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Local Approximation Order of Generalized Multiquadric Radial Basis Functions Interpolation

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ABSTRACT

This paper discusses local interpolation by the generalized multiquadric radial basis functions. The convergence rate for local scattered data interpolation by the generalized multiquadric radial basis function has been presented. The multiquadric interpolant is presented in Lagrange form and used to prove the convergence of the multiquadric interpolation.

INTRODUCTION

Data interpolation is a technique used in data analysis to estimate values between known data points. It involves filling in missing data points or estimating values at points where data is not directly measured. The aim of interpolation is to create a continuous representation of a dataset, which can be useful for various purposes like smoothing out irregularities in data, creating visualizations, and making predictions (Amidror, 2002). Data interpolation can be global or local. Local interpolation uses a sample of known data points to estimate the unknown value. It fits the specified order polynomial using points only within the defined neighbourhood. Local interpolation techniques apply a single mathematical function repeatedly to subsets of the total set of observed data points, then link these regional surfaces to create a composite surface covering the whole study area (Steffensen, 2006).

Scattered data interpolation, especially on irregular domains or higher dimensional geometry, is an important problem in science and engineering. Methods such as trigonometric, algebraic, and spline interpolations have been employed for a wide range of problems but have not been so efficient in higher dimensions or scattered nodes problems on irregular domains. Radial basis function interpolation is an alternative method for such problems (Chen and Cao, 2021; Kazem and Hatam, 2017).

The origin of radial basis functions can be traced back to Hardy (1971) when he introduced RBF multiquadric to solve surface fitting on topography and irregular surfaces. Thorough investigation by Hardy (1971) led to the discovery of the multiquadric. Hardy's multiquadric interpolation scheme was unnoticed till 1979 when a mathematician, Richard Franke compared various methods of solving the scatter data interpolation problem which he found Hardy's multiquadric method to be the most impressive. It is consistently best or near best in terms of accuracy, and always results in visually pleasant surfaces. He found also that the system matrix of the method was invertible, and the method was well posed (Madych and Nelson, 1988; Micchelli, 1984; Meinguet, 1979).

Micchelli (1984) developed the theory behind the multiquadric method. He proved that the system matrix for the multiquadric method was invertible. Kansa (1990) was the first to apply the multiquadric method to solve differential equations. Madych and Nelson (1990) showed the spectral convergence rate of multiquadric interpolation. Since Kansa's discovery, research in RBF methods, and particularly the

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multiquadric has grown rapidly. Recently, RBF methods have gained attention in scientific computing and engineering applications such as function interpolation, numerical solutions to partial differential equations, and multivariate scattered data processing. The main advantage of this method are spectral convergence rates that can be achieved using infinitely smooth functions, geometric flexibility, and ease of implementation (Aràndiga *et al.*, 2020; Rippa, 1999).

The generalized multiquadric RBFs take the form

$$\phi(r) = (1 + (\varepsilon r)^2)^{\beta}, \quad \beta = \cdots, -\frac{3}{2}, -\frac{1}{2}, \ \frac{1}{2}, \frac{3}{2}, \cdots$$

The generalized multiquadric RBF covers a wide range of infinitely differentiable RBFs including the Hardy's multiquadric function with $\beta = \frac{1}{2}$, the inverse multiquadric function with $\beta = -\frac{1}{2}$, and the inverse quadrics with $\beta = -1$. For convenience, we will refer to multiquadric functions with $\beta > 0$ as the generalized multiquadric functions while those with $\beta < 0$ as the generalized inverse multiquadric (GIMQ) functions.

For $\beta < 0$, the GIMQ function is strictly positive definite, and for $0 < \beta < 1$, the generalized multiquadric function is conditionally positive definite of order one. In either case, the system matrix for the interpolation problem is invertible. For $\beta > 1$, the generalized multiquadric function is conditionally positive definite of order $\lceil \beta \rceil$ (where $\lceil \cdot \rceil$ denotes the ceiling function), and to show that the system matrix is invertible, it is necessary to attach low order polynomials to the RBF interpolant (Alipanah, 2016; Chenoweth and Sarra, 2009).

