Comparison Theorems for Weak Topologies (1)

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ABSTRACT

Weak topology on a nonempty set X is defined as the smallest or weakest topology on X with respect to which a given (fixed) family of functions on X is continuous.

Let τ_w be a weak topology generated on a nonempty set X by a family $\{f_\alpha,:\alpha \in \Delta\}$ of functions, together with a corresponding family $\{(X_\alpha, \tau_\alpha):\alpha \in \Delta\}$ of topological spaces. If for some $\alpha_0 \in \Delta$, $\tau_{\alpha 0}$ on $X_{\alpha 0}$ is not the indiscrete topology and $f_{\alpha 0}$ meets certain requirements, then there exists another topology τ_{w1} on X such that τ_{w1} is strictly weaker than τ_w and f_α is τ_{w1}-continuous, for all $\alpha \in \Delta$. Here in Part 1 of our Comparison Theorems for Weak Topologies it is observed that: (a) the new topology τ_{w1} on X deserves to be called a weak topology (with respect to the fixed family of functions) in its own right. Hence we call τ_{w1} a strictly weaker weak topology on X, than τ_w; (b) the usual weak, weak star, and product topologies have chains of pairwise strictly comparable (respectively) weak, weak star, and product topologies around them.

All the necessary and sufficient conditions for the existence of τ_{w1} in relation to τ_w are established. Ample examples are given to illustrate (at appropriate places) the various issues discussed.

Key Words: Topology, Weak Topology, Weak Topological System, Strictly Weaker Weak Topologies, Pairwise Strictly Comparable Weak Topologies

Mathematics Subjects Classification (MSC) 2020: 54A05, 54A10

INTRODUCTION

The word ***Topology*** in Mathematics is comparable to the meaning of the word ***Topography*** in either Geography or Geology. While *topography* means *'the observable nature of a landscape'* in both geography and geology, *topology* means *the nature of the mathematical landscape* upon which mathematical activities and operations (such as analysis of functions) can be performed. For example, if the topography of a land area is good, we can consider building a residential house on that piece of land; but if the land piece is an area of active volcano or a waterlogged swampy place, no one may think of going there to build a house. In mathematics, the effectiveness or efficacy of a *function* that we may define on a set depends on the topology of both the domain set and the range set. One mathematical set can have several *topologies* (plural of topology). For instance, a three-element set has 29 topologies, a 4-element set has exactly 355 topologies, a 7-element set has 9,535,241 topologies, and so on. Then mathematically speaking, what is a topology? Our first definition answers this question.

The topography of a geographical space determines the socioeconomic activity that can be carried out on that space. Similarly, the topology of a mathematical space (a set) determines the kind of mathematical activity that can be carried out on the space. Turn the focus around and ask the question: Can the socioeconomic activity (or the mathematical activity) on a geographical area (resp. mathematical space) determine the topography (the topology) of the geographical area (of the mathematical set)? The answer is 'Yes'. Our aim in this study is not only to use constructive approach and practical examples to show that several weak topologies can be determined by a fixed family of functions on their common domain, but also to show how these topologies compare with one another.

The title of this research paper is: ***Comparison Theorems for Weak Topologies (1)***. The bracketed (1) here means that this research culminates into more than one article and that this manuscript is the number 1 in the entire three-part series of research work.

Definition 1.1 *Let X be a nonempty set, and let τ be a collection of subsets of X in such a way that τ contains the empty set, the entire set X , the intersection of any finite number of sets in τ and contains the union of any number of sets in τ. Then τ is called a topology on X.*

Definition 1.2 *If τ_1 and τ_2 are two topologies on X such that (say) every set in τ_1 is in τ_2 , then we say that the topology τ_1 is weaker than the topology τ_2. If τ_1 is weaker than τ_2 and there exists a set in τ_2 which is not in τ_1 we say that τ_1 is strictly weaker than τ_2.*

Definition 1.3 *If τ_w is the weak topology on X generated by the family $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}$ of topological spaces, together with the family $\{f_\alpha\}_{\alpha\in\Delta}$ of functions, we shall call the triple $[(X,\tau_w), \{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ a weak topological system.*

Definition 1.4 *A product topological system is a triple $[(X, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha\in\Delta}$ of a topological product space (X, τ_p), a family of topological spaces $\{(X_\alpha,\tau_\alpha)\}$ which, together with the family $\{p_\alpha\}$ of projection maps, induce the product topology τ_p on $\bar{X}$.*

We observe that every product topological system is a weak topological system but the converse is not true.

