Numerical Solution of Delay Differential Equations with Heronian Implicit Runge-Kutta Method

Adegoke Stephen Olaniyan[1*], Moshood Tolulope Kazeem[2], Azeez Adebayo Aweda[3], Oladayo Ibukunoluwa Oladimeji[4]

[1,2,3]Department of Mathematics, Lagos State University, Ojo, Lagos

[4]Department of Mathematics, Federal College of Education, Akoka, Lagos

DOI: https://doi.org/10.51584/IJRIAS.2024.912030

Received: 11 December 2024; Accepted: 18 December 2024; Published: 14 January 2025

ABSTRACT

In recent years, there has been growing interest in the numerical solution of Delay Differential Equations (DDEs). This is due to the fact that DDEs provides a good means of modelling many phenomena in diverse application fields ranging from physical sciences, economy, medicine, education to electrodynamics. Hence, the increased attention in the numerical solutions to such problems becomes a necessity. The purpose of this study is to present a numerical method that uses a polynomial interpolating function when solving DDEs. In this paper, Heronian Implicit Runge-Kutta method is considered for the solution of DDEs while the delay term is being estimated with the aid of Hermite Interpolation of the third order. The Stability analysis of this method is considered, and its efficiency is represented and compared with some numerical examples. It is evident from the obtained results of the numerical examples that this numerical method alongside the polynomial interpolating function, which was used to approximate the delay term, is suitable for solving DDEs.

Keywords: Delay Differential Equations, Hermite Interpolation, Heronian Implicit Runge-Kutta Method

INTRODUCTION

Delay differential equations (DDEs) are very important areas of study in applied sciences because they model system where the change in the state depends on past values. They provide a powerful means of modelling many phenomena in diverse fields that recent studies confirmed their important roles in explaining many different effects. Infact, when ordinary differential equation (ODE) based models fail to capture these dynamics accurately, DDEs provide a more robust framework for describing such systems. An O.D.E is of the form:

$$y'(t) = f\big(t, y(t)\big), \quad y(t_0) = c. \tag{1}$$

Delay differential equations (DDEs) extend ordinary differential equations by incorporating time delays which is very crucial in modelling real world processes where feedback, reactions or changes do not occur instantaneously. First order DDE can be written as

$$\begin{aligned} y'(t) &= f\Big(t, y(t), y\Big(t-\tau\big(t, y(t)\big)\Big)\Big), && t > t_0 \\ y(t) &= \phi(t), && t \le t_0 \end{aligned} \tag{2}$$

where $\tau\big(t, y(t)\big)$ is referred to as the delay and $\phi(t)$ the initial function. Due to the complex nature of these equations, analytical investigations have become very difficult and therefore one has to rely mostly on some numerical methods. Several numerical methods have been used for solving DDEs of which the delay terms were approximated with different interpolating polynomial functions. In particular, Runge-Kutta methods for solving ODEs have adapted for the solution of DDEs in recent advances. Kumar D. and Pushpam I.K. [5] adapted a two-stage multiderivative of order 4 developed by Akanbi M.A. [9] to provide solution for DDEs. Here the Lagrange interpolation was applied for estimating the delay term. Numerical treatment of DDE by Runge-Kutta method was used to solve DDE using Hermite interpolation in Ismail F. & Ali A. [6]. The numerical results based on these methods were compared and the Q-stability region of the methods were presented. Shaalini J. V. & Kanaga A. E. [8] presented a new one-step technique to solve DDE by using a non-linear polynomial interpolating function-Lagrange interpolation function to approximate the delay argument. However, in this paper Heronian implicit Runge-Kutta method developed in Olaniyan *et al* [3] will adapted to solve DDE while Hermite polynomial approximate the delay term. The stability property of the method for DDE is considered.