In this paper, we will concentrate on generalized multiquadric RBFs of the form

$$\phi(r) = (1 + (\varepsilon r)^2)^{\beta}, \quad \beta = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

The advantages of the generalized multiquadric RBFs over other RBFs like the Thin Plate Splines are: they are infinitely differentiable, they can conveniently interpolate scattered data in many dimensions and have been found to produce the best results when applied on scattered data, and they contain a shape parameter which have great effects on the accuracy of the solution. Interpolation by the generalized multiquadric RBF produces pleasant surfaces, and a system matrix that is invertible and well posed (Issa *et al.*, 2020; Luga *et al.*, 2019). The generalized multiquadric RBFs have been applied by many researchers to develop numerical methods for the solution of partial differential equations (PDEs) (Bustamante *et al.*, 2010; Misra and Kumar, 2013; Luga and Alechenu, 2019).

The rest of the paper is organized as follows: section 2 discusses radial basis function interpolation and the generalized multiquadric RBF interpolation, as well as the Lagrange representation of the multiquadric interpolant. In section 3, the local approximation order of the generalized multiquadric interpolant is presented. Section 4 is the conclusion.

RADIAL BASIS FUNCTION INTERPOLATION

Suppose we are given data in the form $(\boldsymbol{x}_i, \boldsymbol{f}_i)$, where, $\boldsymbol{f}_i = f(\boldsymbol{x}_i)$, $i = 1, 2, \ldots, n$. Our goal is to find an interpolant $s(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^d$, satisfying

(2.1) $$s(\boldsymbol{x_i}) = \boldsymbol{f}_i, \quad i = 1, 2, \ldots, n.$$

For a strictly positive definite radial basis function interpolant, it is required that $s(\boldsymbol{x})$ be a linear

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combination of translates of $\phi(\boldsymbol{x})$, that is

(2.2) $$s(\boldsymbol{x}) = \sum_{i=1}^{n} c_i \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|), \quad \boldsymbol{x} \in \mathbb{R}^d.$$

For a conditionally positive definite RBF interpolant, a low order polynomial $p \in \Pi_{m-1}^d$, is added to Eq. (2.2). This means that we let $s(\boldsymbol{x})$ have the form

(2.3) $$s(\boldsymbol{x}) = \sum_{i=1}^{n} c_i \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|) + \sum_{j=1}^{q} d_j p_j(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d,$$

where Π_{m-1}^d is the space of polynomials from $\mathbb{R}^d$ to $\mathbb{R}$ of degree at most $m-1$, $q \in \mathbb{N}$ is the degree of Π_{m-1}^d, $\|\cdot\|$ is the Euclidean norm and ϕ is the radial basis function, with the additional constraints

(2.4) $$\sum_{i=1}^{n} c_i p_j(\boldsymbol{x}_i) = 0, \quad j = 1, 2, \dots, q.$$

Adding the extra constraints and the polynomial conditions to the interpolant, we have the system of linear equations

(2.5) $$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix},$$

$$A \cdot \boldsymbol{b} = \boldsymbol{f}_X,$$

alternatively

(2.6) $$\Phi \boldsymbol{c} + P\boldsymbol{d} = \boldsymbol{f},$$

(2.7) $$P^T \boldsymbol{c} = \boldsymbol{0},$$

$$P = \left(p_j(\boldsymbol{x}_i)\right)_{1 \le i \le n, 1 \le j \le q}.$$

We call m the order of the radial basis function (Fasshauer, 2007; Iske, 2003).

To guarantee the existence of a solution to Eq. (2.6) and Eq. (2.7), we require the matrix Φ, for any finite set $X = \{x_1, x_2, \dots, x_m\}$ of interpolation points, to be positive definite on the linear subspace of $\mathbb{R}^d$ containing all vectors $\boldsymbol{c} \in \mathbb{R}^n$ satisfying Eq. (2.7). This can be restated as

(2.8) $$\boldsymbol{c}^T \cdot \Phi \cdot \boldsymbol{c} > 0 \text{ for all } X \text{ and } c \in \mathbb{R}^N \setminus \{0\} \text{ with } P^T \cdot \boldsymbol{c} = 0.$$

Definition 1: Conditionally positive definite radial function

A continuous radial function $\phi: [0, \infty) \to \mathbb{R}$ is said to be conditionally positive definite of order m on $\mathbb{R}^d$ if Eq. (2.8) holds for all possible choices of finite set points $X \subset \mathbb{R}^d$.

