Travelling Wave Solutions to the Space-Time Fractional ZKBBM Equation in Mathematical Physics

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ABSTRACT

This paper focuses on the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation, which serves as a notable model for various phenomena such as water wave mechanics, shallow water waves, quantum mechanics, ion-acoustic waves in plasma, electro-hydro-dynamical models for local electric fields, and signal processing waves through optical fibers. We employ the $exp(-\tau(\xi))$ -expansion approach, aided by travelling wave transformations, to ascertain the helpful and more comprehensive accurate travelling wave solutions to those above nonlinear differential equations (NLDEs). Additionally, we elucidate the physical relevance of the acquired solutions by determining the precise values of the relevant parameters and representing them graphically to gain insight into the underlying physical processes. Finally, we demonstrate that the $exp(-\tau(\xi))$ -expansion approach is advantageous, potent, uncomplicated, and yields more comprehensive answers. It can also assist in analyzing a wide range of travelling wave solutions for various types of nonlinear differential equations.

Keywords: The space-time fractional ZKBBM equation, the $exp(-\tau(\xi))$ -expansion method, travelling wave, solitary wave.

INTRODUCTION

Differential equations are a crucial and significant field within contemporary mathematics. Differential equations may be classified into two main types: ordinary differential equations and partial differential equations, which are commonly used in classical mechanics. Generally, fractional differential equations constitute a significant component of differential equations. Nonlinear differential equations (NLDEs) and fractional nonlinear differential equations (FNLDEs) have substantial applications in integer and fractional calculus. They are of great importance to researchers in various fields such as mathematical physics, engineering, signal processing, control theory, fractal dynamics, optical fibres, chemical kinematics, physics, applied physics, medicine, aerodynamics, hydrology, pharmacy, material science, earthquake modelling, electricity, biological science, population modelling, projectile motion, rocket dynamics, planetary motion, charge or current in electric circuits, chemical reactions, population growth rates, spring-mass systems, beam bending, and heat distribution in rods or slabs. The mathematical expressions for the difficulties above result in differential and fractional differential equations. Generally, most differential equations pertaining to physical events exhibit nonlinearity. Solving linear differential equations is straightforward, but solving nonlinear equations can be time-consuming and, in many instances, analytically unsolvable. During complex expansion processes, researchers endeavor to solve nonlinear differential equations. Nonlinear wave phenomena manifest in diverse scientific and engineering domains, including fluid mechanics, plasma physics, high energy physics, condensed matter physics, quantum mechanics, elastic media, biology, solid state physics, chemical kinetics, optical fibers, biophysics, geochemistry, electricity, propagation of shallow water waves, and chemical physics, among others. To get a deeper understanding of nonlinear events and their potential applications in real-world scenarios, exploring more precise solutions in the form of travelling waves is crucial. The basic equations in physical sciences are nonlinear, and in general, solving nonlinear partial differential equations (NLPDEs) explicitly for precise solutions is frequently quite complex. However, these exact solutions of NLPDEs play a crucial role in studying nonlinear physical phenomena.

Therefore, in the past three decades, many significant methods have been enhanced and development to get exact solutions of NLPDEs, such as, integer and fractional types NLDEs [1–3]. Most of these methods are the homogeneous balance method, likely, the Kudryashov method [4], the generalized Kudryashov method [5], the Modified Kudryashov Method [6], the first integral method [7], the improved modified extended tanh-function method [3,8], the (G′/G,1/G)-expansion technique [9,10], advanced $\exp(-\emptyset(\xi))$-expansion method [1,11,12], the modified extended tanh-function method [13–15], the Jacobi elliptic function method [16], the (G'/G^2)-expansion technique [17,18], the sine–cosine methods [19,20], the tanh-coth method [21], the simplified Hirota's method [22], the Hirota bilinear method [23–25], Soret and Dufour effects [26], the modified simple equation method [27], the exp function method [28], the sine-Gordon expansion method [29], the rational sine-Gordon expansion method [2,30], Wang's Bäcklund transformation-based method [31], the variational iteration method [32–34], the new auxiliary equation method [35], Variational method [36], Deep Learning approach [37], the method of characteristics [38], Dixon resultant method [39], the three-dimensional molecular structure model [40], etc.