MATERIALS AND METHODS

Implicit Methods for DDE

Let us consider a 2-stage Heronian implicit Runge-Kutta method for solving equation (1) given as

$$y_{n+1} = y_n - h\Phi_H(y_n;h) \tag{3}$$

where

$$\Phi_H(y_n;h) = \left(\frac{K_1+K_2+\sqrt{K_1K_2}}{3}\right) \tag{4}$$

and

$$K_r = f\left(y + h\sum_{s=1}^{R} b_{rs}\, K_s\right) \tag{5}$$

For $R = 2$, equation (5) becomes

$$K_r = f\big(y + h(b_{r1}K_1 + b_{r2}K_2)\big) \tag{6}$$

When adapted to DDE, we get

$$\begin{aligned} y_{n+1} &= y_n + h\left(\frac{K_1+K_2+\sqrt{K_1K_2}}{3}\right) \\ K_r &= f\big(y + h(b_{r1}K_1 + b_{r2}K_2), y(t + c_r h - \tau)\big) \end{aligned} \tag{7}$$

The delay term represented by $y(t + c_r h - \tau)$ requires interpolation to approximate the value. Various interpolation techniques have been used as mentioned in the literature. In this paper, the delay term is approximated by the help of Hermite interpolation and the number of support points must be adapted to the order of the method.

STABILITY ANALYSIS OF THE METHOD

The analysis of the stability of numerical methods for solving DDEs depends majorly on two factors which are the delay term and test equation. The most common used linear test equation which will be considered in this paper, is of the form:

$$\begin{aligned} &y'(t) = \lambda y(t) + \Phi y(t-\tau), \qquad t \geq 0 \\ &y'(t) = \mu y(t) \end{aligned} \tag{8}$$

where λ and Φ are constants and μ is continuous. This test equation (8) then becomes

$$\begin{aligned} &y'(t) = \Phi y(t-\tau), \qquad t \geq 0 \\ &y'(t) = \mu y(t) \end{aligned} \tag{9}$$

if λ is equal to zero.

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume IX Issue XII December 2024

The following basic definitions was introduced by Barwell [12] to establish the concept of stability of numerical methods for solving DDE.

Definition 1: Given a numerical method for solving DDEs, the P-stability region of the method is the set S_P of pairs of (a, b), such that the numerical solution of (8) asymptotically vanishes for the step lengths h satisfying $h = \frac{\tau}{m}$, where m is a positive integer.

Definition 2: Let $\Phi \in C$, Q-stability region of the method is the set S_Q of (a, b), such that the numerical solution of (9) asymptotically vanishes for the step lengths h satisfying $h = \frac{\tau}{m}$; where m is a positive integer, $a = h\lambda$ and $b = h\mu$.

According to Olaniyan *et al* [3] the characteristics polynomial of heronian implicit Runge-Kutta (HIRK) scheme obtained was same as the implicit Runge-Kutta method. Hence, we conclude that by applying the HIRK scheme to equation (8) and making use of hermite interpolation for the delay term from the point t_n to t_{n+1} we will have the same characteristics polynomial of implicit Runge-Kutta method obtained in Lambert [7]. The characteristics polynomial is therefore given as

$$\left(1 - \frac{a}{2m} + \frac{a^2}{12m^2}\right)\xi^{2m+1} - \left(1 + \frac{a}{2m} + \frac{a^2}{12m^2}\right)\xi^{2m} - \frac{b}{2m}\left(1 - \frac{a}{2m}\right)\xi^{m+1} - \frac{b}{2m}\left(1 + \frac{a}{2m}\right)\xi^{m} + \frac{b^2}{12m^2}\xi - \frac{b^2}{12m^2} = 0$$

If for which a and b are real, the P-stability region of HIRK is obtained and given in Figure 1. For Q-stability region of HIRK, $a = 0$ and b is complex.

