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A Two-Warehouse Ordering Policy for Non-Instantaneous Deteriorating Items with two Phase Demand Rates, Two-Tiered Pricing and Shortages Under Trade Credit Policy

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ABSTRACT

The classical inventory models for non-instantaneous deteriorating items tacitly assumed that the selling price before and after deterioration sets in is the same. However, when items start decaying, the vendor might resolved to decrease the selling price in order to boost additional sales, reduces the cost of holding stock, attracts new clients and reduces lost due to deterioration. In this research, the vendor's best refill approach for non-instantaneous decaying goods with two-phase demand rates, two-storage facilities, and shortages under an allowable payment delay has been determined. The unit selling price before deterioration sets in is greater than that after deterioration sets. Though a constant consumption rate is considered as soon as deterioration begun, the consumption rate before items starts decaying supposed to be time-dependent quadratic. Shortages are permitted depending on how long it will take before next replenishment. The model determines the best cycle time, optimal order quantity, and optimal time at which the inventory level in the owned ware-house reaches zero in order to increase the overall profit per unit of time. The solution's existence and uniqueness are both checked by establishing the necessary and sufficient conditions. The model is validated by conducting some numerical experiments, after which sensitivity analysis is carried out which offer some managerial insights.

Keywords: Two-warehouse, Two Phase Demand Rates, Two-tiered Pricing, Shortages, Trade Credit Policy.

INTRODUCTION

Inventory models for non-instantaneous deteriorating items, in some cases, assumed that the unit selling price before and after deterioration sets in is the same. However, in real practice, the unit selling price before and after deterioration sets in differs and this assumption need to be considered in developing inventory policies for non-instantaneous deteriorating items, where the objective function is to maximize the total profit of the inventory system. Tsao and Sheen (2008) presented dynamic pricing, promotion and replenishment policies for deteriorating items when a delay in payment is permissible. Lee and Hsu (2009) developed a two warehouse production model for instantaneous deteriorating inventory items with time-dependent demands rates, a finite replenishment rate within a finite planning horizon. Tsao (2010) developed two-phase pricing and inventory decisions for deteriorating and fashion goods under permissible delay in payment. The demand rate vary with price or time, shortages are allowed and partially backlogged, and the objective function is to maximize profit. Chen and Kang (2010) developed integrated inventory models considering the two-level trade credit policy and a price negotiation scheme, in which customers' demand is sensitive to the buyers' price. Wang *et al.* (2015) proposed a dynamic pricing inventory model for non-instantaneous deteriorating items, where the objective function is to maximize the total profit per unit time, and both uniform pricing and two-stage pricing models are developed and a comparative study between dynamic and uniform pricing shows the advantage of dynamic pricing over uniform pricing. Tsao *et al.* (2017) developed two-level pricing and ordering policies for non-instantaneous deteriorating items with price-sensitive demand rates under permissible delay in payment. Babangida and Baraya (2021) developed an EOQ model for non-instantaneous deteriorating items with two-phase demand rates and two level pricing strategies under trade credit policy. The cost of holding items in the stock is assumed to be constant and shortages are not allowed. Pang *et al*. (2022) develop an inventory model

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for perishable items with two-stage pricing. Moreover, some related studies on inventory models with two-phase or two-period pricing strategies can be found in Dye (2012), Dye and Hsieh (2013), Sainathan (2013), Chang *et al*. (2015). Herbon (2015) and so on.

Most inventory models are developed based on the assumption for a single warehouse with unlimited capacity. However, this assumption is debatable in most business set up. Retailer may purchase large quantity of items at a time as result of price discount (for bulk purchase), quantity discount, fear of inflation, uncertainty in demand, stock outs and so on. These items may not be stoked in the existing storage called owned warehouse (OW) with limited capacity due to its bulkiness. The retailer may rent another store called the rented warehouse (RW). Items are moved from the rented warehouse to owned warehouse and sold. This is because the holding cost in rented warehouse is assumed to be higher than that in owned warehouse due to better preserving facility which lower deterioration rate. Hence it is more economical to consume the items of rented warehouse earlier. Chandra et al. (2017) developed ordering policies for non-instantaneous deteriorating items with price dependent demand and two storage facilities under trade credit policy, where shortages are allowed and completely backlogged and the objective function is to maximize profit. Palanivel et al. (2016) developed a two-warehouse economic order quantity model for non-instantaneously deteriorating items with stock-dependent demand under the effects of inflation and the time value of money is presented. Also in this model, shortages are allowed and partially backlogged. The backlogging rate is dependent on the waiting time for the next replenishment. The objective of this model is to minimize the total inventory cost of the retailer by finding the optimal intervals and the optimal order quantity. Udayakumar and Geetha (2018) developed an EOQ model for non-instantaneous deteriorating items with constant demand rate and two levels of storage under trade credit policy, where shortages are not allowed and the objective function is to minimize cost. Babangida and Baraya developed an inventory model for non-instantaneous deteriorating items with time dependent quadratic demand, two storage facilities and shortages under trade credit policy. The demand rate before deterioration sets in is assumed to be time dependent quadratic and that is considered as a constant after deterioration sets in. Shortages are allowed and completely backlogged, and the model determine length of time at which the inventory level reaches zero in OW, cycle length and order quantity simultaneously such that total variable cost has a minimum value.

In the classical inventory model, shortages are not allowed. However, sometimes customers' demands cannot be fulfilled by the supplier from the current stocks, this situation is known as stock out or shortage condition. In real-life situations, stock out is unavoidable due to various uncertainties. According to Sharma (2003), allowing shortages to occur increases cycle length, spread the ordering cost over a long time and hence reducing the total variable cost. Choudhury *et al.* (2013) developed an inventory model for non-instantaneous deteriorating items with stock-dependent demand rate, time-varying holding cost and shortages that are completely backlogged.

However, when shortages occur, one cannot be certain that all customers are willing to wait for a backorder due to customers' impatient and dynamic nature of human beings. When shortages occur, some customers whose needs are not critical at that time may wait for the back-orders to be fulfilled, while others may opt to buy from other sellers. Consequently, the opportunity cost due to lost sales should be considered.

For most items, such as fashionable goods, electronics, automobiles and its spare parts, photographic films, seasonal products and so on, the length of the waiting time for the next replenishment would determine whether the backlogging will be accepted or not. Therefore, the backlogging rate should be variable and depend on the waiting time for the next replenishment. That is, the longer the waiting time, the lower the backlogging rate will be and vice versa. Sarkar and Sarkar (2013) developed an inventory model for deteriorating items with stock-dependent demand rate and time-varying deterioration. Shortages are allowed and partially backlogged; the backlogging rate depends on the waiting time for the next replenishment. Dutta and Kumar (2015) discussed a time-dependent partially backlogged inventory model for deteriorating items with a time-varying demand rate and holding cost. Nath and Sen (2022) developed a partially backlogged inventory model for time-deteriorating items using penalty cost and time-dependent holding cost. Babangida and Baraya (2022) developed an EOQ model for non-instantaneous deteriorating items with two-phase demand rates, linear holding cost and time-dependent partial backlogging rate under trade credit policy. Babangida et al (2023) developed retailer's ideal replenishment strategy for non-instantaneous decaying goods

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with two-phase demand rates, two storage facilities, and shortages under a permissible payment delay. Whether or not the backlog will be accepted depends on how long it will be until the next replenishment. As a result, the backlogging rate fluctuates and depends on how long it takes for the next refill.

It could be observed from the above reviews that non-instantaneous deterioration, two-phase demand rates, two-ware house, two-level pricing strategies, shortages and trade credit policy are the most realistic features to consider in developing inventory policies for items, such as electronics, automobiles, seasonal products, fashionable goods, and so on.

This research work investigates EOQ Models for non-instantaneous deteriorating items with two-phase demand rates, two-tiered pricing strategy and time dependent partial backlogging rates under trade credit policy. The demand rate before deterioration sets in is assumed to be time-dependent quadratic after which it is considered as constant up to when the inventory is completely depleted. It is also assumed that the unit selling price before and after deterioration sets in is not the same. The necessary and sufficient conditions for optimality of the solution have been established. The optimal length of time at which the inventory level reaches zero in OW, cycle length and order quantity that maximizes the total profit per unit time will be determined. Some numerical examples have been given to illustrate the theoretical results of the models. Sensitivity analysis of some parameters of the proposed models had been carried out on the decision variables and suggestions towards maximizing the total profit have also been given.

Model Description and Formulation

This section provides the model notation, assumptions, and formulation.

Notation and Assumptions

Notation

OC The ordering cost per order.

PC The purchasing cost per unit per unit time ($/unit/ year).

SP_1 Unit selling price during the interval $[0,\ p_d]$

SP_2 Unit selling price during the interval$[p_d,\ P]$, where $\mathrm{SP}_1 > \mathrm{SP}_2 > \mathrm{C}$

$\mathrm{C_b}$ Shortage cost per unit per unit of time.

hc_o The holding cost per unit per unit time in own ware-house ($/unit/ year).

hc_r The holding cost per unit per unit time in rented ware-house ($/unit/ year).

I_c The interest charged in stock by the supplier per Dollar per year ($/unit/year).

I_e The interest earned per Dollar per year ($/unit/year) $(I_c \geq I_e)$.

T The trade credit period (in year) for settling accounts.

ω_o Constant deterioration rate in own ware-house, where $0 < \omega_o < 1$.

ω_r Constant deterioration rate in rented ware-house, where $0 < \omega_r < 1,\ \omega_r < \omega_o$

p_d The length of time in which the product exhibits no deterioration.

p_r Time at which the inventory level reaches zero in rented ware-house.

p_o Time at which the inventory level reaches zero in the owned ware-house.

P The length of the replenishment cycle time (time unit).

R_m The maximum positive inventory level per cycle

R_d Capacity of the owned ware-house

$(R_m - R_d)$ Capacity of rented ware-house

N_m The backorder level during the shortage period.

R The order quantity during the cycle length, where $R = (R_m + N_m)$.

$q_O(p)$ Inventory level in the owned ware-house at any time p, where $0 \leq p \leq P$.

$q_r(p)$ Inventory level in the rented ware-house at any time p, where $0 \leq p \leq P$.

$q_s(p)$ Shortage level at any time p where $p_o \leq p \leq P$.

Assumptions

This model is established under the following assumptions.

1. The replenishment rate is instantaneous.
2. The lead time is zero.
3. A single non-instantaneous decaying item is considered.
4. The own ware-house has a fixed capacity of R_d units; the rented ware-house has capacity of $(R_m - R_d)$.
5. The unit inventory holding cost per unit time in the rented ware-house is higher than that in the owned ware-house and the deterioration rate in the rented ware-house is less than that in the owned ware-house.
6. There is no replacement or repair for deteriorated goods during the period under consideration.
7. Demand before deterioration begins is a quadratic function of time p, and is given by $\eta + \mu p + \sigma p^2$, where $\eta \geq 0, \mu \neq 0, \sigma \neq 0$.
8. Demand rate after deterioration sets in is assumed to be constant and is given by ʉ.
9. During the trade credit period $T\ (0 < T < 1)$, the account is not settled; generated sales revenue is deposited in an interest-bearing account. At the end of the period, the retailer pays off all units bought and starts to pay the capital opportunity cost for the goods in stock.
10. During the stock out phase, shortages are permitted and partially backlogged; the backlogging rate is dynamic and based on how long it takes for the next replenishment; the longer the waiting time, the smaller the backlogging rate will be. The negative inventory backlog rate is calculated as $N(p) = \frac{1}{1+\varsigma(P-p)}$, ς is the backlogging parameter ($0 < \varsigma < 1$)and $(P - p)$ is waiting time $(p_o \leq p \leq P)$, $1 - N(p)$ is the remaining fraction lost.