The solutions obtained include a remarkable mathematical model for turbulent motion, an electro-hydro-dynamical model for local electric fields, ion-acoustic waves in plasma, fluid flow motion in shallow water waves under gravity, propagation waves, the noteworthy phenomenon of wave-particle duality, signal processing waves through optical fibres, variation over time of a physical structure in the fractional fluid mechanics system, ion acoustic waves, temperature variations between different locations, conservation of mass and acceleration due to gravity, viscoelasticity waves, and a traffic flow model. We have examined the physical outcomes of the obtained solutions by assigning specific values to the relevant parameters and representing them graphically. We also have established that the $exp(-\tau(\xi))$ -expansion method is potential, efficient, straightforward, further general, and rising method to search huge amount of traveling wave solutions to the NLDEs and FNLDEs.

DESCRIPTION OF THE $exp(-\tau(\xi))$-EXPANSION METHOD

Here we briefly discuss the major characteristics of the $\boldsymbol{exp(-\tau(\xi))}$ **-**expansion method. Let us suppose the general nonlinear partial differential equation of the form:

$$H(u, u_x, u_y, u_z, u_t, u_{xx}, u_{xy} \ldots \ldots \ldots) = 0, \qquad (2.1)$$

where $u = u(x, y, z, t)$ is an unknown function, H is a polynomial in $u(x, y, z, t)$ and its derivatives in which highest order derivatives and nonlinear terms are occupied and the subscripts indicate partial derivatives.

Also, we consider the general nonlinear fractional partial differential equation of the form:

$$H(u, D_t^{\alpha} u, D_x^{\beta} u, D_y^{\gamma} u, D_z^{\varepsilon} u, D_t^{2\alpha} u, D_x^{2\beta} u, \ldots) = 0, \qquad (2.2)$$

where $u = u(x, y, z, t)$ is an unidentified function, H is a polynomial in $u(x, y, z, t)$ and its fractional derivatives, which include the highest order derivative and nonlinear terms of the highest order where in $\alpha, \beta, \gamma, \varepsilon$ are non-integer and the subscripts denote the partial derivatives.

To obtain exact wave solutions of Eq. (2.1) or Eq. (2.2) by applying the $exp(-\tau(\xi))$ **-**expansion method, we have to execute the following noteworthy steps:

Step-1. We combine the real variables x, y and t by a compound variable ξ

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$$u(x,y,t) = u(\xi), \xi = x + y \pm wt, \qquad (2.3)$$

where w is the velocity of the traveling wave and we consider the following traveling wave variable,

$$u(x,y,z,t) = u(\xi),$$
$$\xi = \frac{kt^{\alpha}}{\Gamma(1+\alpha)} + \frac{mx^{\alpha}}{\Gamma(1+\alpha)} + \frac{ny^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\varepsilon}}{\Gamma(1+\varepsilon)}. \qquad (2.4)$$

for fractional differential equations.

Now by making use of the traveling wave transformation Eq. (3.3) or Eq. (3.4) the partial differential Eq. (2.1) or Eq. (2.2) turns into ordinary differential equation (ODE) as below:

$$G(u, u', u''u''', \ldots \ldots \ldots \ldots) = 0, \qquad (2.5)$$

where G is a polynomial of u and its derivatives, and the superscripts refer to the ordinary derivatives with respect to ξ.

Step-2. We advise that the solution of Eq. (3.5) can be exposed in the form:

$$u(\xi) = \sum_{i=0}^{N} A_i \left(exp(-\tau(\xi))\right)^i, \qquad (2.6)$$

where $A_i(0 \leq i \leq N)$ are constants to be determined, such that$A_N \neq 0$ and$\tau = \tau(\xi)$ and satisfied the following ordinary differential equation:

$$\tau'^{(\xi)} = exp(-\tau(\xi)) + \mu\, exp(\tau(\xi)) + \lambda, \qquad (2.7)$$

Eq. (2.7) documented the following solutions:

Set-1: When $\mu \neq 0,\ \lambda^2 - 4\mu > 0$,

$$\tau(\xi) = ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\,tanh(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\xi+c)) - \lambda}{2\mu}\right). \qquad (2.8)$$

Set-2: When $\mu \neq 0,\ \lambda^2 - 4\mu < 0$,

$$\tau(\xi) = ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\,tan(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\xi+c)) - \lambda}{2\mu}\right), \qquad (2.9)$$