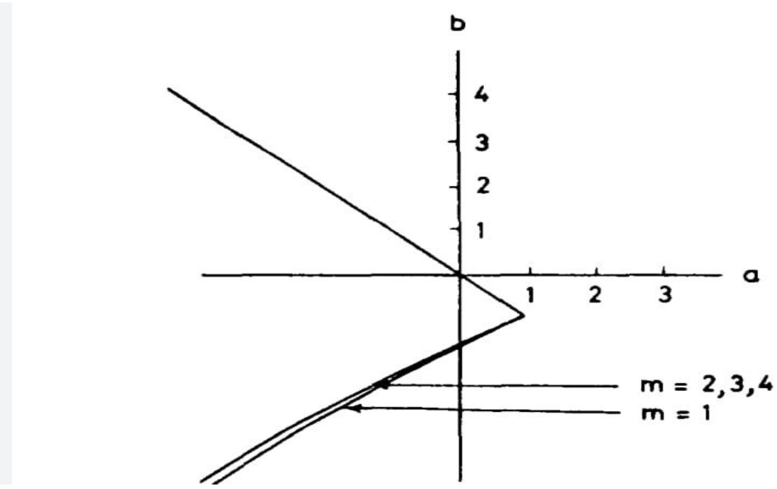

Figure 1: P-stability region of HIRK

Numerical Examples

In this section, HIRK is used to solve some DDEs while the delay terms are evaluated using Hermite interpolation. The numerical results are tabulated and compared with the 2-stage implicit Runge-Kutta method (IRK) and the diagonal implicit Runge-Kutta method (DIRK). The test problems are as follows:

Problem 1: Consider the linear DDE of the form:

$$y' = -24y(t) - e^{-25t}y(t-1), t \geq 0$$

with the initial function:

$$y(t) = e^{-25t}, \quad t \leq 0$$

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume IX Issue XII December 2024

having the exact solution

$$y(t) = e^{-25t}$$

Problem 2: Consider the linear DDE of the form:

$$y' = \left(\frac{-1}{0.03}\right)y(t) + \left(\frac{0.8}{0.03}\right)y(t-1), 0 \leq t \leq 1$$

with the initial function

$$y(t) = \cos t, \quad t \leq 1$$

having the exact solution

$$y(t) = 0.41\cos t + 0.69\sin t + 0.59e^{-33.3t}$$

Problem 3: Consider the linear DDE of the form:

$$y' = 5y(t) + y(t-1)$$

with the initial function

$$y(t) = 5, t \leq 0$$

having the exact solution

$$y(t) = 6e^{5t} - 1$$

The comparison of these methods is performed from absolute error values obtained and are presented in Table 1.

Table 1: Comparison of Absolute Values of Irk, Dirk & Hirk

Time						
	Method	0.2	0.4	0.6	0.8	1.0
Problem 1	IRK	2.59E-06	8.76E-06	1.07E-07	7.93E-07	5.11E-08
	DIRK	7.43E-06	9.55E-06	1.77E-06	4.66E-07	7.96E-07
	HIRK	2.49E-06	7.59E-06	1.66E-07	8.13E-07	6.68E-08
Problem 2	IRK	9.01E-07	3.36E-07	9.91E-06	7.22E-06	5.59E-06
	DIRK	5.53E-07	8.23E-06	1.33E-06	7.88E-05	1.11E-05
	HIRK	9.25E-07	8.53E-07	1.12E-07	9.31E-06	6.73E-06
Problem 3	IRK	1.03E-06	3.03E-06	7.91E-06	9.21E-06	2.77E-07
	DIRK	7.73E-06	5.88E-06	4.01E-06	2.62E-06	1.15E-06
	HIRK	1.12E-06	3.73E-06	8.22E-06	9.66E-06	1.98E-07

CONCLUSION

In this paper, we proposed the numerical solution of DDEs using the Heronian Implicit Runge-Kutta (HIRK) method, with Hermite interpolation employed to approximate the delay terms. The characteristic polynomial and the corresponding stability region of HIRK for DDEs is obtained and is similar to that of other implicit Runge-Kutta methods. In order to ascertain the efficiency of this method, some test problems were solved using the adapted HIRK method and some standard methods of the same stage. It can be seen from the results in Table 1 that the HIRK method is suitable for solving DDEs.

ISSN No. 2454-6194 | DOI: 10.51584/IJRIAS |Volume IX Issue XII December 2024

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