayesian estimates, ultimately leading to more robust findings in empirical research.

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