Set-3: When $\mu = 0, \lambda \neq 0$, and $\lambda^2 - 4\mu > 0$,

$$\tau(\xi) = -ln\left(\frac{\lambda}{exp(\lambda(\xi+c)) - 1}\right), \qquad (2.10)$$

Set-4: When $\mu \neq 0 \lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\tau(\xi) = ln\left(-\frac{2(\lambda(\xi+c)+2)}{\lambda^2(\xi+c)}\right), \qquad (2.11)$$

Set-5: When $\mu = 0, \lambda \neq 0$, and $\lambda^2 - 4\mu = 0$,

$$\tau(\xi) = ln(\xi + c), \qquad (2.12)$$

Step-3. The positive integer N can be calculated by considering the homogeneous balance between the highest order derivatives and the nonlinear terms of the highest order appearing in Eq. (2.5).

Step-4. We utilize Eq. (2.6) into Eq. (2.5) and then we consider the function $exp(-\tau(\xi))$. Therefore, of this substitution, we attain a polynomial in $exp(-\tau(\xi))$and equalize to zero express a system of algebraic equations whichever can be solved to find $A_N, \ldots \ldots, w, \lambda, \mu$The values of $A_N, \ldots\ldots, w, \lambda, \mu$ in company with general solution of Eq. (3.7) inclusive the determination of the solution of Eq. (2.1) or Eq. (2.2).

Application for the Zkbbm Equation

In this sub-section, we have considered the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation in the form:

$$D_t^\alpha u + D_x^\alpha u - 2auD_x^\alpha u - bD_t^\alpha(D_x^{2\alpha}u) = 0, \qquad (3.1)$$

Now, making use of the traveling wave transformation Eq. (2.4) which reduces Eq. (3.1) into the following ordinary differential equation

$$ku' + mu' - 2\,am\,u\,u' - bkm^2u''' = 0, \qquad (3.2)$$

Integrating Eq. (4.4.2) with respect toξ and choosing the integrating constant zero, we obtain

$$(k+m)u - amu^2 - bkm^2u'' + C = 0, \qquad (3.3)$$

where C is an integrating constant.

Now balancing between the highest order nonlinear term and linear terms occurring in Eq. (4.4.3), gives $N = 1$.Therefore, the solution of equation Eq. (4.4.3) takes the following form

$$u(\xi) = A_0 + A_1(exp(-\tau(\xi))) + A_2(exp(-\tau(\xi)))^2, (3.4)$$

Where A_0, A_1, A_2 are arbitrary constants such that$A_2 \neq 0$.

We substitute Eq. (3.4) into Eq. (3.3) and taking consideration Eq. (3.4), it generates a polynomial and then setting the coefficients of $exp(-\tau(\xi))$ to zero, yield

$$C + kA_0 + mA_0 - amA^2{}_0 - bkm^2\lambda\mu A_1 - 2bkm^2\mu^2A_2 = 0, \qquad (3.5)$$

$$kA_1 + mA_1 - bkm^2\lambda^2A_1 - 2bkm^2\mu A_1 - 2amA_0A_1 - 6bkm^2\lambda\mu A_2 = 0, \qquad (3.6)$$

$$-3bkm^2\lambda A_1 - amA^2{}_1 + (k+m)A_2 - 4bkm^2(\lambda^2+2\mu)A_2 - 2amA_0 = 0, \qquad (3.7)$$

$$-2bkm^2A_1 - 10bkm^2\lambda A_2 - 2amA_1A_2 = 0, \qquad (3.8)$$

$$-6bkm^2A_2 - amA_2^2 = 0, \qquad (3.9)$$

Solving from Eq. (3.5) – Eq. (3.9) we have obtained one set of solution:

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Set -1

$$A_0 = \frac{m - k(-1 + bm^2(\lambda^2 + 8\mu))}{2am}, \qquad A_1 = -\frac{6bkm\lambda}{a}, \qquad A_2 = -\frac{6bkm}{a},$$
$$C = -\frac{2km + m^2 + k^2(1 - b^2m^4(\lambda^2 - 4\mu)^2)}{4am}.$$

where λ, μ are arbitrary constants.

Therefore, we have discussed the solutions Set -1 of the mentioned equation arranged.