Formulation of the model

The retailer's ideal replenishment strategy for non-instantaneous decaying commodities with two-phase demand rates, two-storage facilities, and shortages within a permissible payment delay has been taken into consideration in this article. Allowable payment delays encourage retailers to stock up on more goods since they boost sales, increase cash flow, lower the cost of stock holding, draw in new customers, or just retain their

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current ones. When the quantity exceeds the merchant's ware-house capacity, the retailer may choose to rent a ware-house to store the excess inventory. In this inventory system, R_m units of a single product arrive at the inventory at the beginning of the cycle in which R_d units are stored in their own ware-house and the remaining $(R_m - R_d)$ units in a rented ware-house. Thus, in order to find the optimal replenishment policy of the inventory system, two cases of when $p_d < p_r$ and when $p_d > p_r$ are discussed and are as follows.

Case I: $p_d < p_r$ (when items start deteriorating before the inventory level in rented ware-house completely depleted to zero)

Figure 3.1 designates the behaviour of the inventory system. During the time interval $[0,\ p_d]$, the inventory level $q_r(p)$ in rented ware-house is depleting gradually due to market demand only and it is assumed to be a quadratic function of time p whereas in the owned ware-house inventory level remains unchanged. At time interval $[p_d,\ p_r]$ the inventory level $q_r(p)$ in the rented ware-house is depleting due to combined effects of constant market demand rate ȗ and deterioration while the inventory level in the owned ware-houses gets used up due to deterioration only. At time interval $[p_r,\ p_o]$, the inventory level $q_o(p)$ in the owned ware-house depletes to zero due to the combined effects of consumer demand and deterioration. Shortages occur at the time $p = p_o$ and are partially backlogged in the interval $[p_o,\ P]$. The whole process of the inventory system is repeated.

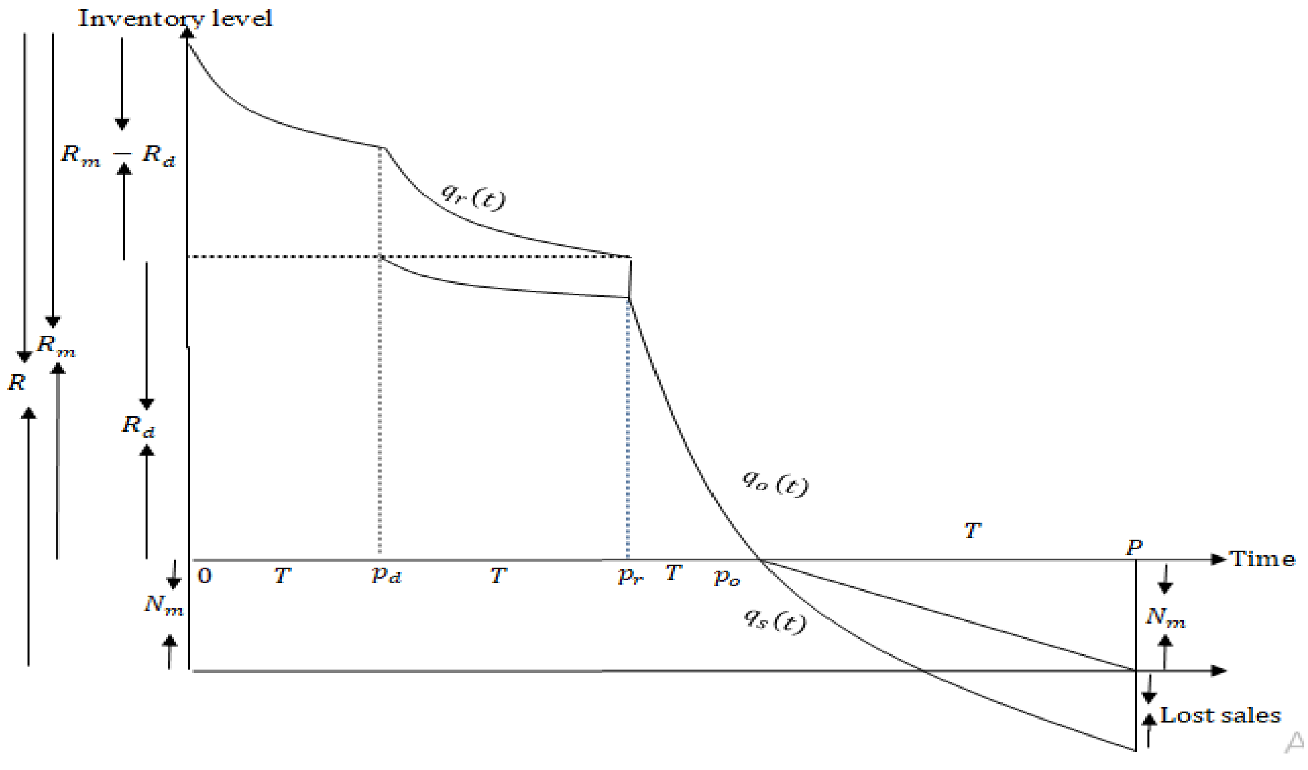

Figure 3.1: Two-ware-house inventory system when $p_d < p_r$

The differential equations that describe the inventory level in both rented ware-house and owned ware-house at any time p over the period $[0,\ P]$ are given by

$$\frac{dq_r(p)}{dp} = -(\eta + \mu p + \sigma p^2), \qquad 0 \le p \le p_d \qquad (1)$$

$$\frac{dq_r(p)}{dp} + \omega_r q_r(p) = -\text{ȗ}, \qquad p_d \le p \le p_r \qquad (2)$$

$$\frac{dq_o(p)}{dp} + \omega_o q_o(p) = 0, \qquad p_d \le p \le p_r \qquad (3)$$

$$\frac{dq_o(p)}{dp} + \omega_o q_o(p) = -ȗ, \qquad p_r \le p \le p_o \qquad (4)$$

$$\frac{dq_s(p)}{dp} = -\frac{ȗ}{1+\varsigma(P-p)}, \qquad p_o \le p \le P \qquad (5)$$

with boundary conditions $q_r(0) = R_m - R_d$, $q_r(p_r) = 0$, $q_o(p_d) = R_d$, $q_o(p_o) = 0$ and $q_s(p_o) = 0$.

The solutions of equations (1), (2), (3), (4) and (5) are as follows

$$q_r(p) = R_m - R_d - \left(\eta p + \mu\frac{p^2}{2} + \sigma\frac{p^3}{3}\right), \qquad 0 \le p \le p_d \qquad (6)$$

$$q_r(p) = \frac{ȗ}{\omega_r}\left(e^{\omega_r(p_r-p)} - 1\right), \qquad p_d \le p \le p_r \qquad (7)$$

$$q_o(p) = R_d e^{\omega_o(p_d-p)}, \qquad p_d \le p \le p_r \qquad (8)$$

$$q_o(p) = \frac{ȗ}{\omega_o}\left(e^{\omega_o(p_o-p)} - 1\right), \qquad p_r \le p \le p_o \qquad (9)$$

$$q_s(p) = -\frac{ȗ}{\varsigma}\left[ln[1+\varsigma(P-p_o)] - ln[1+\varsigma(P-p)]\right], \qquad p_o \le p \le P \qquad (10)$$

Considering continuity of $q_o(p)$ at $p = p_r$, it follows from equations (8) and (9) that

$$R_d = \frac{ȗ}{\omega_o}\left(e^{\omega_o(p_o-p_d)} - e^{\omega_o(p_r-p_d)}\right), \qquad p_o \le p \le P \qquad (11)$$

Considering continuity of $q_r(p)$ at $p = p_d$, it follows from equations (6) and (7) that

$$R_m = \frac{ȗ}{\omega_o}\left(e^{\omega_o(p_o-p_d)} - e^{\omega_o(p_r-p_d)}\right) + \left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) + \frac{ȗ}{\omega_r}\left(e^{\omega_r(p_r-p_d)} - 1\right), \qquad p_o \le p \le P \qquad (12)$$

The maximum backordered units N_m is obtained at $p = P$, and then from equation (10), it follows that

$$N_m = -q_s(P) = \frac{ȗ}{\varsigma}\left[ln[1+\varsigma(P-p_o)]\right] \qquad (13)$$

Consequently, the order size across the entire time period $[0,\ P]$ is

$$R = R_m + N_m = \frac{ȗ}{\omega_o}\left(e^{\omega_o(p_o-p_d)} - e^{\omega_o(p_r-p_d)}\right) + \left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) + \frac{ȗ}{\omega_r}\left(e^{\omega_r(p_r-p_d)} - 1\right) + \frac{ȗ}{\varsigma}\left[ln[1+\varsigma(P-p_o)]\right] \qquad (14)$$

Sale revenue (SR)

$$SR = \mathrm{SP}_1\left[\int_0^{p_d}(\eta + \mu p + \sigma p^2)dt\right] + \mathrm{SP}_2\left[\int_{p_d}^{p_r} ȗ\,dt + \int_{p_r}^{p_o} ȗ\,dt + \int_{p_o}^{P}\frac{ȗ}{1+\varsigma(P-p)}dt\right]$$

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$$= \mathrm{SP}_1\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) + \mathrm{SP}_2 ȕ(p_o - p_d) + \mathrm{SP}_2\frac{ȕ}{\varsigma}\left[ln[1+\varsigma(P-p_o)]\right] \quad (15)$$

(v) Purchasing cost $= \mathrm{PC}Q$

$$= \mathrm{PC}\left[\frac{ȕ}{\omega_o}\left(e^{\omega_o(p_o-p_d)} - e^{\omega_o(p_r-p_d)}\right) + \left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) + \frac{ȕ}{\omega_r}\left(e^{\omega_r(p_r-p_d)} - 1\right) + \frac{ȕ}{\varsigma}\left[ln[1+\varsigma(P-p_o)]\right]\right] \quad (16)$$

The total profit per unit time $ATPPU(p_o, P)$ is given by

$$ATPPU(p_o, P) = \begin{cases} ATPPU_{11}(p_o, P), & \text{Sub} - \text{case } 1.1\text{: } 0 < T \le p_d \\ ATPPU_{12}(p_o, P), & \text{Sub} - \text{case } 1.2\text{: } p_d < T \le p_r \\ ATPPU_{13}(p_o, P), & \text{Sub} - \text{case } 1.3\text{: } p_r < T \le p_o \\ ATPPU_{14}(p_o, P), & \text{Sub} - \text{case } 1.4\text{: } T > p_o \end{cases} \quad (17)$$

where

$\mathrm{ATPPU}_{11}(\mathrm{p_o}, \mathrm{P}) = \frac{1}{\mathrm{P}}$\{Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales - interest charge+ interest earned\}

$$\begin{aligned} = \frac{1}{\mathrm{P}}\Bigg\{ & \mathrm{SP}_1\left[\int_0^{\mathrm{p_d}}(\eta + \mu \mathrm{p} + \sigma \mathrm{p}^2)\mathrm{dt}\right] + \mathrm{SP}_2\left[\int_{\mathrm{p_d}}^{\mathrm{p_r}} ȕ\mathrm{dt} + \int_{\mathrm{p_r}}^{\mathrm{p_o}} ȕ\mathrm{dt} + \int_{\mathrm{p_o}}^{\mathrm{P}}\frac{ȕ}{1+\varsigma(\mathrm{P}-\mathrm{p})}\mathrm{dt}\right] - \mathrm{PCR} - \mathrm{OC} \\ & - \mathrm{hcc_r}\left[\int_0^{\mathrm{p_d}}\mathrm{q_r(p)dp} + \int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_r(p)dp}\right] \\ & - \mathrm{hcc_o}\left[\int_0^{\mathrm{p_d}}\mathrm{q_o(p)dp} + \int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_o(p)dp} + \int_{\mathrm{p_r}}^{\mathrm{p_o}}\mathrm{q_o(p)dp}\right] - \mathrm{C_b}\left[\int_{\mathrm{p_o}}^{\mathrm{P}} -\mathrm{q_s(p)dp}\right] \\ & - \mathrm{C_\pi} ȕ\int_{\mathrm{p_o}}^{\mathrm{P}}\left(1 - \frac{ȕ}{1+\varsigma(\mathrm{P}-\mathrm{p})}\right)\mathrm{dp} \\ & - \mathrm{pcq_c}\left[\int_{\mathrm{T}}^{\mathrm{p_d}}\mathrm{q_r(p)dp} + \int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_r(p)dp} + \int_{\mathrm{T}}^{\mathrm{p_d}}\mathrm{q_o(p)dp} + \int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_o(p)dp} + \int_{\mathrm{p_r}}^{\mathrm{p_o}}\mathrm{q_o(p)dp}\right] \\ & + \mathrm{SP_1 I_e}\left[\int_0^{\mathrm{T}}(\eta + \mu \mathrm{p} + \sigma \mathrm{p}^2)\mathrm{pdp}\right]\Bigg\} \end{aligned}$$

$$\begin{aligned} = \frac{ȕ}{\mathrm{P}}\Bigg\{ & -\frac{1}{2}\mathrm{W_{11}p_o^2} + \mathrm{X_{11}p_o} - \mathrm{Y_{11}} - \frac{(\mathrm{C_b} + \mathrm{C_\pi}\varsigma)}{2}\mathrm{P}^2 - (\mathrm{SP_2} - \mathrm{PC})\frac{\varsigma}{2}\mathrm{p}^2 + (\mathrm{C_b} + \mathrm{C_\pi}\varsigma)\mathrm{p_o P} \\ & + (\mathrm{SP_2} - \mathrm{PC})\varsigma \mathrm{p p_o} + (\mathrm{SP_2} - \mathrm{PC})\mathrm{p}\Bigg\} \quad (18) \end{aligned}$$