Theorem 1 (Iske, 2003)

Let ϕ be a conditionally positive definite radial function. The interpolation problem, Eq. (2.1) has under constraints,

$$p(\boldsymbol{x}_j) = 0,\ \ j = 1, \dots, n \Rightarrow p = 0 \text{ for } p \in \Pi_{m-1}^{d}$$

a unique solution s of the form, Eq. (2.3) provided Eq. (2.8) is satisfied.

Generalized multiquadric radial basis function interpolation

To extend the concept of radial basis function interpolation for finite volume methods, we generalize the concept of interpolation. Let $\Omega = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ denote a set of linear differentials and $f_1, f_2, \dots, f_n \in \mathbb{R}$ are certain given function values, then the RBF approximation, augmented by a polynomial is given by

$$s(\boldsymbol{x}) = \sum_{i=1}^{n} c_i \lambda_i^y \phi(\|\boldsymbol{x} - \boldsymbol{y}\|) + \sum_{j=1}^{q} d_j p_j(\boldsymbol{x})$$

where λ_i^y denotes the functional applied to $\phi(\|\boldsymbol{x} - \boldsymbol{y}\|)$ as a function of y with x fixed. The interpolation problem is

$$\lambda_i(s) = \lambda_i f, \qquad i = 1, 2, \dots, n$$

where $\lambda_i f = \boldsymbol{f}_i,\ i = 1, 2, \dots, n$. Since the problem is underdetermined, we add the constraints

$$\sum_{j=1}^{n} c_j \lambda_i p_j = 0, \quad j = 1,2, \dots, q.$$

In matrix terms, the problem is to find a solution to the $(n + q) \times (n + q)$ system of equations

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix}, \qquad (2.9)$$

where

$$\Phi = \left(\lambda_i^x \lambda_j^y\ \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)\right)_{1 \le i,j \le n} \in \mathbb{R}^{n \times n}$$

$$P = \left(\lambda_i p_j\right)_{1 \le i \le n,\ \ 1 \le j \le q} \in \mathbb{R}^{n \times q}$$

and $\boldsymbol{c} = (c_1, \dots, c_n)^T,\ \boldsymbol{d} = (d_1, \dots, d_m)^T,\ \boldsymbol{f} = (f_i)_{i=1,\dots,n} = (\lambda_i f) \in \mathbb{R}^n$. The unique solvability of the system, Eq. (2.9) is based on the theory of conditionally positive definite matrices.

Lagrange representation of interpolant (Iske, 2003)

Sometimes it is useful to work with the Lagrange representation of the interpolant

$$s(\boldsymbol{x}) = \sum_{i=1}^{n} L_i(\boldsymbol{x}) f(\boldsymbol{x}_i)$$

where the Lagrange basis function $L_1(x), \dots, L_n(x)$ satisfy

$$L_i(\boldsymbol{x}_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j \end{cases}$$

$i, j = 1, 2, \ldots, n$ and so

$$s(\boldsymbol{x}_j) = f(\boldsymbol{x}_j).$$

For a fixed point $\boldsymbol{x} \in \mathbb{R}^d$, the vector $L(\boldsymbol{x}) = \left(L_1(x), \ldots, L_n(x)\right)^T \in \mathbb{R}^n$ and $\mu(\boldsymbol{x}) = \left(\mu_1(x), \ldots, \mu_n(x)\right)^T \in \mathbb{R}^n$ are the unique solution of the linear system

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} L(\boldsymbol{x}) \\ \mu(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} R(\boldsymbol{x}) \\ S(\boldsymbol{x}) \end{bmatrix},$$

where $R(\boldsymbol{x}) = \left(\phi\left(\left\|\boldsymbol{x} - \boldsymbol{x}_j\right\|\right)\right)_{j=1,\ldots,n} \in \mathbb{R}^n$ and $S(\boldsymbol{x}) = \left(p_1(x), \ldots, p_m(x)\right) \in \mathbb{R}^n$.