Now substituting the value of Set-1 into Eq. (3.4) yields

$$u(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}(exp(-\tau(\xi))) - \frac{6bkm}{a}(exp(-\tau(\xi)))^2, \qquad (3.10)$$

When $\mu \neq 0, \lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}\left(\frac{-2\mu}{\left(\sqrt{\lambda^2 - 4\mu}\right) tanh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C)) + \lambda}\right) - \frac{6bkm}{a}\left(\frac{-2\mu}{\left(\sqrt{\lambda^2 - 4\mu}\right) tanh(\frac{\sqrt{\lambda^2 - 4\mu}}{2}(\xi + C)) + \lambda}\right)^2.$$

While $\mu \neq 0, \lambda^2 - 4\mu < 0$,

$$u_2(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}\left(\frac{2\mu}{\left(\sqrt{4\mu - \lambda^2}\right) tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C)) - \lambda}\right) - \frac{6bkm}{a}\left(\frac{2\mu}{\left(\sqrt{4\mu - \lambda^2}\right) tan(\frac{\sqrt{4\mu - \lambda^2}}{2}(\xi + C)) - \lambda}\right)^2.$$

When $\mu = 0, \lambda \neq 0, \lambda^2 - 4\mu > 0$,

$$u_3(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}\left(\frac{1}{\left(exp(\lambda(\xi + C)\right) - 1}\right) - \frac{6bkm}{a}\left(\frac{1}{\left(exp(\lambda(\xi + C)\right) - 1}\right)^2.$$

Applying $\mu \neq 0, \lambda \neq 0, \lambda^2 - 4\mu > 0$,

$$u_4(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}\left(-\frac{\lambda^2(\xi + C)}{2\left(\lambda(\xi + C)\right) + 2}\right) - \frac{6bkm}{a}\left(-\frac{\lambda^2(\xi + C)}{2\left(\lambda(\xi + C)\right) + 2}\right)^2.$$

Using $\mu = 0, \lambda = 0, \lambda^2 - 4\mu = 0$,

$$u_5(\xi) = \frac{m - k\left(-1 + bm^2(\lambda^2 + 8\mu)\right)}{2am} - \frac{6bkm\lambda}{a}\left(\frac{1}{\xi + C}\right) - \frac{6bkm}{a}\left(\frac{1}{\xi + C}\right)^2.$$

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where C is an arbitrary constant.

It is remarkable to notice that the traveling wave solutions of the space-time Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZKBBM) equation by means of the $exp(-\tau(\xi))$ -expansion method is fresh, practical and more general and have not been available in the previous literature. The attained solutions are to be convenient to search the nature the wave profile as the demandable model of the ion-acoustic waves in plasma, the water waves mechanics, the shallow water waves, the quantum mechanics, the electro-hydro-dynamical model for local electric field, the waves of electromagnetic field, the sound waves, the signal processing waves through optical fibers, etc.

RESULT AND DISCUSSION

A. Graphical Presentation

In this section, we have delineated the shape of figures of the obtained solutions to the space-time fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation which are given below:

The shape of the figure of the obtained solution $u_1(\xi)$ for $\lambda = 3, \mu = 3, G = 1.5, m = k = 2, \alpha = .1, a = b = 1$ of the parameters is shown below:

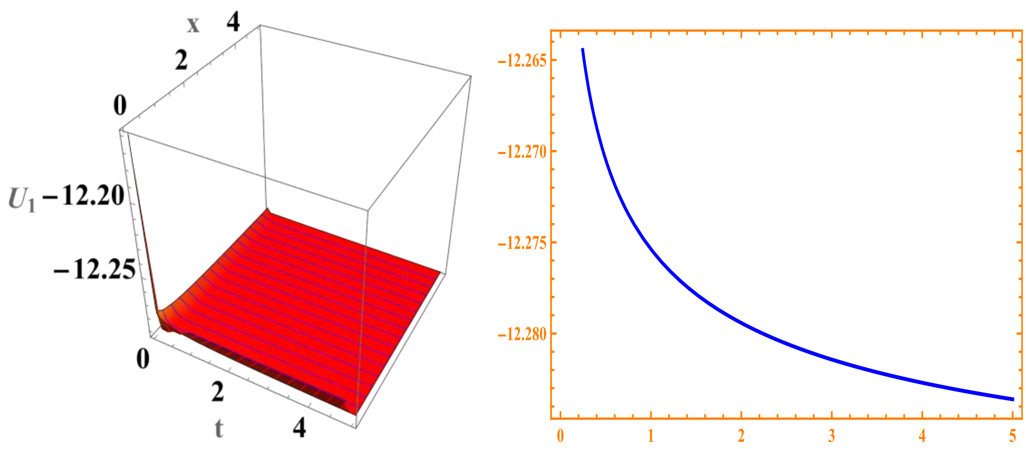

Figure 1. 3D and 2D plot of solution $u_1(\xi)$ within the interval $0 \le x, t \le 5$for 3D and $t = 0$ for 2D.