Where

$$\mathrm{W_{11}} = [\mathrm{hc_o}[\omega_o \mathrm{p_d} + 1] + \mathrm{C}\omega_o + (\mathrm{C_b} + \mathrm{C_\pi}\varsigma) + \mathrm{pcI_c}[(\mathrm{p_d} - \mathrm{T})\omega_o + 1] + (\mathrm{SP_2} - \mathrm{PC})\varsigma]$$

$\mathrm{X_{11}} = \left[\mathrm{hc_o}\omega_o \mathrm{p_d^2} + \mathrm{Cp_d}\omega_o + \mathrm{pcI_c}[\mathrm{p_d}\omega_o(\mathrm{p_d} - \mathrm{T}) + \mathrm{T}]\right]$ and

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$$\begin{aligned}
\mathrm{Y}_{11} = -\frac{1}{\mathrm{u}}\Bigg[&(\mathrm{SP}_1-\mathrm{PC})\left(\eta \mathrm{p_d}+\mu\frac{\mathrm{p_d^2}}{2}+\sigma\frac{\mathrm{p_d^3}}{3}\right)-(\mathrm{SP}_2-\mathrm{PC})\mathrm{up_d}-\frac{\mathrm{PCd}\omega_o\mathrm{p_d^2}}{2}-\mathrm{OC}\\
&-\mathrm{hc_r}\left[\left(\eta\frac{\mathrm{p_d^2}}{2}+\mu\frac{\mathrm{p_d^3}}{3}+\sigma\frac{\mathrm{p_d^4}}{4}\right)+\frac{\mathrm{u}}{2}\{\mathrm{p_r^2}+\omega_r(\mathrm{p_r}-\mathrm{p_d})^2\mathrm{p_d}\}\right]\\
&-\mathrm{hc_o}\left[\frac{\mathrm{u}}{2}\{\omega_o(2\mathrm{p_r}\mathrm{p_d^2}-\mathrm{p_r^2}\mathrm{p_d})-\mathrm{p_r^2}\}\right]\\
&-\mathrm{pcI_c}\Bigg[\frac{\mathrm{u}(\mathrm{p_d}-\mathrm{T})}{2}\{\omega_o(2\mathrm{p_r}\mathrm{p_d}-\mathrm{p_r^2})-2\mathrm{p_d}+\omega_r(\mathrm{p_r}-\mathrm{p_d})^2\}+\frac{\eta}{2}(\mathrm{p_d}-\mathrm{T})^2\\
&+\frac{\mu}{6}(2\mathrm{p_d}+\mathrm{T})(\mathrm{p_d}-\mathrm{T})^2+\frac{\sigma}{12}(3\mathrm{p_d^2}+2\mathrm{p_d}\mathrm{T}+\mathrm{T}^2)(\mathrm{p_d}-\mathrm{T})^2+\frac{\mathrm{u}}{2}\mathrm{p_d^2}\Bigg]\\
&+\mathrm{SP_1I_e}\left(\frac{\eta}{2}\mathrm{T}^2+\frac{\mu}{3}\mathrm{T}^3+\frac{\sigma}{4}\mathrm{T}^4\right)\Bigg].
\end{aligned}$$

$\mathrm{ATPPU}_{12}(\mathrm{p_o},\mathrm{P}) = \frac{1}{\mathrm{P}}${Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales - interest charge+ interest earned}

$$\begin{aligned}
=\frac{1}{\mathrm{P}}\Bigg\{&\mathrm{SP}_1\left[\int_0^{\mathrm{p_d}}(\eta+\mu p+\sigma p^2)dt\right]+\mathrm{SP}_2\left[\int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{u}dt+\int_{\mathrm{p_r}}^{\mathrm{p_o}}\mathrm{u}dt+\int_{\mathrm{p_o}}^{\mathrm{P}}\frac{\mathrm{u}}{1+\varsigma(\mathrm{P}-\mathrm{p})}dt\right]-\mathrm{PCR}-\mathrm{OC}\\
&-\mathrm{hc_r}\left[\int_0^{\mathrm{p_d}}\mathrm{q_r(p)dp}+\int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_r(p)dp}\right]-\mathrm{hc_o}\left[\int_0^{\mathrm{p_d}}\mathrm{q_o(p)dp}+\int_{\mathrm{p_d}}^{\mathrm{p_r}}\mathrm{q_o(p)dp}+\int_{\mathrm{p_r}}^{\mathrm{p_o}}\mathrm{q_o(p)dp}\right]\\
&-\mathrm{C_b}\left[\int_{\mathrm{p_o}}^{\mathrm{P}}-\mathrm{q_s(p)dp}\right]-\mathrm{C_\pi u}\int_{\mathrm{p_o}}^{\mathrm{P}}\left(1-\frac{\mathrm{u}}{1+\varsigma(\mathrm{P}-\mathrm{p})}\right)\mathrm{dp}\\
&-\mathrm{pcI_c}\left[\int_{\mathrm{T}}^{\mathrm{p_r}}\mathrm{q_r(p)dp}+\int_{\mathrm{T}}^{\mathrm{p_r}}\mathrm{q_o(p)dp}+\int_{\mathrm{p_r}}^{\mathrm{p_o}}\mathrm{q_o(p)dp}\right]\\
&+\left[\mathrm{SP_1I_e}\left(\int_0^{\mathrm{p_d}}(\eta+\mu p+\sigma p^2)\mathrm{pdp}\right)+\mathrm{SP_2I_e}\left(\int_{\mathrm{p_d}}^{\mathrm{T}}\mathrm{updp}\right)\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
=\frac{\mathrm{u}}{\mathrm{P}}\Bigg\{&-\frac{1}{2}\mathrm{W}_{12}\mathrm{p_o^2}+\mathrm{X}_{12}\mathrm{p_o}-\mathrm{Y}_{12}-\frac{(\mathrm{C_b}+\mathrm{C_\pi}\varsigma)}{2}\mathrm{P}^2-(\mathrm{SP}_2-\mathrm{PC})\frac{\varsigma}{2}\mathrm{p}^2+(\mathrm{C_b}+\mathrm{C_\pi}\varsigma)\mathrm{p_o}\mathrm{P}+(\mathrm{SP}_2-\mathrm{PC})\varsigma\mathrm{pp_o}\\
&+(\mathrm{SP}_2-\mathrm{PC})\mathrm{p}\Bigg\} \qquad (19)
\end{aligned}$$

where

$\mathrm{W}_{12}=[\mathrm{hc_o}[\omega_o\mathrm{p_d}+1]+\mathrm{PC}\omega_o+(\mathrm{C_b}+\mathrm{C_\pi}\varsigma)+\mathrm{pcI_c}],\ \mathrm{X}_{12}=[\mathrm{hc_o}\omega_o\mathrm{p_d^2}+\mathrm{PCp_d}\omega_o+\mathrm{pcI_cT}]$ and

$$\begin{aligned}
\mathrm{Y}_{12} = -\frac{1}{\mathrm{u}}\Bigg[&(\mathrm{SP}_1-\mathrm{PC})\left(\eta \mathrm{p_d}+\mu\frac{\mathrm{p_d^2}}{2}+\sigma\frac{\mathrm{p_d^3}}{3}\right)-(\mathrm{SP}_2-\mathrm{PC})\mathrm{up_d}-\frac{\mathrm{PCd}\omega_o\mathrm{p_d^2}}{2}-\mathrm{OC}\\
&-\mathrm{hc_r}\left[\left(\eta\frac{\mathrm{p_d^2}}{2}+\mu\frac{\mathrm{p_d^3}}{3}+\sigma\frac{\mathrm{p_d^4}}{4}\right)+\frac{\mathrm{u}}{2}\{\mathrm{p_r^2}+\omega_r(\mathrm{p_r}-\mathrm{p_d})^2\mathrm{p_d}\}\right]\\
&-\mathrm{hc_o}\left[\frac{\mathrm{u}}{2}\{\omega_o(2\mathrm{p_r}\mathrm{p_d^2}-\mathrm{p_r^2}\mathrm{p_d})-\mathrm{p_r^2}\}\right]-\mathrm{pcI_c}\frac{\mathrm{u}}{2}\mathrm{T}^2+\mathrm{SP_1I_e}\left(\eta\frac{\mathrm{p_d^2}}{2}+\mu\frac{\mathrm{p_d^3}}{3}+\sigma\frac{\mathrm{p_d^4}}{4}\right)\\
&+\mathrm{SP_2I_e}\left(\frac{\mathrm{uT}^2}{2}-\frac{\mathrm{up_d^2}}{2}\right)\Bigg].
\end{aligned}$$