We can write the above linear system as

$$A\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{x})$$

by letting

$$A(\boldsymbol{x}) = \begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix}, \ \boldsymbol{v}(\boldsymbol{x}) = \begin{bmatrix} L(\boldsymbol{x}) \\ \mu(\boldsymbol{x}) \end{bmatrix}, \ \boldsymbol{b}(\boldsymbol{x}) = \begin{bmatrix} R(\boldsymbol{x}) \\ S(\boldsymbol{x}) \end{bmatrix}.$$

Therefore

$$\begin{aligned} s(\boldsymbol{x}) &= \langle L(\boldsymbol{x}), \boldsymbol{f} \rangle = \langle \boldsymbol{v}(\boldsymbol{x}), \boldsymbol{f}_X \rangle \\ &= \langle A^{-1} \cdot \boldsymbol{b}(\boldsymbol{x}), \boldsymbol{f}_X \rangle \\ &= \langle \boldsymbol{b}(\boldsymbol{x}), \ A^{-1} \cdot \boldsymbol{f}_X \rangle \\ &= \langle \boldsymbol{b}(\boldsymbol{x}), b \rangle \end{aligned}$$

where $\langle \cdot, \ \cdot \rangle$ denoted the inner product of the Euclidean space $\mathbb{R}^d$ and where we let

$$\boldsymbol{f}_X = \begin{bmatrix} \boldsymbol{f} \\ 0 \end{bmatrix} \in \mathbb{R}^{n+m}, \ b = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} \in \mathbb{R}^{n+m}.$$

LOCAL APPROXIMATION ORDER OF THE GENERALIZED MULTIQUADRIC INTERPOLANT

Following the approach of Iske (2003), the convergence rate of the generalized multiquadric RBF interpolation and the approximation order of the local multiquadric RBF interpolant is thus, given.

As regards the approximation order of the generalized multiquadric interpolant, we consider solving, for some fixed point $x_0 \in \mathbb{R}^d$ and any $h > 0$

(3.1) $$u(x_0 + hx_i) = s^h(x_0 + hx_i), \quad 1 \leq i \leq n$$

where $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ is a Π_{m-1}^d – unisolvent point set of moderate size, that is, n is small. Moreover, s^h denotes the unique generalized multiquadric interpolant of the form

(3.2) $$s^h(hx) = \sum_{i=1}^{n} c_i^h \phi_\beta(\|h\boldsymbol{x} - h\boldsymbol{x}_i\|) + \sum_{j=1}^{q} d_j p_j(hx)$$

Definition 2

Let s^h denotes the generalized multiquadric interpolant, using ϕ_β satisfying Eq. (3.2). We say that the approximation order of local generalized multiquadric interpolation at $x_0 \in \mathbb{R}^d$ and with respect to the function space $\mathcal{F}$ is p if and only if for any $u \in \mathcal{F}$

$$u(x_0 + hx) - s^h(x_0 + hx) = 0(h^p), h \to 0$$

holds for any $x \in \mathbb{R}^d$, and any finite Π_{m-1}^d – unisolvent point set $X \subset \mathbb{R}^d$.

We let $x_0 = 0$. Now $c^h = ({c_1}^h, \dots, {c_n}^h) \in \mathbb{R}^n$, $d^h = \left({d_1}^h, \dots, {d_q}^h\right) \in \mathbb{R}^q$ can be found by solving the linear system

$$\begin{bmatrix} A_h & P_h \\ {P_h}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{c}^h \\ \boldsymbol{d}^h \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_{|hx} \\ 0 \end{bmatrix}$$

where we let

$$A_h = \left(\phi_\beta\left(\|h\boldsymbol{x}_i - h\boldsymbol{x}_j\|\right)\right)_{1 \le i,j \le n} \in \mathbb{R}^{n \times n},$$

$$P_h = \left(p_j(x_i)\right)_{1 \le i \le n, 1 \le j \le q} \in \mathbb{R}^{n \times q},$$

$$\boldsymbol{u}_{|hx} = \left(u(hx_i)\right)_{1 \le i \le n} \in \mathbb{R}^n.$$