The shape of the figure of the obtained solution $u_2(\xi)$ for $\lambda = 3, \mu = 3, G = 1.5, m = k = 2, \alpha = .1, a = b = 1$ of the parameters is shown below:

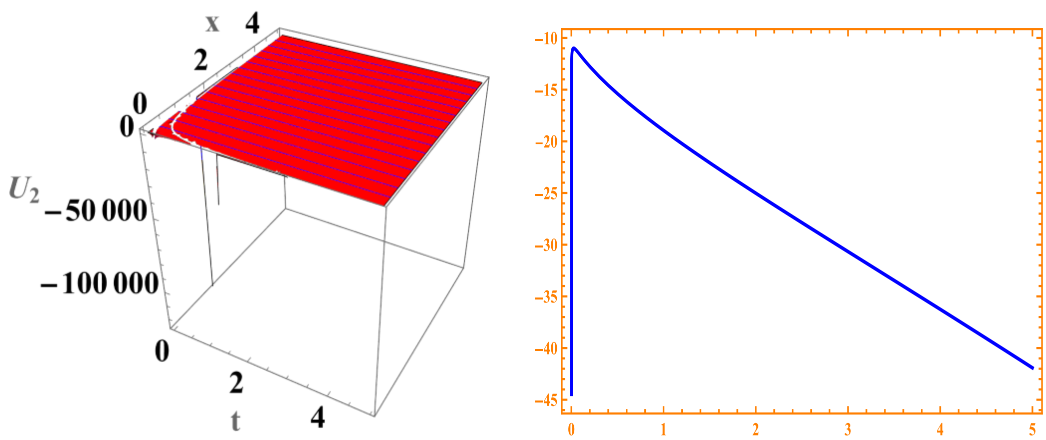

Figure 2. 3D and 2D plot of solution $u_2(\xi)$ within the interval $0 \le x, t \le 5$for 3D and $t = 0$ for 2D.

The shape of the figure of the obtained solution $u_3(\xi)$ for $\lambda = 3, \mu = 3, G = 1.5, m = k = 2, \alpha = .1, a = b = 1$; of the parameters is shown below:

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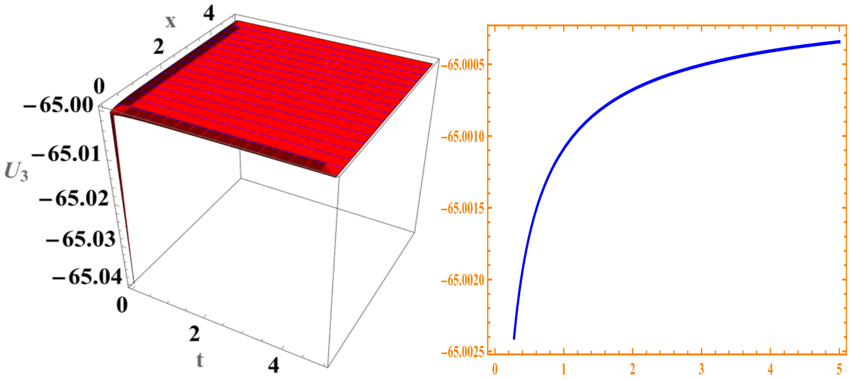

Figure 3. 3D and 2D plot of solution $u_3(\xi)$ within the interval $0 \leq x, t \leq 5$for 3D and $t = 0$ for 2D.