$ATPPU_{13}(p_o,P)=\frac{1}{P}${Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales - interest charge+ interest earned}

$$
\begin{aligned}
&= \frac{1}{P}\Bigg\{\mathrm{SP}_1\left[\int_0^{p_d}(\eta + \mu p + \sigma p^2)dt\right] + \mathrm{SP}_2\left[\int_{p_d}^{p_r}\text{ȗ}dt + \int_{p_r}^{p_o}\text{ȗ}dt + \int_{p_o}^{P}\frac{\text{ȗ}}{1+\varsigma(P-p)}dt\right] - PCR - OC \\
&\quad - hc_r\left[\int_0^{p_d}q_r(p)dp + \int_{p_d}^{p_r}q_r(p)dp\right] \\
&\quad - hc_o\left[\int_0^{p_d}q_o(p)dp + \int_{p_d}^{p_r}q_o(p)dp + \int_{p_r}^{p_o}q_o(p)dp\right] - C_b\left[\int_{p_o}^{P}-q_s(p)dp\right] \\
&\quad - C_\pi\text{ȗ}\int_{p_o}^{P}\left(1 - \frac{\text{ȗ}}{1+\varsigma(P-p)}\right)dp - pcI_c\left[\int_T^{p_o}q_o(p)dp\right] \\
&\quad + \mathrm{SP}_1 I_e\left[\int_0^{p_d}(\eta + \mu p + \sigma p^2)pdp\right] + \mathrm{SP}_2 I_e\left[\int_{p_d}^{p_r}\text{ȗ}pdp + \int_{p_r}^{T}\text{ȗ}pdp\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\text{ȗ}}{P}\Bigg\{-\frac{1}{2}W_{13}p_o^2 + \mathrm{X}_{13}p_o - Y_{13} - \frac{(C_b + C_\pi\varsigma)}{2}P^2 - (SP_2 - PC)\frac{\varsigma}{2}p^2 + (C_b + C_\pi\varsigma)p_oP \\
&\quad + (SP_2 - PC)\varsigma p p_o + (SP_2 - PC)p\Bigg\} \qquad (20)
\end{aligned}
$$

where

$W_{13} = [hc_o[\omega_o p_d + 1] + PC\omega_o + (C_b + C_\pi\varsigma) + pcI_c], \mathrm{X}_{13} = [hc_o\omega_o p_d^2 + PCp_d\omega_o + pcI_cT]$ and

$$
\begin{aligned}
Y_{13} &= \frac{1}{\text{ȗ}}\Bigg[(SP_1 - PC)\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) - (SP_2 - PC)\text{ȗ}p_d - \frac{PCd\omega_o p_d^2}{2} - OC \\
&\quad - hc_r\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + \frac{\text{ȗ}}{2}\{p_r^2 + \omega_r(p_r - p_d)^2p_d\}\right] \\
&\quad - hc_o\left[\frac{\text{ȗ}}{2}\{\omega_o(2p_rp_d^2 - p_r^2p_d) - p_r^2\}\right] - pcI_c\frac{\text{ȗ}}{2}T^2 + SP_1I_e\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) \\
&\quad + SP_2I_e\left(\frac{\text{ȗ}T^2}{2} - \frac{\text{ȗ}p_d^2}{2}\right)\Bigg].
\end{aligned}
$$

and

$ATPPU_{14}(p_o, P) = \frac{1}{P}${Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales + interest earned}

$$
\begin{aligned}
&= \frac{1}{P}\Bigg\{\mathrm{SP}_1\left[\int_0^{p_d}(\eta + \mu p + \sigma p^2)dt\right] + \mathrm{SP}_2\left[\int_{p_d}^{p_r}\text{ȗ}dt + \int_{p_r}^{p_o}\text{ȗ}dt + \int_{p_o}^{P}\frac{\text{ȗ}}{1+\varsigma(P-p)}dt\right] - PCR - OC \\
&\quad - hc_r\left[\int_0^{p_d}q_r(p)dp + \int_{p_d}^{p_r}q_r(p)dp\right] \\
&\quad - hc_o\left[\int_0^{p_d}q_o(p)dp + \int_{p_d}^{p_r}q_o(p)dp + \int_{p_r}^{p_o}q_o(p)dp\right] - C_b\left[\int_{p_o}^{P}-q_s(p)dp\right] \\
&\quad - C_\pi\text{ȗ}\int_{p_o}^{P}\left(1 - \frac{\text{ȗ}}{1+\varsigma(P-p)}\right)dp \\
&\quad + \mathrm{SP}_1 I_e\left(\int_0^{p_d}(\eta + \mu p + \sigma p^2)pdp + (T - p_o)\int_0^{p_d}(\eta + \mu p + \sigma p^2)dp\right) \\
&\quad + \mathrm{SP}_2 I_e\left(\int_{p_d}^{p_r}\text{ȗ}pdp + (T - p_o)\int_{p_d}^{p_r}\text{ȗ}dp + \int_{p_r}^{p_o}\text{ȗ}pdp + (T - p_o)\int_{p_r}^{p_o}\text{ȗ}dp\right)\Bigg\}
\end{aligned}
$$

$$= \frac{ȕ}{P}\left\{-\frac{1}{2}W_{14}p_o^2 + \mathrm{X}_{14}p_o - Y_{14} - \frac{(C_b + C_\pi ς)}{2}P^2 - (SP_2 - PC)\frac{ς}{2}p^2 + (C_b + C_\pi ς)p_o P + (SP_2 - PC)ςpp_o + (SP_2 - PC)p\right\} \quad (21)$$

where

$W_{14} = [hc_o[\omega_o p_d + 1] + PC\omega_o + (C_b + C_\pi ς) + SP_2 I_e], \mathrm{X}_{14} = \left[hc_o\omega_o p_d^2 + PCp_d\omega_o + SP_2 I_e(p_d + T) - \frac{1}{ȕ}\left[SP_1 I_e\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right)\right]\right]$ and

$$Y_{14} = \frac{1}{ȕ}\left[(SP_1 - PC)\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) - (SP_2 - PC)ȕp_d - \frac{PCd\omega_o p_d^2}{2} - OC - hc_r\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + \frac{ȕ}{2}\{p_r^2 + \omega_r(p_r - p_d)^2 p_d\}\right] - hc_o\left[\frac{ȕ}{2}\{\omega_o(2p_r p_d^2 - p_r^2 p_d) - p_r^2\}\right] + SP_1 I_e\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + T\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right)\right] - SP_2 I_e\frac{ȕ}{2}(2T + p_d)p_d\right].$$

Optimal Decision

The necessary and sufficient conditions are developed to determine the best ordering policies that optimizes the total profit per unit time. The necessary condition for the total profit per unit time $ATPPU_{ij}(p_o, P)$ to be minimum are $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial p_o} = 0$ and $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial P} = 0$ for $i = 1$ when $p_r > p_d$ and $j = 1, 2, 3, 4$. The value of (p_o, P) obtained from $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial p_o} = 0$ and $\frac{\partial ATPPU_{i\,j}(p_o,P)}{\partial P} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial p_o^2}\right)\left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial P^2}\right) - \left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial p_o\,\partial P}\right)^2\right\} > 0$ is satisfied which gives a minimum for the total profit per unit time $ATPPU_{ij}(p_o, P)$.

Optimality condition for sub-case 1.1: $0 < M \le t_d$

The necessary conditions for the total profit in equation (18) to be the minimum are $\frac{\partial ATPPU_{11}(p_o,P)}{\partial p_o} = 0$ and $\frac{\partial ATPPU_{11}(p_o,P)}{\partial P} = 0$, which give

$$\frac{\partial ATPPU_{11}(p_o, P)}{\partial p_o} = \frac{ȕ}{P}\{-W_{11}p_o + \mathrm{X}_{11} + (C_b + C_\pi ς)P + (SP_2 - PC)ςp\} = 0 \quad (22)$$

and

$$P = \frac{1}{(C_b + C_\pi ς) + (SP_2 - PC)ς}(W_{11}p_o - \mathrm{X}_{11}) \quad (23)$$

Note that

$$W_{11}p_o - \mathrm{X}_{11} = \left[hc_o(p_d\omega_o(p_o - p_d) + p_o) + PC\omega_o(p_o - p_d) + (C_b + C_\pi ς)p_o + pcI_c\big((p_o - T)\omega_o + \omega_o(p_d - T)(p_o - p_d)\big)\right] > 0$$

since $(p_d - T) \ge 0$, $(p_o - T), (p_o - p_d) > 0$

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Similarly

$$\frac{\partial ATPPU_{11}(p_o, P)}{\partial P} = -\frac{\text{ȕ}}{P^2}\left\{-\frac{1}{2}W_{11}p_o^2 + \mathrm{X}_{11}p_o - Y_{11} + \frac{(C_b + C_\pi\varsigma)}{2}P^2 + (SP_2 - PC)\frac{\varsigma}{2}p^2\right\} \quad (24)$$

P from equation (23) is substituted into equation (24) which yields

$$W_{11}\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)p_o^2 - 2\mathrm{X}_{11}(W_{11} - (C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma)p_o \\ - \left(2\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)Y_{11} - \mathrm{X}_{11}^2\right) = 0 \quad (25)$$

Let $\Delta_{11} = W_{11}\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)p_d^2 - 2\mathrm{X}_{11}\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)p_d - \left(2\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)Y_{11} - \mathrm{X}_{11}^2\right)$, then the outcome shown below is attained.

Lemma 1.1

(i) If $\Delta_{11} \leq 0$, then the solution of $p_o \in [p_d, \infty)$ (say p_{o11}^*) which satisfies equation (25) not only exists but also is unique.

(ii) If $\Delta_{11} > 0$, then the solution of $p_o \in [p_d, \infty)$ which satisfies equation (25) does not exist.

Proof of (i): From equation (25), a new function $F_{11}(p_o)$ is defined as follows

$$F_{11}(p_o) = W_{11}\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)p_o^2 - 2\mathrm{X}_{11}\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)p_o \\ - \left(2\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)Y_{11} - \mathrm{X}_{11}^2\right), \quad p_o \in [p_d, \infty) \quad (26)$$

Taking the first-order derivative of $F_{11}(p_o)$ with respect to $p_o \in [p_d, \infty)$ yields

$$\frac{F_{11}(p_o)}{dp_o} = 2(W_{11}p_o - \mathrm{X}_{11})\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right) > 0$$

Because $(W_{11}p_o - \mathrm{X}_{11}) > 0$ and $\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right) = \left[hc_o[\omega_o p_d + 1] + C\omega_o + pcI_c[(p_d - T)\omega_o + 1]\right] > 0$

Hence $F_{11}(p_o)$ is a strictly increasing of p_o in the interval $[p_d, \infty)$. Moreover, $\lim_{p_o \to \infty} F_{11}(p_o) = \infty$ and $F_{11}(p_d) = \Delta_{11} \leq 0$. Therefore, by applying intermediate value theorem, there exists a unique p_o say $p_{11}^* \in [p_d, \infty)$ such that $F_{11}(p_{o11}^*) = 0$. Hence p_{o11}^* is the unique solution of equation (23). Thus, the value of p_o (denoted by p_{o11}^*) can be found from equation (23) and is given by

$$p_{o11}^* = \frac{\mathrm{X}_{11}}{W_{11}} + \frac{1}{W_{11}}\sqrt{\frac{(2W_{11}Y_{11} - \mathrm{X}_{11}^2)\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}{\left(W_{11} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)}} \quad (27)$$

Once p_{o11}^* is obtained, then the value of P (denoted by P_{11}^*) can be found from equation (21) and is given by

$$P_{11}^* = \frac{1}{\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}(W_{11}p_{o11}^* - \mathrm{X}_{11}) \quad (28)$$

Equations (27) and (28) give the optimal values of p_{o11}^* and P_{11}^* for the cost function in equation (18) only if X_{11} satisfies the inequality given in equation (29)

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$$\mathrm{X}_{11}^2 < 2W_{11}Y_{11} \tag{29}$$

Proof of (ii): If $\Delta_{11}> 0$, then from equation (26), $F_{11}(p_o) > 0$. Since $F_{11}(p_o)$ is a strictly increasing function of $p_o \in [p_d, \infty)$, $F_{11}(p_o) > 0$ for all $p_o \in [p_d, \infty)$. Thus, a value of $p_o \in [p_d, \infty)$ cannot be found such that $F_{11}(p_o) = 0$. This completes the proof.

Theorem 1.1

(i) If $\Delta_{11}\leq 0$, then the total profit $ATPPU_{11}(p_o, P)$ is convex and reaches its global minimum at the point $(p_{o11}^*,\ P_{11}^*)$, where $(p_{o11}^*,\ P_{11}^*)$ is the point which satisfies equations (25) and (22).