We can write the above system as

$$A \cdot b^h = u_h$$

that is, for notational brevity, we let

$$A_h = \begin{bmatrix} A_h & P_h \\ {P_h}^T & 0 \end{bmatrix}, \ \ b^h = \begin{bmatrix} \boldsymbol{c}^h \\ \boldsymbol{d}^h \end{bmatrix}, \ \ u_h = \begin{bmatrix} \boldsymbol{u}_{|hx} \\ 0 \end{bmatrix}.$$

We assume that the interpolant s^h has a Lagrange type representation

$$s^h(hx) = \sum_{i=1}^{n} L_i^h(hx) u(hx_i)$$

where

$$\sum_{i=1}^{n} L_i^h(hx) p(hx_i) = p(hx) \ \text{ for all } p \in \Pi_{m-1}^d .$$

For $x \in \mathbb{R}^d$, the vector $L^h(hx) = \left(L_1^h(hx), \dots, L_n^h(hx)\right)^T \in \mathbb{R}^n$ together with $\mu^h(hx) = \left(\mu_1^h(hx), \dots, \mu_q^h(hx)\right)^T \in \mathbb{R}^q$ is the unique solution of the linear system

(3.3) $$\begin{bmatrix} A_h & P_h \\ {P_h}^T & 0 \end{bmatrix} \begin{bmatrix} L^h(hx) \\ \mu^h(hx) \end{bmatrix} = \begin{bmatrix} R_h(hx) \\ S_h(hx) \end{bmatrix}$$

where

$$R_h(hx) = \left(\phi_\beta(\|h\boldsymbol{x} - h\boldsymbol{x}_j\|)\right)_{1 \le j \le n} \in \mathbb{R}^h,$$

$$S^h(hx) = \left(p_1(hx), \dots, p_q(hx)\right) \in \mathbb{R}^q$$

and can be written as

$$A_h \cdot v^h(hx) = b_h(hx).$$

Now,

$$D^\alpha s^h(hx) = \sum_{i=1}^{n} D^\alpha L_i^h(hx) u(hx_i)$$

where $D^\alpha L_i^h(hx)$ and $D^\alpha \mu^h(hx)$ are the unique solution of the linear system

$$\begin{bmatrix} A_h & P_h \\ {P_h}^T & 0 \end{bmatrix} \begin{bmatrix} D^\alpha L^h(hx) \\ D^\alpha \mu^h(hx) \end{bmatrix} = \begin{bmatrix} D^\alpha R_h(hx) \\ D^\alpha S_h(hx) \end{bmatrix}.$$

Let $h = 1$, then

(3.4) $$\sum_{j=1}^{n} L_j^1(x) \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) + \sum_{k=1}^{q} \mu_k^1(x) p_k(x_j) = \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|)$$

and

$$\sum_{j=1}^{n} \left(D^\alpha L_j^1(x)\right) \phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) + \sum_{k=1}^{q} \left(D^\alpha \mu_k^1(x)\right) p_k(x_j) = D^\alpha \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|).$$

Note that if

$$\phi(\|\boldsymbol{x} - \boldsymbol{y}\|) = [1 + \|\boldsymbol{x} - \boldsymbol{y}\|^2]^\beta,$$

$$\phi(\|h\boldsymbol{x} - h\boldsymbol{y}\|) = [1 + h^2\|\boldsymbol{x} - \boldsymbol{y}\|^2]^\beta = \left[h^2\left(\frac{1}{h^2} + \|\boldsymbol{x} - \boldsymbol{y}\|^2\right)\right]^\beta,$$

$$= h^{2\beta}(c + \|\boldsymbol{x} - \boldsymbol{y}\|^2)^\beta \quad \text{where } c = \frac{1}{h^2},$$

$$= h^{2\beta}\tilde{\phi}(\|\boldsymbol{x} - \boldsymbol{y}\|) \quad \text{where } \tilde{\phi}(\|\boldsymbol{x} - \boldsymbol{y}\|) = (c + \|\boldsymbol{x} - \boldsymbol{y}\|^2)^\beta .$$

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(3.5) $$p_l(x) = \sum_{j=1}^{n} L_j^1(x) p_l(x_j), \quad l = 1, 2, \dots, q.$$