The shape of the figure of the obtained solution $u_4(\xi)$ for $\lambda = 3, \mu = 3, G = 1.5, m = k = 2, \alpha = .1, a = b = 1$ of the parameters is shown below:

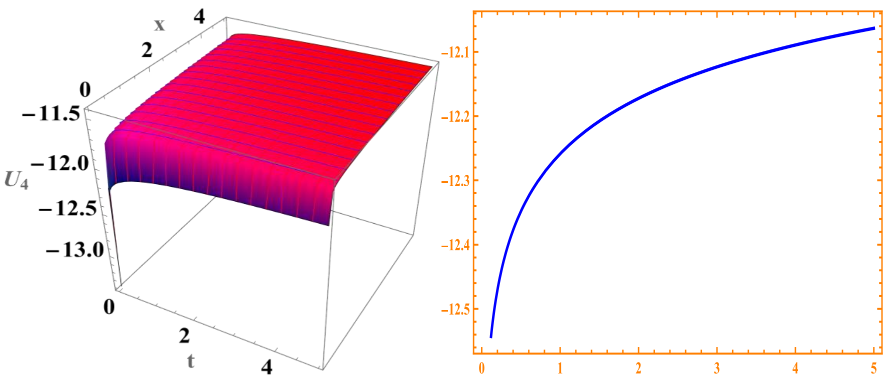

Figure 4. 3D and 2D plot of solution $u_4(\xi)$ within the interval $0 \leq x, t \leq 5$for 3D and $t = 0$ for 2D.

The shape of the figure of the obtained solution $u_5(\xi)$ for $\lambda = 3, \mu = 3, G = 1.5, m = k = 2, \alpha = .1, a = b = 1$ of the parameters is shown below:

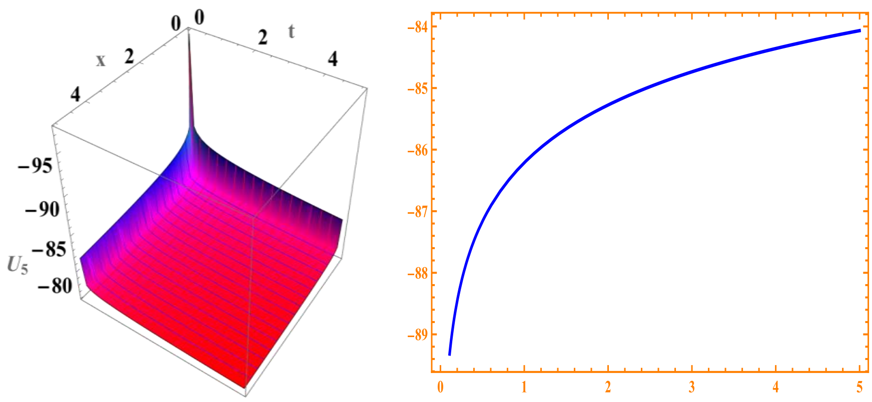

Figure 5. 3D and 2D plot of solution $u_5(\xi)$ within the interval $0 \leq x, t \leq 5$for 3D and $t = 0$ for 2D.

B. Discussion and Physical Implications

It is remarkable to observe that the obtained solutions play a significant role in revealing the obscurity of the tangible events. On the other hand, by using the $exp(-\tau(\xi))$ -expansion method, we have obtained five solutions which are new, distinct and useful and have not been found by the fractional sub-equation method. Furthermore, the different choices of the integral constant from Eq. (3.8) – Eq. (3.12) give many exact wave solutions by the help of the $exp(-\tau(\xi))$ -expansion method. The attained solutions by this method might be useful to analyze the physical significance for its definite values of the parameters and help us to know the internal matters. Therefore, we have observed that the $exp(-\tau(\xi))$ -expansion method is simple, much effective, useful and more general and give huge amount of new exact travelling wave solutions than the fractional sub-equation method.

The ZKBBM equation is a sophisticated mathematical model employed to elucidate diverse physical processes in fluid dynamics and plasma physics, specifically when wave propagation occurs. Including space-time fractional derivatives into this equation makes it possible to represent anomalous diffusion or dispersion. These phenomena are commonly seen in physical systems with non-locality and memory effects. Let's explore the physical consequences of the answers to this equation:

Anomalous diffusion and dispersion: Fractional derivatives introduce a more comprehensive type of diffusion and dispersion, distinct from the conventional integer-order derivatives. The ZKBBM equation enables the representation of intricate wave propagation phenomena, including super diffusion (characterized by faster particle spreading than in regular diffusion) and sub diffusion (characterized by slower particle spreading). This refers to physical scenarios in which the characteristics of the medium result in the wave encountering different levels of resistance, resulting in more precise depictions of real-life occurrences. Effects that occur outside of a specific location or region. The presence of fractional derivatives indicates that the wave propagation is affected by the complete past behavior of the system rather than simply its local characteristics. Non-locality is essential in several physical systems, such as turbulent flows, where interactions across scales are critical. Plasma physics uses fractional models to represent the long-range interactions between charged particles accurately.