(ii) If $\Delta_1> 0$, then the total profit $ATPPU_{11}(p_o, P)$ has a minimum value at the point $(p_{o11}^*,\ P_{11}^*)$ where $p_{o11}^* = p_d$ and $P_{11}^* = \frac{1}{((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma)}(W_{11}p_d - \mathrm{X}_{11})$

Proof of (i): When $\Delta_{11}\leq 0$, it is observed that p_{o11}^* and P_{11}^* are the unique solutions of equations (25) and (22) from Lemma l.1(i). Taking the second derivative of $ATPPU_{11}(p_o, P)$ with respect to p_o and P, and then finding the values of these functions at the point $(p_{o11}^*,\ P_{11}^*)$ yields

$$\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial p_o^2}\right|_{(p_{o11}^*,\ P_{11}^*)} = \frac{\text{ȕ}}{P_{11}^*}W_{11} > 0$$

$$\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial p_o \partial P}\right|_{(p_{o11}^*,\ P_{11}^*)} = -\frac{\text{ȕ}}{P_{11}^*}\big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big)$$

$$\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial P^2}\right|_{(p_{o11}^*,\ P_{11}^*)} = \frac{\text{ȕ}}{P_{11}^*}\big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big) > 0$$

and

$$\left(\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial p_o^2}\right|_{(p_{o11}^*,\ P_{11}^*)}\right)\left(\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial P^2}\right|_{(p_{o11}^*,\ P_{11}^*)}\right) - \left(\left.\frac{\partial^2 ATPPU_{11}(p_o,\ P)}{\partial p_o \partial P}\right|_{(p_{o11}^*,\ P_{11}^*)}\right)^2$$
$$= \frac{\text{ȕ}^2\big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big)}{P_{11}^{*2}}\big[hc_o[\omega_o p_d + 1] + PC\omega_o + pcI_c[(p_d - T)\omega_o + 1]\big] > 0 \tag{30}$$

It is therefore concluded from equation (28) and Lemma 1.1 that $ATPPU_{11}(p_{o11}^*,\ P_{11}^*)$ is convex and $(p_{o11}^*,\ P_{11}^*)$ is the global minimum point of $ATPPU_{11}(p_o,\ P)$. Hence the values of p_o and P in equations (27) and (28) are optimal.

Proof of (ii): When $\Delta_{11}> 0$, then $F_{11}(p_o) > 0$ for all $p_o \in [p_d,\ \infty)$. Thus, $\frac{\partial ATPPU_{11}(p_o,\ P)}{\partial P} = \frac{F_{11}(p_o)}{P^2} > 0$ for all $p_o \in [p_d, \infty)$ which implies $ATPPU_{11}(p_o,\ P)$ is an increasing function of T. Thus $ATPPU_{11}(p_o,\ P)$ has a minimum value when T is minimum. Therefore, $ATPPU_{11}(p_o,\ P)$ has a minimum value at the point $(p_{o11}^*,\ P_{11}^*)$ where $p_{o11}^* = p_d$ and $P_{11}^* = \frac{1}{((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma)}(W_{11}p_d - \mathrm{X}_{11})$. This completes the proof.

Optimality condition for sub-case 1.2: $t_d < M \leq t_r$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o12}^*) and P (denoted by P_{12}^*) are given by

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$$p_{o12}^{*}=\frac{X_{12}}{W_{12}}+\frac{1}{W_{12}}\sqrt{\frac{(2W_{12}Y_{12}-X_{12}^{2})\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}{\left(W_{12}-\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)\right)}} \tag{31}$$

and

$$P_{12}^{*}=\frac{1}{\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}(W_{12}p_{o12}^{*}-X_{12}) \tag{32}$$

Equations (31) and (32) give the optimal values of p_{o12}^{*} and P_{12}^{*} for the cost function in equation (17) only if X_{12} satisfies the inequality given in equation (33)

$$X_{12}^{2}<2W_{12}Y_{12} \tag{33}$$

Optimality condition for sub-case 1.3: $t_r < T \le t_o$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o13}^{*}) and P (denoted by P_{13}^{*}) are given by

$$p_{o13}^{*}=\frac{X_{13}}{W_{13}}+\frac{1}{W_{13}}\sqrt{\frac{(2W_{13}Y_{13}-X_{13}^{2})\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}{\left(W_{13}-\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)\right)}} \tag{34}$$

$$P_{13}^{*}=\frac{1}{\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}(W_{13}p_{o13}^{*}-X_{13}) \tag{35}$$

Equations (34) and (35) give the optimal values of p_{o13}^{*} and P_{13}^{*} for the cost function in equation (18) only if X_{13} satisfies the inequality given in equation (36)

$$X_{13}^{2}<2W_{13}Y_{13} \tag{36}$$

Optimality condition for sub-case 1.4: $M > t_o$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o14}^{*}) and P (denoted by P_{14}^{*}) are given by

$$p_{o14}^{*}=\frac{X_{14}}{W_{14}}+\frac{1}{W_{14}}\sqrt{\frac{(2W_{14}Y_{14}-X_{14}^{2})\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}{\left(W_{14}-\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)\right)}} \tag{37}$$

$$P_{14}^{*}=\frac{1}{\left((C_b+C_\pi\varsigma)+(SP_2-PC)\varsigma\right)}(W_{14}p_{o14}^{*}-X_{14}) \tag{38}$$

Equations (37) and (38) give the optimal values of p_{o14}^{*} and P_{14}^{*} for the cost function in equation (19) only if X_{14} satisfies the inequality given in equation (39)

$$X_{14}^{2}<2W_{14}Y_{14} \tag{39}$$

Thus, the economic order quantity (EOR) corresponding to the optimal cycle length P^{*} will be computed as follows:

$EOR^{*}=$The maximum inventory $+$the backordered units during the shortage period.

$$= \frac{ᶙ}{\omega_o}\left(e^{\omega_o(p_o^*-p_d)} - e^{\omega_o(p_r-p_d)}\right) + \left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) + \frac{ᶙ}{\omega_r}\left(e^{\omega_r(p_r-p_d)} - 1\right) + \frac{ᶙ}{\varsigma}\left[ln[1+\varsigma(P^* - p_o^*)]\right] \quad (40)$$

Numerical Examples

This section provides some numerical examples to illustrate the model established.

Example 3.1.1 (Sub-case 1.1)

Given the following input parameters: $OC = \$550$/order, $PC = \$55$/unit/year, $S = \$75$/unit/year, $hc_o = \$10$/unit/year, $hc_r = \$17$/unit/year, $\omega_o = 0.08$ units/year, $\omega_r = 0.04$ units/year, $\eta = 1080$ units, $\mu = 280$ units, $\sigma = 25$ units, $ᶙ = 550$ units, $p_d = 0.2971$ year (109 days), $p_r = 0.3126$ year (114 days), $C_b = \$25$/unit/year, $C_\pi = \$10$/unit/year, $\varsigma = 0.75$, $T = 0.0884$ year (32 days), $I_c = 0.11$, $I_e = 0.09$. It is observed that $T \le p_d$, $\Delta_{11} = -47.1059 < 0$, $X_{11}^2 = 0.6696$, $2W_{11}Y_{11} = 116.1989$ and hence $X_{11}^2 < 2W_{11}Y_{11}$. Substituting the above values in equations $(27), (28), (18)$ and (40), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o11}^* = 0.5247$ year (191 days), $P_{11}^* = 0.7002$ year (256 days), $ATPPU_{11}(p_{o11}^*, P_{11}^*) = \$2775.5607$ per year and $EOR_{11}^* = 767.4678$ units per year.

Example 3.1.2 (Sub-case 1.2)

The data are adopted as in Example 3.1.1 apart from $T = 0.2998$ year (110 days). It is observed that $T > p_d$, $\Delta_{12} = -55.6558 < 0$, $X_{12}^2 = 3.5659$, $2W_{12}Y_{12} = 109.0987$ and hence $X_{12}^2 < 2W_{12}Y_{12}$. Substituting the above values in equations $(31), (32), (19)$ and (40), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o12}^* = 0.4654$ year (170 days), $P_{12}^* = 0.5857$ year (214 days), $ATPPU_{12}(p_{o12}^*, P_{12}^*) = \$2372.4383$ per year and $EOR_{12}^* = 993.5549$ units per year.

Example 3.1.3 (Sub-case 1.3)

The data are adopted as in Example 3.1.1 apart from $T = 0.32764$ year (120 days). It is observed that $T > p_r$, $\Delta_{13} = -44.0696 < 0$, $X_{13}^2 = 2.4398$, $2W_{13}Y_{13} = 94.0811$ and hence $X_{13}^2 < 2W_{13}Y_{13}$. Substituting the above values in equations $(34), (35), (20)$ and (40), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o13}^* = 0.4250$ year (155 days), $P_{13}^* = 0.5689$ year (208 days), $ATPPU_{13}(p_{o13}^*, P_{13}^*) = \$1853.7399$ per year and $EOR_{13}^* = 1011.6565$ units per year.

Example 3.1.4 (Sub-case 1.4)

The data are adopted as in Example 3.1.1 apart from $T = 0.3859$ year (141 days). It is observed that $\Delta_{14a} = -46.6787 < 0$, $\Delta_{14b} = 8.9876 > 0$, $X_{14}^2 = 1.0959$, $2W_{14}Y_{14} = 68.0493$. Here $\Delta_{14a} \le 0 \le \Delta_{14b}$ and $X_{14}^2 < 2W_{14}Y_{14}$. Substituting the above values in equations $(37), (38), (21)$ and (40), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o14}^* = 0.3588$ year (131 days), $P_{14}^* = 0.4716$ year (172 days), $ATPPU_{14}(p_{o14}^*, P_{14}^*) = \$2598.7098$ per year and $EOR_{14}^* = 1198.6876$ units per year. It is also seen that $T > p_o$.

Case II: when $t_d > t_r$ (Deterioration starts after the inventory level in the rented ware-house becomes zero)

Figure 3.2 designates behaviours of the inventory system. During the time interval $[0,\ p_r]$, the inventory level $q_r(p)$ in the rented ware-house is depleting gradually due to market demand only and it is assumed to be a

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quadratic function of time p whereas in the owned ware-house inventory level remains unchanged. At time interval $[p_r,\ p_d]$ the inventory level $q_o(p)$ in the owned ware-house is depleting due to demand from the consumers and is also assumed to be a quadratic function of time p. At time interval $[p_d,\ p_o]$, the inventory level $q_o(p)$ in the owned ware-house depletes to zero due to the combined effects of demand from the consumers and deterioration. Shortages occur at the time $p = p_o$ and are partially backlogged in the interval $[p_o, P]$. The whole process of the inventory is repeated.

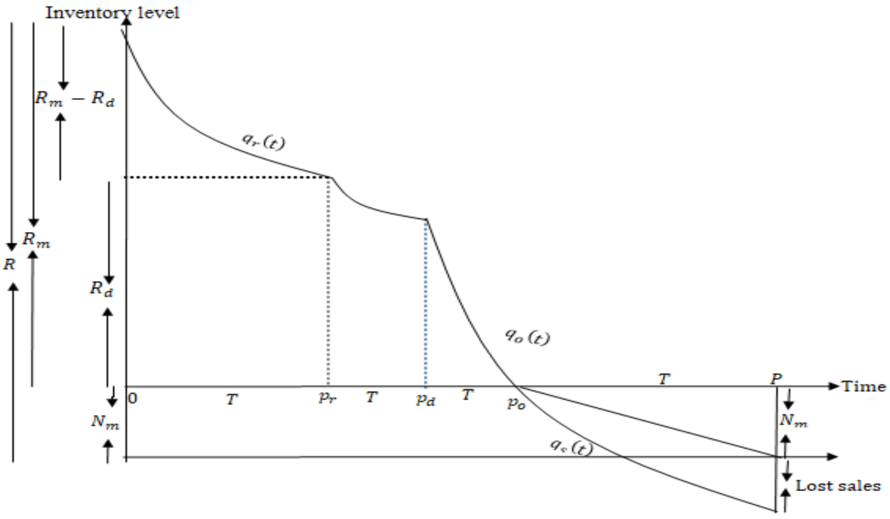

Figure 3.2: Two-ware-house inventory system when $p_d > p_r$

The differential equations that describe the inventory level in both rented ware-house and owned ware-house at any time p over the period $[0,\ P]$ are given by

$$\frac{dq_r(p)}{dp} = -(\eta + \mu p + \sigma p^2), \qquad 0 \le p \le\ p_r \qquad (41)$$

$$\frac{dq_o(p)}{dp} = -(\eta + \mu p + \sigma p^2), \qquad p_r \le p \le\ p_d \qquad (42)$$

$$\frac{dq_o(p)}{dp} + \omega_o q_o(p) = -\breve{u}, \qquad p_d \le p \le\ p_o \qquad (43)$$

$$\frac{dq_s(p)}{dp} = -\frac{\breve{u}}{1 + \varsigma(P - p)}, \qquad p_o \le p \le\ P \qquad (44)$$

with boundary conditions $q_r(p_r) = 0$, $q_o(p_r) =\ R_d$, $q_o(p_o) = 0$ and $q_s(p_o) = 0$.