Definition 1.5 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. The weak topology τ_w is called an indiscrete weak topology (or a minimal weak topology)*[1] *if the family of functions in this system cannot generate a strictly weaker weak topology than τ_w, on X.*

Definition 1.6 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. The weak topology τ_w is called a discrete weak topology (or a maximal weak topology)*[2] *if the family of functions in this system cannot generate a strictly stronger weak topology than τ_w, on X.*

MAIN RESULTS—SOME PRELIMINARY DEVELOPMENTS

Lemma 2.1 *Let Ψ and Φ be two nonempty subsets of the power set 2^X of a nonempty set X such that (say) Ψ is a proper subfamily of Φ. If f is a 1-1 function mapping into all the elements of Φ, and there exists an element of the domain of f mapped into an element of Φ not in Ψ, then $S_1 = \{f^{-1}(G) : G \in \Psi\}$ is a proper subfamily of $S_2 = \{f^{-1}(G) : G \in \Phi\}$.*

Proof:

Ψ is a proper subfamily of Φ. So, there exists $G_0 \in \Phi \ni G_0 \notin \Psi$, and $G \in \Phi \forall G \in \Psi$. Therefore from this hypothesis $S_1 = \{f^{-1}(G) : G \in \Psi\}$ is a subfamily of $S_2 = \{f^{-1}(G) : G \in \Phi\}$ and—since $G_0 \notin \Psi$ and f is 1-1—in particular the set $f^{-1}(G_0) \notin S_1$ (for otherwise we will have a contradiction). This means that S_1 is a proper subfamily of S_2. ⊙

Remark

If f is not 1-1, S_1 may equal S_2 even though Ψ is a proper subfamily of Φ. See examples 1 and 2 below. And if f

[1] As we shall see later, an *indiscrete weak topology* on X may not equal what may, now, be called the ordinary indiscrete topology $\{X, \emptyset\}$ of X.

[2] As we shall see later, a *discrete weak topology* on X may be strictly weaker than what may be called the ordinary discrete topology $\{X, 2^X\}$ of X.

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is 1-1 and there is no element of the domain of f mapped into an element of Φ not in Ψ, then again S_1 may equal S_2; this is illustrated in example 3.

Example 1:

Let $X = \{a, b, c\}$, $\Psi = \{\emptyset, X, \{a\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$, and let E be any nonempty set with cardinality greater than 1. Let f: $E \to X$ be a map such that $f(E) = \{a\}$. Then $S_1 = \{f^{-1}(G): G \in \Psi\} = \{\emptyset, E\}$ and $S_2 = \{f^{-1}(G): G \in \Phi\} = \{\emptyset, E\}$. That is, $S_1 = S_2$.

Example 2:

Let $E = \{1,2,3.4,5,6\}$, $X = \{a, b, c, d\}$ and Let g: $E \to X$ be a map such that $g(1) = a = g(4)$; $g(3) = b = g(5)$. Let $\Psi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c\}, \{a, b, c\}\}$. Then Ψ is a proper subfamily of Φ but $S_1 = \{g^{-1}(G): G \in \Psi\} = \{\emptyset, \{1,3,4,5\}, \{1,4\}, \{3,5\}\}$ and $S_2 = \{g^{-1}(G): G \in \Phi\} = \{\emptyset, \{1,3,4,5\}, \{1,4\}, \{3,5\}\}$. So $S_1 = S_2$. We see that $g^{-1}(X) = g^{-1}(\{a, b\}) = g^{-1}(\{a, b, c\}) = \{1,3,4,5\}$.

Example 3:

Let $X = \{a, b, c\}$, $\Psi = \{\emptyset, X, \{a\}\}$ and $\Phi = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Let f: $E \to X$ be a map such that $f(E) = \{a\}$, where E is a singleton. Then f is 1-1 but $S_1 = \{f^{-1}(G): G \in \Psi\} = \{\emptyset, E\}$ and $S_2 = \{f^{-1}(G): G \in \Phi\} = \{\emptyset, E\}$. That is, $S_1 = S_2$.