Wave Damping and Dispersion Relations:

The ZKBBM equation is commonly used to represent the equilibrium between nonlinearity, dispersion, and dissipation in the propagation of waves. When fractional derivatives are incorporated, the equilibrium of this system undergoes a transformation, which impacts the dissipation and dispersion of waves throughout time and space. Understanding the mechanisms of energy and information transmission in a medium is crucial since it directly impacts the design of materials and systems that depend on regulated wave propagation.

Elaborate Wave Patterns: The solutions to the space-time fractional ZKBBM equation can display a more diverse range of wave shapes compared to their integer-order counterparts. This encompasses solitary waves, shock waves, and several other nonlinear waveforms. These intricate wave formations directly relate to physical systems with similar characteristics, such as coastal and ocean engineering, atmospheric research, and even biological systems.

Practical Implementation: Plasma Physics: The equation may accurately represent wave events in magnetized plasma by considering the fractional kinetics of particle interactions. Fluid dynamics is a branch of physics that may provide a detailed description of waves in shallow water, considering the effects of anomalous dispersion caused by the features of the medium. Signal Processing: Fractional models are employed in signal processing to account for non-local effects and memory, making them useful for building and comprehending wave-based communication systems. The solutions to the space-time fractional ZKBBM equation offer a more detailed and precise explanation of wave phenomena in different physical situations. This is achieved by considering the influences of abnormal diffusion, non-local interactions, and intricate wave structures. By studying the underlying physics of systems that display these behaviors, we may better comprehend and contribute to improving models and technology in several domains, including fluid dynamics and plasma physics.

The $\exp(-\tau(\xi))$-expansion method is a technique that involves searching for solutions in the form of an exponential function with an argument that includes a new variable ξ. This technique is employed to acquire precise solutions for nonlinear differential equations.

Advantages:

1. The technique, reassuringly, is generally easy to implement, often using algebraic operations and familiar transformations.
2. It can provide precise solutions for many nonlinear equations, especially those that describe wave processes.
3. It can be adjusted to accommodate a wide range of nonlinear equations, including fractions.

Limitation:

1. It may not be universally applicable to all categories of nonlinear differential equations. The efficacy of the procedure frequently relies on the structure of the equation.
2. The solutions obtained are usually restricted to exponential forms, which may not encompass all potential behaviors of the system.
3. The approach necessitates the selection of suitable parameters and transformations, which can occasionally be challenging to determine.

CONCLUSION

This study focuses on identifying novel and significant precise traveling wave solutions of the space-time fractional ZKBBM problem. We do this by employing the $exp(-\tau(\xi))$ -expansion approach and utilizing traveling wave transformations. Most of the solutions obtained are expressed as trigonometric, hyperbolic, and rational functions. The achieved solutions may be applicable to various physical phenomena, such as fluid motion in shallow water waves, gravity-driven water waves in the long-wave regime, ion acoustic waves in plasma, quantum mechanics, electro-hydro-dynamical models for local electric fields, and signal processing through optics. In addition, we have examined the physical implications of the answers discovered by representing them graphically. Various well-known forms of solution shapes are analyzed, including kink-shaped wave solutions, singular kink-shaped wave solutions, singular periodic-shaped wave solutions, and so on. The obtained answers demonstrated that the $exp(-\tau(\xi))$ -expansion approach is simple, effective, robust, and versatile. It can be applied to analyze precise wave solutions for various nonlinear differential equations (NLDEs) and fractional nonlinear differential equations (FNLDEs) encountered in diverse areas of mathematics and engineering. In order to acquire these particular answers, there are often challenges in determining the balance number and the most effective approach to use. The existing methods are not always easy and clear, and they do not provide a direct solution to the Nonlinear Differential Equations (NLDEs) and Fractional Nonlinear Differential Equations (FNLDEs). Our future work involves applying the $exp(-\tau(\xi))$ -expansion method and other methods to solve NLDEs and FNLDEs for balance number. Additionally, we aim to utilize these methods to identify complex and highly nonlinear DEs for non-integer balance number, without the need for a new transformation.

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