The solutions of equations (41), (42), (43) and (44) are as follows

$$q_r(p) = \eta(p_r - p) + \frac{\mu}{2}(p_r^2 - p^2) + \frac{\sigma}{3}(p_r^3 - p^3), \qquad 0 \le p \le\ p_r \qquad (45)$$

$$q_o(p) = R_d + \eta(p_r - p) + \frac{\mu}{2}(p_r^2 - p^2) + \frac{\sigma}{3}(p_r^3 - p^3), \qquad p_r \le p \le\ p_d \qquad (46)$$

$$q_o(p) = \frac{\breve{u}}{\omega_o}\left(e^{\omega_o(p_o - p)} - 1\right), \qquad p_d \le p \le\ p_o \qquad (47)$$

$$q_s(p)\ = -\frac{\breve{u}}{\varsigma}\left[ln[1 + \varsigma(P - p_o)] - ln[1 + \varsigma(P - p)]\right], \qquad p_o \le p \le\ P \qquad (48)$$

Considering continuity of $q_o(p)$ at $p = p_d$, it follows from equations (46) and (47) that

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$$R_d = \eta(p_d - p_r) + \frac{\mu}{2}(p_d^2 - p_r^2) + \frac{\sigma}{3}(p_d^3 - p_r^3) + \frac{\text{ȕ}}{\omega_o}\left(e^{\omega_o(p_o - p_d)} - 1\right) \qquad (49)$$

Now, at $p = 0$ when $q_r(p) = R_m - R_d$ and solving equation (45) to get the maximum inventory level per cycle as

$$R_m = \eta p_d + \frac{\mu}{2}p_d^2 + \frac{\sigma}{3}p_d^3 + \frac{\text{ȕ}}{\omega_o}\left(e^{\omega_o(p_o - p_d)} - 1\right) \qquad (50)$$

The maximum backordered units N_m is obtained at $p = P$, and then from equation (48), it follows that

$$N_m = -q_s(P) = \frac{\text{ȕ}}{\varsigma}\left[ln[1 + \varsigma(P - p_o)]\right] \qquad (51)$$

Thus the order size during total time interval $[0,\ P]$ is

$$R = ATPPU + N_m = \eta p_d + \frac{\mu}{2}p_d^2 + \frac{\sigma}{3}p_d^3 + \frac{\text{ȕ}}{\omega_o}\left(e^{\omega_o(p_o - p_d)} - 1\right) + \frac{\text{ȕ}}{\varsigma}\left[ln[1 + \varsigma(P - p_o)]\right], \qquad p_o \leq p \leq P \qquad (52)$$

The total profit per unit time $ATPPU\left(p_{o},P\right)$ is given by

$$ATPPU\left(p_{o},P\right) = \begin{cases} ATPPU_{21}\left(p_{o},P\right), & \text{Sub} - \text{case } 2.1 \qquad 0 < T \leq p_r \\ ATPPU_{22}\left(p_{o},P\right), & \text{Sub} - \text{case } 2.2 \qquad p_r < T \leq p_d \\ ATPPU_{23}\left(p_{o},P\right), & \text{Sub} - \text{case } 2.3 \qquad p_d < T \leq p_o \\ ATPPU_{24}\left(p_{o},P\right), & \text{Sub} - \text{case } 2.4 \qquad T > p_o \end{cases} \qquad (53)$$

where

$ATPPU_{21}(p_o, P) = \frac{1}{P}${Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales -interest charge+ interest earned}

$$= \frac{1}{P}\Bigg\{\mathrm{SP}_1\left[\int_0^{p_r}(\eta + \mu p + \sigma p^2)dt\right] + \mathrm{SP}_2\left[\int_{p_r}^{p_d}\text{ȕ}dt + \int_{p_d}^{p_o}\text{ȕ}dt + \int_{p_o}^{P}\frac{\text{ȕ}}{1 + \varsigma(P - p)}dt\right] - PCR - OC$$
$$- hcc_r\left[\int_0^{p_r}q_r(p)dp + \int_{p_r}^{p_d}q_r(p)dp\right]$$
$$- hcc_o\left[\int_0^{p_r}q_o(p)dp + \int_{p_r}^{p_d}q_o(p)dp + \int_{p_d}^{p_o}q_o(p)dp\right] - C_b\left[\int_{p_o}^{P}-q_s(p)dp\right]$$
$$- C_\pi\text{ȕ}\int_{p_o}^{P}\left(1 - \frac{\text{ȕ}}{1 + \varsigma(P - p)}\right)dp$$
$$- pcq_c\left[\int_T^{p_r}q_r(p)dp + \int_{p_r}^{T}q_r(p)dp + \int_T^{p_d}q_o(p)dp + \int_{p_r}^{p_d}q_o(p)dp + \int_{p_d}^{p_o}q_o(p)dp\right]$$
$$+ \mathrm{SP}_1 I_e\left[\int_0^T(\eta + \mu p + \sigma p^2)pdp\right]\Bigg\}$$

$$= \frac{\text{ȕ}}{P}\Bigg\{-\frac{1}{2}W_{21}p_o^2 + \mathrm{X}_{21}p_o - Y_{21} - \frac{(C_b + C_\pi\varsigma)}{2}P^2 - (SP_2 - PC)\frac{\varsigma}{2}p^2 + (C_b + C_\pi\varsigma)p_oP$$
$$+ (SP_2 - PC)\varsigma p p_o + (SP_2 - PC)p\Bigg\} \qquad (54)$$

Where

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$$W_{21} = [hc_o[\omega_o p_d + 1] + C\omega_o + (C_b + C_\pi \varsigma) + pcI_c[(p_d - T)\omega_o + 1] + (SP_2 - PC)\varsigma]$$

$X_{21} = \left[hc_o\omega_o p_d^2 + Cp_d\omega_o + pcI_c[p_d\omega_o(p_d - T) + T]\right]$ and

$$\begin{aligned} Y_{21} = -\frac{1}{\breve{u}}\Bigg[&(SP_1 - PC)\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) - (SP_2 - PC)\breve{u}p_d - \frac{PCd\omega_o p_d^2}{2} - OC \\ &- hc_r\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + \frac{\breve{u}}{2}\{p_d^2 + \omega_r(p_d - p_r)^2 p_r\}\right] \\ &- hc_o\left[\frac{\breve{u}}{2}\{\omega_o(2p_d p_r^2 - p_d^2 p_r) - p_d^2\}\right] \\ &- pcI_c\Bigg[\frac{\breve{u}(p_d - T)}{2}\{\omega_o(2p_r p_d - p_d^2) - 2p_r + \omega_r(p_d - p_r)^2\} + \frac{\eta}{2}(p_d - T)^2 \\ &+ \frac{\mu}{6}(2p_d + T)(p_d - T)^2 + \frac{\sigma}{12}(3p_d^2 + 2p_d T + T^2)(p_d - T)^2 + \frac{\breve{u}}{2}p_d^2\Bigg] \\ &+ \mathrm{SP}_1 I_e\left(\frac{\eta}{2}T^2 + \frac{\mu}{3}T^3 + \frac{\sigma}{4}T^4\right)\Bigg]. \end{aligned}$$

$ATPPU_{22}(p_o, P) = \frac{1}{P}${Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales - interest charge+ interest earned}

$$\begin{aligned} = \frac{1}{P}\Bigg\{&\mathrm{SP}_1\left[\int_0^{p_r}(\eta + \mu p + \sigma p^2)dt\right] + \mathrm{SP}_2\left[\int_{p_r}^{p_d}\breve{u}dt + \int_{p_d}^{p_o}\breve{u}dt + \int_{p_o}^{P}\frac{\breve{u}}{1 + \varsigma(P - p)}dt\right] - PCR - OC \\ &- hcc_r\left[\int_0^{p_r}q_r(p)dp + \int_{p_r}^{p_d}q_r(p)dp\right] \\ &- hcc_o\left[\int_0^{p_r}q_o(p)dp + \int_{p_r}^{p_d}q_o(p)dp + \int_{p_d}^{p_o}q_o(p)dp\right] - C_b\left[\int_{p_o}^{P} -q_s(p)dp\right] \\ &- C_\pi\breve{u}\int_{p_o}^{P}\left(1 - \frac{\breve{u}}{1 + \varsigma(P - p)}\right)dp \\ &- pcI_c\left[\int_{p_r}^{T}q_r(p)dp + \int_{T}^{p_d}q_o(p)dp + \int_{p_d}^{p_o}q_o(p)dp\right] \\ &+ \left[\mathrm{SP}_1 I_e\left(\int_0^{p_r}(\eta + \mu p + \sigma p^2)pdp + \int_{p_r}^{T}(\eta + \mu p + \sigma p^2)pdp\right)\right]\Bigg\} \end{aligned}$$

$$\begin{aligned} = \frac{\breve{u}}{P}\Bigg\{&-\frac{1}{2}W_{22}p_o^2 + X_{22}p_o - Y_{22} - \frac{(C_b + C_\pi\varsigma)}{2}P^2 - (SP_2 - PC)\frac{\varsigma}{2}p^2 + (C_b + C_\pi\varsigma)p_o P \\ &+ (SP_2 - PC)\varsigma p p_o + (SP_2 - PC)p\Bigg\} \qquad (55) \end{aligned}$$

where

$W_{22} = [hc_o[\omega_o p_d + 1] + PC\omega_o + (C_b + C_\pi\varsigma) + pcI_c]$, $X_{22} = [hc_o\omega_o p_d^2 + PCp_d\omega_o + pcI_c T]$ and

$$\begin{aligned} Y_{22} = -\frac{1}{\breve{u}}\Bigg[&(SP_1 - PC)\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) - (SP_2 - PC)\breve{u}p_d - \frac{PCd\omega_o p_d^2}{2} - OC \\ &- hc_r\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + \frac{\breve{u}}{2}\{p_d^2 + \omega_r(p_d - p_r)^2 p_r\}\right] \\ &- hc_o\left[\frac{\breve{u}}{2}\{\omega_o(2p_d p_r^2 - p_d^2 r) - p_d^2\}\right] - pcI_c\frac{\breve{u}}{2}T^2 + SP_1 I_e\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right)\Bigg]. \end{aligned}$$