From Eq. (3.4)

(3.6) $$\sum_{j=1}^{n} L_j^1(x)\phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|) + \sum_{k=1}^{q} \mu_k^1(x) p_k(x_i) = \phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|)$$

$$\sum_{j=1}^{n} L_j^1(x)\left[h^{2\beta}\phi(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|)\right] + h^{2\beta}\sum_{k=1}^{q} \mu_k^1(x) p(x_i) = h^{2\beta}\phi(\|\boldsymbol{x} - \boldsymbol{x}_i\|)$$

Let $t(x - y) = \left[\tilde{\phi}(\|\boldsymbol{x} - \boldsymbol{y}\|) - \phi(\|\boldsymbol{x} - \boldsymbol{y}\|)\right]$ and

$$\sum_{j=1}^{n} L_j^1(x) t(\boldsymbol{x}_i - \boldsymbol{x}_j) = t(\boldsymbol{x}_i - \boldsymbol{x})$$

(3.7) $$h^{2\beta}\sum_{j=1}^{n} L_j^1(x) t(\boldsymbol{x}_i - \boldsymbol{x}_j) = h^{2\beta} t(\boldsymbol{x}_i - \boldsymbol{x})$$

Adding Eq. (3.6) and Eq. (3.7) gives us

$$\sum_{j=1}^{n} \left(L_j^1(x)\right)\left[h^{2\beta}\tilde{\phi}(\|\boldsymbol{x} - \boldsymbol{y}\|)\right] + \sum_{k=1}^{q} (h^{2\beta}\mu_k^1) p(x_i) = h^{2\beta}\tilde{\phi}(\|\boldsymbol{x} - \boldsymbol{x}_i\|).$$

Now, if we define $\tilde{u}_k^h$ as

$$\tilde{u}_k^h = h^{2\beta}\mu_k^1, \;\; k = 1, 2, \dots, q,$$

then, for each $\boldsymbol{x} \in \mathbb{R}^d$, the vector

$$\begin{pmatrix} \left(L_j^1(x)\right)_{j=1,\dots,n} \\ \left(\tilde{u}_k^h\right)_{k=1,\dots,q} \end{pmatrix}$$

solves the linear system, Eq. (3.3). Since the solution is unique,

$$L_j^h(x) = L_j^1(x), \;\; j = 1, 2, \dots, n$$

or

$$L^h(x) = L^1(x).$$

We now draw conclusions on the approximation order of the local multiquadric interpolation with respect to C^p. To this end, let us consider $u \in C^p$, the p^{th} order Taylor polynomial of u and hx is given as

$$T_p(y) = \sum_{|\alpha|<p} \frac{1}{\alpha!} D^\alpha u(hx)(y-hx)^\alpha$$

where $p = \lceil \beta \rceil$. By using

$$u(hx) = T_p(y) - \sum_{0<|\alpha|<p} \frac{1}{\alpha!} D^\alpha u(hx)(y-hx)^\alpha$$

so that

$$u(hx) = T_p(hx_i) - \sum_{0<|\alpha|<p} \frac{1}{\alpha!} D^\alpha u(hx)(hx_i-hx)^\alpha.$$

This leads to

$$u(hx) - s(hx) = \sum_{i=1}^{n} L_i^h(hx)\left[T_p(hx_i) - f(hx_i)\right].$$

Now, by our result, the Lebesgue constant

$$\Lambda = \sup_{h\to 0} \sum_{i=1}^{n} \left|L_i^h(hx)\right| = \sum_{i=1}^{n} \left|L_i^1(hx)\right|$$

is bounded so that

$$|u(hx) - s(hx)| = O(h^p), \;\; h \to 0.$$

Therefore, the approximation order of the generalized multiquadric interpolant using ϕ_β with respect to C^p is p, where $p = \lceil \beta \rceil$.

CONCLUSION

The local approximation order of the generalized multiquadric radial basis function interpolation has been presented. Firstly, the generalized multiquadric radial basis function has been defined, then radial basis function interpolation has been described with particular interest to the generalized multiquadric radial basis function interpolation. The Lagrange representation of the interpolant is also presented and used to prove the convergence of the multiquadric interpolation.

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