Example 4:

Let E, X, Ψ and Φ all be as defined in example 2 and let g: $E \to X$ be a map defined by $g(1) = a$, $g(3) = b$, $g(4) = c$, $g(5) = d$. We now have $g^{-1}(\emptyset) = \emptyset$, $g^{-1}(X) = \{1,3,4,5\}$, $g^{-1}(\{a\}) = \{1\}$, $g^{-1}(\{b\}) = \{3\}$ and $g^{-1}(\{a, b\}) = \{1,3\}$. Therefore, $S_1 = \{g^{-1}(G): G \in \Psi\} = \{\emptyset, \{1,3,4,5\}, \{1\}, \{3\}, \{1,3\}\}$.

Now $g^{-1}(\{c\}) = 4$ and $g^{-1}(\{a, b, c\}) = \{1,3,4\}$. Hence $S_2 = \{g^{-1}(G): G \in \Phi\} = \{\emptyset, \{1,3,4,5\}, \{1\}, \{3\}, \{1,3\}, \{4\}, \{1,3,4\}\}$. We now see that S_1 is a proper subfamily of S_2.

Henceforth whenever we mention 1-1 function in a weak topological system we shall assume that it meets the conditions of lemma 2.1; except otherwise stated.

Proposition 2.1 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. For some $\alpha_o \in \Delta$, arbitrary but fixed, let τ_o be a topology on X_{α_o} such that τ_o is strictly weaker than τ_{α_o}. If (for this fixed $\alpha_o \in \Delta$) $f_{\alpha 0}$ is 1-1, then $\exists\tau_{w1}$, a topology on X, such that (i) $\tau_{w1} < \tau_w$ and (ii) f_α is continuous with respect to τ_{w1}, for all $\alpha \in \Delta$.*

Proof:

Let

$$S_1 = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta, \alpha \neq \alpha_0\} \bigcup \{f_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \tau_0\}$$

and let

$$S_2 = \{f_\alpha^{-1}(G_{\alpha_i}) : G_{\alpha_i} \in \tau_\alpha, \alpha \in \Delta\}.$$

Then by lemma 1, S_1 is a proper subfamily of S_2 since τ_o is strictly weaker than τ_{α_o} and f_{α_o} is 1-1. We know that S_2 is a sub-base for τ_w; and similarly, since τ_o is a topology on X_{α_o}, S_1 is a sub-base for another topology τ_{w_1} on X. As S_1 is a proper subfamily of S_2, there exists at least one set, say G, in S_2 such that $G \notin S_1$. It follows those finite intersections of sets in S_2 (that is, base for τ_w) contains at least one set G more than the finite intersections of the sets in S_1 (which is a base for τ_{w_1}). Hence the topology τ_{w_1} is weaker than τ_w by at least one set G. That is, τ_{w_1} is strictly weaker than τ_w. We also observe that f_α is τ_{w_1} -continuous, for each $\alpha \in \Delta$.

⊙

Observations:

1. The proposition above and the lemma 2.1 that facilitated its proof relied heavily on the existence of *just one* 1-1 function in a weak topological system, not on the existence of τ_o; since every non-indiscrete topology has a strictly weaker topology.

2. Two weak topologies almost always the only ones of interest (so-called *the* weak and *the* weak star topologies) to many authors are about linear maps on linear spaces. The questions now vis-a-vis the proposition 2.1 here are Is every linear map a 1-1 function? The answer is 'No'. Projection maps are linear but not 1-1.

   Does there exist linear maps which are 1-1? Answer: 'Yes'. The identity maps are linear and 1-1.

   Is every 1-1 map linear? Answer: 'No'. The function $f(x) = x^3$ is 1-1 but not linear.

3. Since there exist linear maps which are 1-1 and since the usual weak and weak star topologies are general statements about linear maps, proposition 2.1 implies that these topologies have strictly weaker weak or weak star topologies.

4. Among the results represented by the exposition of this paper is the fact that the usual weak and weak star topologies, among others, have chains of pairwise strictly comparable weaker weak topologies.

Corollary 2.1 *The usual weak and weak star topologies have chains of pairwise strictly comparable weaker weak or weak star topologies.*

Proof:

Since these topologies are weak topologies generated, on sets (linear spaces precisely), by all the linear maps on such sets, since some linear maps (namely, the identity maps) are 1-1 functions, Proposition 2.1 ensures this result.

⊙

It may appear by now that it is only when a function f is 1-1 that S_1 would be a proper subfamily of S_2 given that Ψ is a proper subfamily of Φ. This is not so. In fact, f being 1-1 is only a sufficient condition for S_1 to be a proper subfamily of S_2 (given that Ψ is a proper subfamily of Φ) but it is not a necessary condition. The following example illustrates this.