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$ATPPU_{23}(p_o,P) = = \frac{1}{P}$ {Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales - interest charge+ interest earned}

$$= \frac{1}{P}\Bigg\{\mathrm{SP}_1\left[\int_0^{p_r}(\eta+\mu p+\sigma p^2)dt\right]+\mathrm{SP}_2\left[\int_{p_r}^{p_d}\text{ȗ}dt+\int_{p_d}^{p_o}\text{ȗ}dt+\int_{p_o}^{P}\frac{\text{ȗ}}{1+\varsigma(P-p)}dt\right]-PCR-OC$$
$$-hcc_r\left[\int_0^{p_r}q_r(p)dp+\int_{p_r}^{p_d}q_r(p)dp\right]-hcc_o\left[\int_0^{p_r}q_o(p)dp+\int_{p_r}^{p_d}q_o(p)dp+\int_{p_d}^{p_o}q_o(p)dp\right]$$
$$-C_b\left[\int_{p_o}^{P}-q_s(p)dp\right]-C_\pi\text{ȗ}\int_{p_o}^{P}\left(1-\frac{\text{ȗ}}{1+\varsigma(P-p)}\right)dp-pcI_c\left[\int_T^{p_o}q_o(p)dp\right]$$
$$+\mathrm{SP}_1I_e\left[\int_0^{p_d}(\eta+\mu p+\sigma p^2)pdp+\int_{p_r}^{p_d}(\eta+\mu p+\sigma p^2)pdp\right]+\mathrm{SP}_2I_e\left[\int_{p_r}^{T}\text{ȗ}pdp\right]\Bigg\}$$

$$=\frac{\text{ȗ}}{P}\Bigg\{-\frac{1}{2}W_{23}p_o^2+\mathrm{X}_{23}p_o-Y_{23}-\frac{(C_b+C_\pi\varsigma)}{2}P^2-(SP_2-PC)\frac{\varsigma}{2}p^2+(C_b+C_\pi\varsigma)p_oP+(SP_2-PC)\varsigma pp_o$$
$$+(SP_2-PC)p\Bigg\}\ (56)$$

where

$W_{23}=[hc_o[\omega_op_d+1]+PC\omega_o+(C_b+C_\pi\varsigma)+pcI_c], \mathrm{X}_{23}=\left[hc_o\omega_op_d^2+PCp_d\omega_o+pcI_cT\right]$ and

$$Y_{23}=\frac{1}{\text{ȗ}}\Bigg[(SP_1-PC)\left(\eta p_d+\mu\frac{p_d^2}{2}+\sigma\frac{p_d^3}{3}\right)-(SP_2-PC)\text{ȗ}p_d-\frac{PCd\omega_op_d^2}{2}-OC$$
$$-hc_r\left[\left(\eta\frac{p_d^2}{2}+\mu\frac{p_d^3}{3}+\sigma\frac{p_d^4}{4}\right)+\frac{\text{ȗ}}{2}\{p_r^2+\omega_r(p_d-p_r)^2p_r\}\right]-hc_o\left[\frac{\text{ȗ}}{2}\{\omega_o(2p_dp_r^2-p_d^2p_r)-p_d^2\}\right]$$
$$-pcI_c\frac{\text{ȗ}}{2}T^2+SP_1I_e\left(\eta\frac{p_d^2}{2}+\mu\frac{p_d^3}{3}+\sigma\frac{p_d^4}{4}\right)+SP_2I_e\left(\frac{\text{ȗ}T^2}{2}-\frac{\text{ȗ}p_d^2}{2}\right)\Bigg].$$

and

$ATPPU_{24}(p_o,P) = \frac{1}{P}$ {Sale Revenue- Purchasing cost -Ordering cost - inventory holding cost for rented warehouse -inventory holding cost for owned warehouse - backordered cost-cost of lost sales + interest earned}

$$= \frac{1}{P}\Bigg\{\mathrm{SP}_1\left[\int_0^{p_r}(\eta+\mu p+\sigma p^2)dt\right]+\mathrm{SP}_2\left[\int_{p_r}^{p_d}\text{ȗ}dt+\int_{p_d}^{p_o}\text{ȗ}dt+\int_{p_o}^{P}\frac{\text{ȗ}}{1+\varsigma(P-p)}dt\right]-PCR-OC$$
$$-hcc_r\left[\int_0^{p_r}q_r(p)dp+\int_{p_r}^{p_d}q_r(p)dp\right]$$
$$-hcc_o\left[\int_0^{p_r}q_o(p)dp+\int_{p_r}^{p_d}q_o(p)dp+\int_{p_d}^{p_o}q_o(p)dp\right]-C_b\left[\int_{p_o}^{P}-q_s(p)dp\right]$$
$$-C_\pi\text{ȗ}\int_{p_o}^{P}\left(1-\frac{\text{ȗ}}{1+\varsigma(P-p)}\right)dp$$
$$+\mathrm{SP}_1I_e\left(\int_0^{p_r}(\eta+\mu p+\sigma p^2)pdp+(T-p_o)\int_0^{p_r}(\eta+\mu p+\sigma p^2)dp\right)$$
$$+\mathrm{SP}_1I_e\left(\int_{p_d}^{p_r}(\eta+\mu p+\sigma p^2)pdp+(T-p_o)\int_{p_r}^{p_d}(\eta+\mu p+\sigma p^2)dp+\int_{p_d}^{p_o}\text{ȗ}pdp\right.$$
$$\left.+(T-p_o)\int_{p_d}^{p_o}\text{ȗ}dp\right)\Bigg\}$$

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$$= \frac{ɩ}{P}\left\{-\frac{1}{2}W_{24}p_o^2 + \mathrm{X}_{24}p_o - Y_{24} - \frac{(C_b + C_\pi\varsigma)}{2}P^2 - (SP_2 - PC)\frac{\varsigma}{2}p^2 + (C_b + C_\pi\varsigma)p_oP + (SP_2 - PC)\varsigma p p_o + (SP_2 - PC)p\right\} \quad (57)$$

where

$W_{24} = [hc_o[\omega_o p_d + 1] + PC\omega_o + (C_b + C_\pi\varsigma) + SP_2I_e]$, $\mathrm{X}_{24} = \left[hc_o\omega_o p_d^2 + PCp_d\omega_o + SP_2I_e(p_d + T) - \frac{1}{ɩ}\left[SP_1I_e\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right)\right]\right]$ and

$$Y_{24} = \frac{1}{ɩ}\Bigg[(SP_1 - PC)\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right) - (SP_2 - PC)ɩp_d - \frac{PCd\omega_o p_d^2}{2} - OC - hc_r\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + \frac{ɩ}{2}\{p_d^2 + \omega_r(p_d - p_r)^2p_r\}\right] - hc_o\left[\frac{ɩ}{2}\{\omega_o(2p_dp_r^2 - p_d^2p_r) - p_d^2\}\right] + SP_1I_e\left[\left(\eta\frac{p_d^2}{2} + \mu\frac{p_d^3}{3} + \sigma\frac{p_d^4}{4}\right) + T\left(\eta p_d + \mu\frac{p_d^2}{2} + \sigma\frac{p_d^3}{3}\right)\right] - SP_2I_e\frac{ɩ}{2}(2T + p_d)p_d\Bigg].$$

Optimal Decision

In order to find the optimal ordering policies that minimize the total profit per unit time, the necessary and sufficient conditions are established. The necessary condition for the total profit per unit time $ATPPU_{ij}(p_o, P)$ to be minimum are $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial p_o} = 0$ and $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial P} = 0$ for $i = 2$ when $p_d > p_r$ and $j = 1, 2, 3, 4$. The value of (p_o, P) obtained from $\frac{\partial ATPPU_{ij}(p_o,P)}{\partial p_o} = 0$ and $\frac{\partial ATPPU_{i\ j}(p_o,P)}{\partial P} = 0$ and for which the sufficient condition $\left\{\left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial p_o^2}\right)\left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial P^2}\right) - \left(\frac{\partial^2 ATPPU_{ij}(p_o,P)}{\partial p_o\,\partial P}\right)^2\right\} > 0$ is satisfied gives a minimum for the total profit per unit time $ATPPU_{ij}(p_o, P)$.

Optimality condition for sub-case 2.1: $0 < T \le t_r$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o12}^*) and P (denoted by P_{12}^*) are given by

$$p_{o21}^* = \frac{\mathrm{X}_{21}}{W_{21}} + \frac{1}{W_{21}}\sqrt{\frac{(2W_{21}Y_{21} - \mathrm{X}_{21}^2)\big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big)}{\big(W_{21} - \big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big)\big)}} \quad (58)$$

$$P_{21}^* = \frac{1}{\big((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\big)}(W_{21}p_{o21}^* - \mathrm{X}_{21}) \quad (59)$$

Equations (72) and (73) give the optimal values of p_{o21}^* and P_{21}^* for the cost function in equation (54) only if X_{21} satisfies the inequality given in equation (74)

$$\mathrm{X}_{21}^2 < 2W_{21}Y_{21} \quad (60)$$

Optimality condition for sub-case 2.2: $t_r < T \leq t_d$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o12}^*) and P (denoted by P_{12}^*) are given by

$$p_{o22}^* = \frac{X_{22}}{W_{22}} + \frac{1}{W_{22}}\sqrt{\frac{(2W_{22}Y_{22} - X_{22}^2)\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}{\left(W_{22} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)}} \tag{61}$$

$$P_{22}^* = \frac{1}{\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}(W_{22}p_{o22}^* - X_{22}) \tag{62}$$

Equations (61) and (62) give the optimal values of p_{o22}^* and P_{22}^* for the cost function in equation (55) only if X_{22} satisfies the inequality given in equation (63)

$$X_{22}^2 < 2W_{22}Y_{22} \tag{63}$$

Optimality condition for sub-case 2.3: $t_d < T \leq t_o$

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o12}^*) and P (denoted by P_{12}^*) are given by

$$p_{o23}^* = \frac{X_{23}}{W_{23}} + \frac{1}{W_{23}}\sqrt{\frac{(2W_{23}Y_{23} - X_{23}^2)\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}{\left(W_{23} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)}} \tag{64}$$

$$P_{23}^* = \frac{1}{\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}(W_{23}p_{o23}^* - X_{23}) \tag{65}$$

Equations (64) and (65) give the optimal values of p_{o23}^* and $P_{23}^*$for the cost function in equation (56) only if X_{23} satisfies the inequality given in equation (66)

$$X_{23}^2 < 2W_{23}Y_{23} \tag{66}$$

Optimality condition for sub-case 2.4: $M > t_o$.

Applying the same procedure as in sub-case 1.1, the value of p_o (denoted by p_{o12}^*) and P (denoted by P_{12}^*) are given by

$$p_{o24}^* = \frac{X_{24}}{W_{24}} + \frac{1}{W_{24}}\sqrt{\frac{(2W_{24}Y_{24} - X_{24}^2)\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}{\left(W_{24} - \left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)\right)}} \tag{67}$$

$$P_{24}^* = \frac{1}{\left((C_b + C_\pi\varsigma) + (SP_2 - PC)\varsigma\right)}(W_{24}p_{o24}^* - X_{24}) \tag{68}$$

Equations (67) and (58) give the optimal values of p_{o24}^* and P_{24}^* for the cost function in equation (57) only if X_{24} satisfies the inequality given in equation (69).

$$X_{24}^2 < 2W_{24}Y_{24} \tag{69}$$

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Thus, the economic order quantity (EOR) corresponding to the optimal cycle length P^* will be computed as follows:

EOR^* =The maximum inventory +the backordered units during the shortage period.

$$= \eta p_d + \frac{\mu}{2} p_d^2 + \frac{\sigma}{3} p_d^3 + \frac{ɩ}{\omega_o}\left(e^{\omega_o(p_o^* - p_d)} - 1\right) + \frac{ɩ}{\varsigma}\left[ln[1 + \varsigma(P^* - p_o^*)]\right] \qquad (70)$$

Numerical Examples

This section provides some numerical examples to illustrate the model established.

Example 3.2.1 (Sub-case 2.1)

The data are adopted as in Example 3.1.1 apart from $p_d = 0.3201$ year (117 days). It is observed that $T \le p_r$, $\Delta_{21} = -85.2345 < 0$, $X_{21}^2 = 0.8771$, $2W_{21}Y_{21} = 143.3421$ and hence $X_{21}^2 < 2W_{21}Y_{21}$. Substituting the above values in equations $(58), (59), (54)$ and (70), the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o21}^* = 0.4527$ year (165 days), $P_{21}^* = 0.6153$ year (225 days), $ATPPU_{21}(p_{o21}^*, P_{21}^*) =$ \$1996.9865 per year and $EOR_{21}^* = 701.8182$ units per year.

Example 3.2.2 (Sub-case 2.2)

The data are adopted as in Example 3.2.1 apart from $T = 0.3152$ year (115 days). It is observed that $T > p_r$, $\Delta_{22} = -82.4679 < 0$, $X_{22}^2 = 7.6981$, $2W_{22}Y_{22} = 115.8946$ and hence $X_{22}^2 < 2W_{22}Y_{22}$. Substituting the above values in equations $(61), (62), (55)$ and $((70)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o22}^* = 0.4321$ year (158 days), $P_{22}^* = 0.5538$ year (202 days), $ATPPU_{22}(p_{o22}^*, P_{22}^*) =$ \$1798.7077 per year and $EOR_{22}^* = 876.6575$ units per year.

Example 3.2.3 (Sub-case 2.3)

The data are adopted as in Example 3.2.1 apart from $T = 0.3444$ year (126 days). It is observed that $T > p_d$, $\Delta_{23} = -87.0998 < 0$, $X_{23}^2 = 8.9098$, $2W_{23}Y_{23} = 106.7717$ and hence $X_{23}^2 < 2W_{23}Y_{23}$. Substituting the above values in equations $(64), (65), (56)$ and $((70)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o23}^* = 0.4357$ year (159 days), $P_{23}^* = 0.5821$ year (213 days), $ATPPU_{23}(p_{o23}^*, P_{23}^*) =$ \$1721.3664 per year and $EOR_{23}^* = 887.6543$ units per year.

Example 3.2.4 (Sub-case 2.4)

The data are adopted as in Example 3.2.1 apart from $T = 0.4109$ year (150 days). It is observed that $\Delta_{24a} = -168.4024 < 0$, $\Delta_{24b} = 19.0177 > 0$, $X_{24}^2 = 01.0998$, $2W_{24}Y_{24} = 85.6297$. Here hence $\Delta_{24a} \le 0 \le \Delta_{24b}$ and $X_{24}^2 < 2W_{24}Y_{24}$. Substituting the above values in equations $(67), (68), (57)$ and $((70)$, the value of optimal time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and economic order quantity are respectively obtained as follows: $p_{o24}^* = 0.3908$ year (143 days), $P_{24}^* = 0.4878$ year (178 days), $ATPPU_{24}(p_{o24}^*, P_{24}^*) =$ \$1676.0974 per year and $EOR_{24}^* = 912.9875$ units per year. It is also seen that $T > p_o$.

Therefore,

$$ATPPU(p_o^*, P^*) = Max\begin{pmatrix} ATPPU_{11}(p_{o11}^*, P_{11}^*), ATPPU_{12}(p_{o12}^*, P_{12}^*), ATPPU_{12}(p_{o13}^*, P_{13}^*), ATPPU_{14}(p_{o14}^*, P_{14}^*), \\ ATPPU_{21}(p_{o21}^*, P_{21}^*), ATPPU_{22}(p_{o22}^*, P_{22}^*), ATPPU_{23}(p_{o23}^*, P_{23}^*), ATPPU_{24}(p_{o24}^*, P_{24}^*) \end{pmatrix} = ATPPU_{14}(p_{o14}^*, P_{14}^*) = \$2598.7098 \text{ per year}$$

Thus, the optimal solution is: $p_o^* = 0.3588$ year (131 days), $P^* = 0.4716$ year (172 days), $ATPPU(p_o^*, P^*) =$ \$2598.7098 per year and $EOQ^* = 1198.6876$

Sensitivity Analysis

The sensitivity analysis related in conjunction with different parameters is performed by changing each of the parameters from -20% to 20% taking one parameter at a time and keeping the remaining parameters unchanged. The effects of these parameters on length of time at which the inventory level reaches zero in the owned ware-house, cycle length, total profit and the economic order quantity per cycle for the optimal solution has been presented in the figures below.

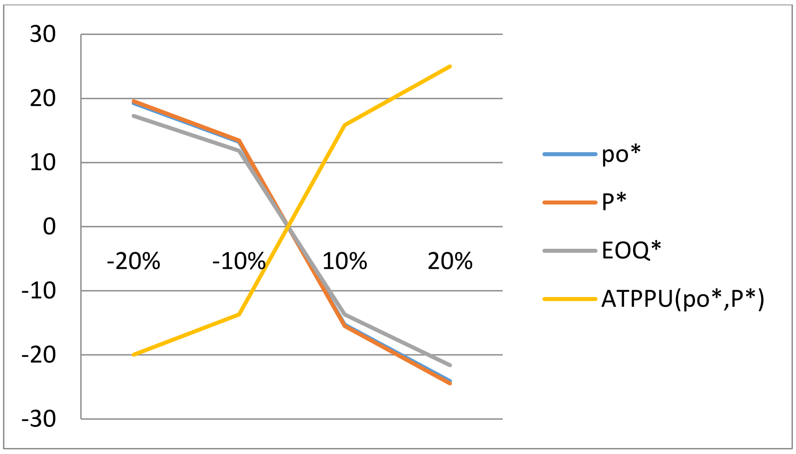

Figure 3.3: effect of change of selling price before deterioration sets in (SP_1) on decision variables

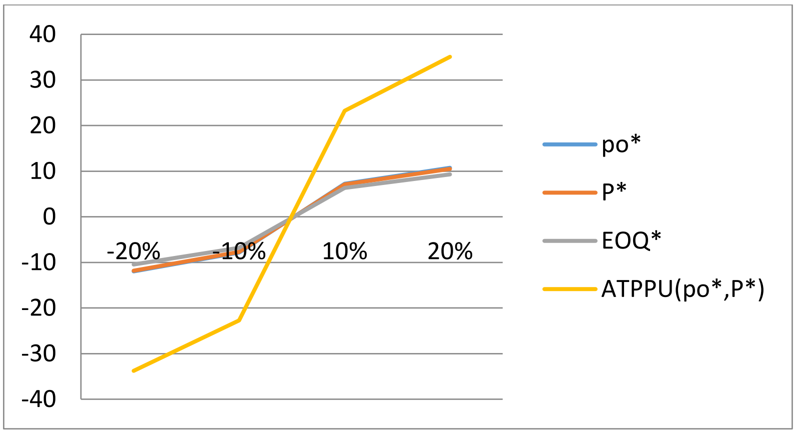

Figure 3.4: effect of change of selling price after deterioration sets in (SP_2) on decision variables

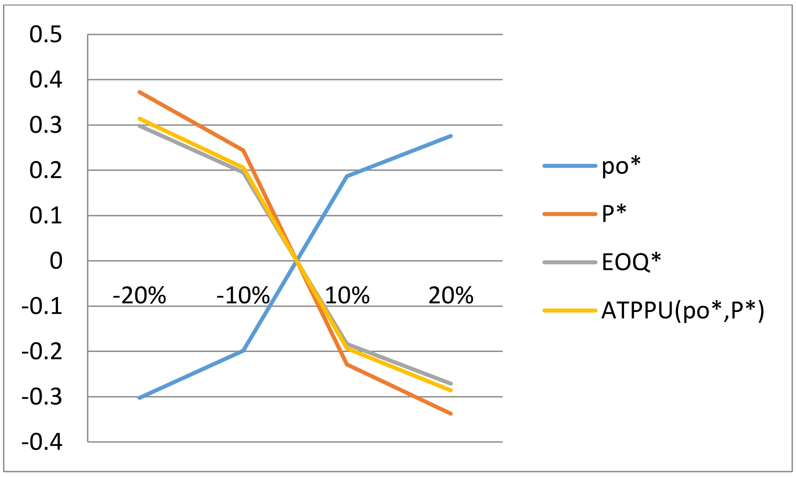

Figure 3.5: effect of change of shortage cost (C_b) on decision variables

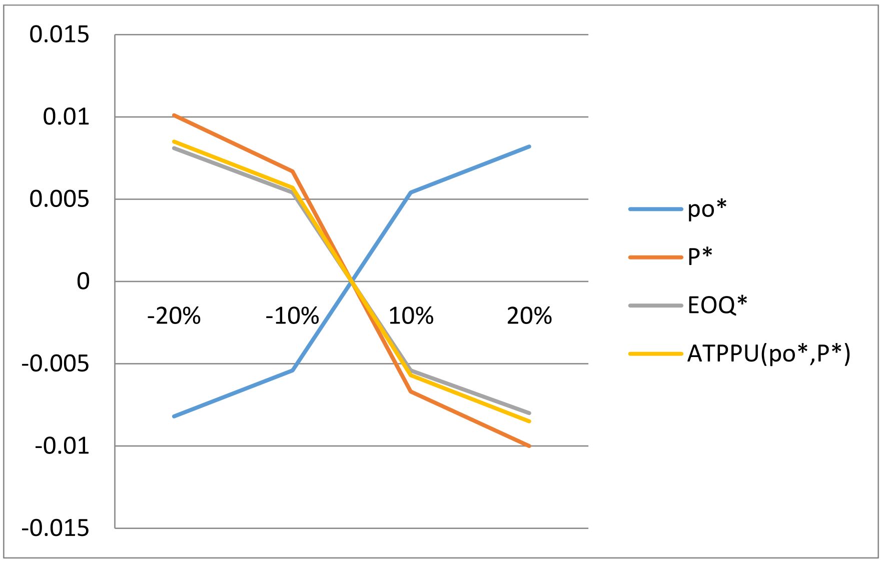

Figure 3.6: effect of change of lost sales (C_π) on decision variables

RESULTS AND DISCUSSION

Based on the computational results shown in Tables and figures above, the following managerial insights are obtained.

(i) From figure 3.3, it is apparently seen that as the unit selling price before deterioration sets in (SP_1) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (p_o^*), cycle length (P^*) and order quantity (EOQ^*) decrease while the total profit$\left(ATPPU(p_o^*,\ P^*)\right)$ increases and vice versa. This implies that as the selling price increases the retailer will order less quantity to enjoy the benefits of trade credit more frequently.

(ii) From figure 3.4, it is evidently seen that as the unit selling price after deterioration sets in (SP_2) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (p_o^*), cycle length (P^*), order quantity (EOQ^*) and the total profit $\left(ATPPU(p_o^*,\ P^*)\right)$ increase and vice versa. This implies that as the selling price is increasing the retailer maximizes higher profit.

(iii) From figure 3.5, it is clearly seen that as the shortage cost (C_b) increases the total profit $\left(ATPPU(p_o^*,\ P^*)\right)$, the economic order quantity (EOQ^*), the optimal cycle length (P^*) decreases while the the optimal time at which the inventory level reaches zero in the owned ware-house (p_o^*), increases.

(iv) From figure 3.6, it evidently seen that as the unit cost of lost sales per unit (C_π) increases, the optimal time at which the inventory level reaches zero in the owned ware-house (p_o^*), also increases while cycle length (P^*), order quantity (EOQ^*) and the total profit $\left(ATPPU(p_o^*,\ P^*)\right)$ decrease.This implies that the retailer should order less quantity when the unit cost of lost sales is high.

CONCLUSION

In this research, a two-warehouse ordering policy for non-instantaneous deteriorating items with two phase demand rates, two-tiered pricing and shortages under trade credit policy has been established. The demand rate before deterioration sets in is assumed to be a time-dependent quadratic function after which it is considered as a constant function up to when the inventory is completely used up. Shortages considered which are partially backlogged. The length of the waiting time would determine whether backlogging will be accepted or not, hence, the backlogging rate is variable and depends on the waiting time for the next replenishment. The optimal time at which the inventory level reaches zero in the owned ware-house, cycle length and order quantity that minimizes total profit has been determined. Some numerical examples have been given to demonstrate the assumed set of results of the model. Then Sensitivity analyses of some model parameters on optimal solutions have been performed and finally, suggestions toward minimizing the total profit of the inventory system have been provided. The model can be extended by taking more realistic assumptions such as variable deterioration, inflation rates, reliability of goods and so on.

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