Example 5:

Let E, X, Ψ and Φ all be as given in examples 2 and 4 above. Let h: $E \to X$ be a map defined by $h(1) = a$, $h(2) = c$, $h(3) = b$, $h(4) = a$ and $h(5) = b$. Then we see that

$S_1 = \{h^{-1}(G): G \in \Psi\} = \{h^{-1}(\emptyset), h^{-1}(X), h^{-1}(\{a\}), h^{-1}(\{b\}), h^{-1}(\{a, b\})\} =$

$\{\emptyset, \{1,2,3,4,5\}, \{1,4\}, \{3,5\}, \{1,3,4,5\}\}$. And that

$S_2 = \{h^{-1}(G): G \in \Phi\}$

$= \{h^{-1}(\emptyset), h^{-1}(X), h^{-1}(\{a\}), h^{-1}(\{b\}), h^{-1}(\{a, b\}), h^{-1}(\{c\}), h^{-1}(\{a, b, c\})\} = \{\emptyset, \{1,2,3,4,5\}, \{1,4\}, \{3,5\}, \{1,3,4,5\}, \{2\}\}$.

We observe that $card(S_1) = 5$ and $card(S_2) = 6$; that $S_1 \subset S_2$ and that $S_1 \neq S_2$. A more general form of lemma 2.1

can therefore be stated as follows.

Lemma 2.2 *Let Ψ and Φ be two nonempty subsets of the power set 2^X of a nonempty set X such that Ψ is a proper subfamily of Φ. If f is a function mapping into each element of Φ, and there exists $G_0 \in \Phi - \Psi$ such that $f^{-1}(G_0) \neq f^{-1}(G)$, $\forall G \in \Psi$, then $S_1 = \{f^{-1}(G): G \in \Psi\}$ is a proper subfamily of $S_2 = \{f^{-1}(G): G \in \Phi\}$.*

Proof:

Since $\exists G_0 \in \Phi$, $\ni f^{-1}(G_0) \neq f^{-1}(G)$, $\forall G \in \Psi$ and since $\Psi \subset \Phi$ it follows that the collection $S_1 = \{f^{-1}(G): G \in \Psi\}$ is a proper subfamily of $S_2 = \{f^{-1}(G): G \in \Phi\}$.

⊙

We can now also obtain a more general form of proposition 2.1.

Proposition 2.2 *Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system. For some $\alpha_0 \in \Delta$, arbitrary but fixed, let τ_0 be a topology on $X_{\alpha 0}$ such that τ_0 is strictly weaker than $\tau_{\alpha 0}$. If $\exists G_0 \in \tau_{\alpha 0}$ such that*

$f_{\alpha_0}^{-1}(G_0) \neq f_{\alpha_0}^{-1}(G), \forall G \in \tau_0$,

then $\exists \tau_{w1}$, a topology on X, such that (i) $\tau_{w1} < \tau_w$ and (ii)

f_α is continuous with respect to τ_{w1}, for all $\alpha \in \Delta$.

Proof:

Since $\exists G_0 \in \tau_{\alpha 0}$ such that $f_{\alpha_0}^{-1}(G_0) \neq f_{\alpha_0}^{-1}(G), \forall G \in \tau_0$, it follows that $G_0 \in \tau_{\alpha 0} - \tau_0$ and (by lemma 2) in particular

$S_1 = \{f_{\alpha_0}^{-1}(G) : G \in \tau_0\}$

is a proper subfamily of

$S_2 = \{f_{\alpha_0}^{-1}(G) : G \in \tau_{\alpha_0}\}$.

Clearly elements of S_2 are among the sub-basic sets of τ_w and, since τ_0 is a topology, S_1 is also a subset of a sub-base for another topology τ_{w1} on X, strictly weaker than τ_w. Since $[(X, \tau_{w1}), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ is a weak topological system, f_α is τ_{w1}-continuous, $\forall \alpha \in \Delta$.

⊙

Remark:

Proposition 2.2 implies that even a product topology can have a strictly weaker product topology.

EXAMPLE 6:

Let $X_1 = \{a, b\} = X_2$ be two sets and let $\overline{X} = X_1 \times X_2 = \{(a, a), (a, b), (b, a), (b, b)\}$.

Let the projection maps on $\overline{X}$ be defined in the usual way $p_i : \overline{X} \to X_i, 1 \leq i \leq 2$ by $p_i\{(x, y)\} = x$, if $i = 1$ and $p_i\{(x, y)\} = y$, if $i = 2$. Let both factor spaces of $\overline{X}$ be endowed with the topology $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the product topology τ_p on $\overline{X}$ is $\tau_p = 2^{\overline{X}}$, the power set of $\overline{X}$; which is a family of 16 subsets of $\overline{X}$.

If we now let a factor space of $\overline{X}$, say X_1, be endowed with a topology τ_0 strictly weaker than τ such that $\exists G_0 \in \tau$ and such that $p_1^{-1}(G_0) \neq p_1^{-1}(G), \forall G \in \tau_0$ we shall get a strictly weaker product topology τ_{p_1}, on $\overline{X}$, than τ_p.

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To see this, let τ_0 on X_1 be $\tau_0 = \{\emptyset, X_1, \{a\}\}$. Then (with the topology of X_2 still being τ) the product topology now on $\overline{X}$ would be $\tau_{p_1} = \{\phi, \overline{X}, \{(a,a),(a,b)\}, \{(a,a),(b,a)\}, \{(a,b),(b,b)\}, \{(a,a)\}, \{(a,b)\}, \{(a,a),(a,b),(b,a)\}, \{(a,a),(a,b),(b,b)\}\}$ a family of only 9 subsets of $\overline{X}$.

It can also be verified easily that both projection maps p_1 and p_2 are continuous with respect to τ_{p_1} if τ_0 and τ are endowed on X_1 and X_2 respectively.

Note:

Example 6 actually represents a general phenomenon in product topological systems; namely that if $[(X, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha\in\Delta}$ is a product topological system, and there exists $\alpha_0 \in \Delta$ such that τ_{α_0} has a strictly weaker topology τ_0, on X_{α_0} then there exists a strictly weaker product topology τ_{p1} than τ_p on X^{-} with respect to which all the projection maps are continuous. We shall give a formal proof of this later, but for now, let's have another lemma.

Lemma 2.3 *Let p_α: $\overline{X} \to X_\alpha$ be a projection map of a Cartesian product set onto a factor space. If x_{α_1} and x_{α_2} are two different elements of X_α, then $p_\alpha^{-1}(x_{\alpha_1}) \neq p_\alpha^{-1}(x_{\alpha_2})$.*

Proof:

Since projection maps count coordinates and return them to respective (or corresponding) factor spaces, we have

$p_\alpha^{-1}(x_{\alpha_1}) = \{\bar{x} \in \bar{X} : p_\alpha(\bar{x}) = x_{\alpha_1}\} = \{(x_r)_{r\in\Delta} \in \bar{X} : x_\alpha = x_{\alpha_1}\}$.

Also

$p_\alpha^{-1}(x_{\alpha_2}) = \{\bar{x} \in \bar{X} : p_\alpha(\bar{x}) = x_{\alpha_1}\} = \{(x_r)_{r\in\Delta} \in \bar{X} : x_\alpha = x_{\alpha_2}\}$.

As tuples (or vectors) are equal if and only if their corresponding components are equal, and since $x_{\alpha1} \neq x_{\alpha2}$, we must have $p_\alpha^{-1}(x_{\alpha_1}) \cap p_\alpha^{-1}(x_{\alpha_2}) = \emptyset$; that is, $p_\alpha^{-1}(x_{\alpha_1})$) and $p_\alpha^{-1}(x_{\alpha_2})$) have no element in common. As both $p_\alpha^{-1}(x_{\alpha_1})$) and $p_\alpha^{-1}(x_{\alpha_2})$) are nonempty, it follows that $p_\alpha^{-1}(x_{\alpha_1}) \neq p_\alpha^{-1}(x_{\alpha_2})$.

⊙

Corollary 2.2 *Let $p_\alpha : \overline{X} \to X_\alpha$ be a projection mapping. If A and B are two nonempty subsets of X_α such that (say) A is a proper subset of B, then $p_\alpha^{-1}(A) \subset p_\alpha^{-1}(B)$ and $p_\alpha^{-1}(A) \neq p_\alpha^{-1}(B)$; that is, $p_\alpha^{-1}(A)$ is a proper subset of $p_\alpha^{-1}(B)$.*

Proof:

Since $A \subset B$ and $A \neq B$, $\exists b_0 \in B \ni b_0 \notin A$. This implies that $b_0 \neq a$, $\forall a \in A$. This implies (by lemma 2.3) that $p_\alpha^{-1}(b_0) \neq p_\alpha^{-1}(a)$, $\forall a \in A$. This implies that $p_\alpha^{-1}(b_0) \notin \{p_\alpha^{-1}(a) : a \in A\} = p_\alpha^{-1}(A)$.

But $\{p_\alpha^{-1}(a) : a \in A\} \subset \{p_\alpha^{-1}(b) : b \in B\}$, because $A \subset B$. And we also know that $p_\alpha^{-1}(b_0) \in \{p_\alpha^{-1}(b) : b \in B\}$ as $b_0 \in B$. Hence $p_\alpha^{-1}(A)$ *is a proper subset of* $p_\alpha^{-1}(B)$.

⊙

Corollary 2.3 *Let $p_\alpha : \overline{X} \to X_\alpha$ be a projection mapping and let Ψ and Φ be two nonempty subsets of the power set 2^{X_α} of X_α. If Ψ is a proper subfamily of Φ, then $S_1 = \{p_\alpha^{-1}(G) : G \in \Psi\}$ is a proper subfamily of*

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$S_2 = \{p_\alpha^{-1}(G) : G \in \Phi\}$.

Proof:

Clearly $S_1 = \{p_\alpha^{-1}(G) : G \in \Psi\}$ is a subfamily of $S_2 = \{p_\alpha^{-1}(G) : G \in \Phi\}$, from hypothesis. We only show that $S_1 \neq S_2$. Let $G_0 \in \Phi - \Psi$. Since each set is the union of its own elements, we have

$$p_\alpha^{-1}(G_0) = \bigcup_{g \in G_0} p_\alpha^{-1}(g) \neq \bigcup_{g \in G} p_\alpha^{-1}(g) = p_\alpha^{-1}(G), \forall G \in \Psi$$

This implies that $p_\alpha^{-1}(G_0) \neq p_\alpha^{-1}(G), \forall G \in \Psi$. This implies that $p_\alpha^{-1}(G_0) \notin S_1$ and since $p_\alpha^{-1}(G_0) \notin S_2$, it follows that $S_1 \neq S_2$. That is, S_1 is a proper subfamily of S_2.

⊙

Proposition 2.3 *Let* $[(X, \tau_p), \{(X_\alpha, \tau_\alpha)\}, \{p_\alpha\}]_{\alpha \in \Delta}$ *be a product topological system. If (for some* $\alpha_0 \in \Delta$*)* $\tau_{\alpha 0}$ *has a strictly weaker topology* τ_0*, on* $X_{\alpha 0}$*, then the product topology* τ_p *on* $\overline{X}$ *has a strictly weaker product topology,* τ_{p_1}.

Proof:

From hypothesis τ_0 is a proper subfamily of τ_{α_0}. By corollary 2.3, $S_1 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_0\}$ *is a proper subfamily of* $S_2 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_{\alpha_0}\}$. Since τ_0 is a topology on X_{α_0}, it follows that $S_1 = \{p_{\alpha_0}^{-1}(G) : G \in \tau_0\}$ is part of a sub-base for a product topology τ_{p_1} on $\overline{X}$ (with the topologies of the other factor spaces unchanged). Since S_2 is part of a sub-base for τ_p and since S_1 is a proper subfamily of S_2, τ_{p_1} is strictly weaker than τ_p.

⊙

Remark:

1. It is now clearer that the condition of 1-1-ness in proposition 2.1 is only a sufficient, but not necessary, requirement for a strictly weaker weak topology to be obtained, given that the topology of a range space has a strictly weaker topology.

2. The reasoning in propositions 2.1 and 2.2 implies that if τ_1 is strictly weaker than τ_0, τ_2 strictly weaker than τ_1, and so on, then there exist correspondingly weak topologies τ_{w2}, τ_{w3}, etc., on X, such that $\tau_w > \tau w1 > \tau w2 > \tau w3 > \cdots$.

3. If we have a weak topological system $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha \in \Delta}, \{f_\alpha\}_{\alpha \in \Delta}]$, one question is whether we can always find another topology τ_{w1} on X such that $\tau_w > \tau_{w1}$ and such that each function in the family is continuous? That is, does τ_{w1} always exist for every weak topology τ_w? Another question (if it be found that τ_{w1} does not exist for all weak topologies τ_w) is whether we can characterize such weak topologies τ_w for which we can find such τ_{w1}. And yet another question is: What (if any) topological property can τ_w transmit to, or induce on τ_{w1}? This last question can be seen as property inheritance question—and it is as important here as it is in human society. These questions and more are what we shall be looking at in the next section.

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