Comparison Theorems for Weak Topologies (3)

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ABSTRACT

Weak topology on a nonempty set X is defined as the smallest or weakest topology on X with respect to which a given (fixed) family of functions on X is continuous.

In an effort to construct and analyze strictly stronger weak topologies, only four weak topologies (namely, the Arens topology, the Mackey topology, *the weakened topology*, and the strong topology) have been compared before now. Also, all these weak topologies were constructed with an eye on only the polars of linear spaces (i.e. with a focus on only linear functionals on linear spaces). To that extent, the study of strictly stronger weak topologies *before now* can be described as a study of linear topological spaces. Here in Part 3 of our Comparison Theorems for Weak Topologies:

1. We established and clarified the place of our comparison theorems in the context of *the* weakened topology, the Arens topology, the Mackey topology, and the strong topology that have up to now been compared.

2. We showed that there are many other weak topologies between the four already compared weak topologies—constructible even by the use of polars. In particular, we showed that we can we find a weak topology stronger than the one hitherto known and referred to as *the strong topology*.

3. Then we showed that if two weak topologies, generated by one family of functions on a set, are strictly comparable, then there exist in a range space two strictly comparable topologies which induce the weak topologies.

4. The rather simplistic view that" ***The weak topology is Hausdorff***" has been held by many authors for very long time. We changed that narrative here, as we stated and proved (with examples) that:" Some weak topologies are Hausdorff while others are not."

Key Words: Topology, Weak Topology, Weak Topological System, Product Topological System, Chain of Topologies, Strictly Weaker Weak Topologies, Pairwise Strictly Comparable Weak Topologies

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Strictly Stronger Weak Topologies?

So far we have only been looking at the possibility of getting strictly weaker weak topologies when requisite conditions are met. We now look at the possibility of obtaining strictly stronger weak topologies.

This particular idea has been explored before by other researchers; however, the development here is a more extension and generalization of the approach adopted before (by others) in getting strictly stronger weak topologies. For instance, the only (four) weak topologies constructed and compared before were achieved by using polars of subsets of normed linear spaces. In that sense it has been more or less a study of only *normed linear spaces*.

We showed the link between constructing weak topologies by the use of polars and constructing them by the general method which we have since adopted. Then we proved that the general method adopted here is indeed general enough as it encompasses (what may now be called) the polar method. Then we showed that between the four already compared weak topologies there exist many other weak topologies—constructible even by the use of polars. Finally, we proved that if two weak topologies (generated by one family of functions) on a set are strictly comparable, then there exist in a range space two strictly comparable topologies which induce the weak topologies. This last exposition is a converse way of proving the earlier assertion that the polar method is part of the general method (exposed by us) of constructing weak topologies.

In making this inquiry, we follow our tradition and do not assume that there is a linear structure or a norm on the set X.

It is a well-known proposition that if B is a family of subsets of a given set X, then B is a base for *some* topology on X if (a) $\bigcup_{G\in B} G = X$; and (b) if

$G_1, G_2 \in B$, then $G_1 \cap G_2 \in B$. This fact will be crucially used in what follows.

Proposition 1.1 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* τ_w *is not the discrete topology on X, and there exists* $\alpha_0 \in \Delta$ *such that* τ_{α_0} *is not the discrete topology on* X_{α_0} *then there exists a topology* τ_{w+1} *on X such that (i)* $\tau_w < \tau_{w+1}$*; and (ii)* f_α *is* τ_{w+1}*-continuous, for all* $\alpha \in \Delta$.

Proof:

Since $\tau_{\alpha_0} < 2^{X\alpha}0$ there exists $G_0 \subset X_{\alpha_0}$ such that $G_0 \notin X_{\alpha_0}$. Let $S_0 = \tau_{\alpha_0} \cup \{G_0\}$ and let B_0 = {finite intersections of elements of S_0}. Then we see that

(a) $\bigcup_{G\in B_0} G = X_{\propto_0}$; and (b) if $G_1, G_2 \in B_0$, then $G_1 \cap G_2 \in B_0$.

Hence B_0 is a base for *some* topology τ_0 on X_{α_0}. It is clear that $\tau_{\alpha_0} < \tau_0$, and if we replace τ_{α_0} with τ_0 in the weak topological system we shall get a weak topology τ_{w+1}, on X generated by this fixed family of functions. Then it is easy to verify that (i) $\tau_w < \tau_{w+1}$; and that (ii) each f_α is continuous with respect to τ_{w+1}.

Theorem 1.1 *Let* $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ *be a weak topological system. If* τ_w *is a non-trivial topology on X, that is* $\{\emptyset, X\} < \tau_w < 2^X$, *then* τ_w *is in the middle (midst) of weak topologies on X, some strictly weaker and some strictly stronger than* τ_w, *in that* $\{\emptyset, X\} < \cdots < \tau_{w1} < \tau_w < \tau_{w+1} < \cdots < 2^X$.

Proof:

This is the conclusive meaning of all the foregoing results.

Relationship with Existing Results

As we have pointed out, researchers have in the past constructed weak topologies on normed linear spaces by using polars of sets. Let E be a normed linear space, E^* its algebraic dual, and E' its topological dual. What was called *the* weak topology $\sigma(E, E^*)$, on E (or $\sigma(E^*, E)$, on E^*) is the topology on E (or on E^*) made up of polars of finite subsets of E^* (or of E). And what was called *the weakened topology* $\sigma(E, E')$, on E (or $\sigma(E', E)$, on E') is the topology on E (or on E') formed by taking polars of finite subsets of E' (or of E). (See [3], pages 88 and 89.)

A particular issue was then called the Mackey problem and it is as follows: If we have a dual system (E, E'), how may we characterize those topologies on E compatible with the duality (E, E')—in the sense that every element of E' is not only continuous but in addition can be represented by an element of E—and which are locally convex? It was remarked ([3], page 504) that such topologies do exist and that the weakest of them is $\sigma(E, E')$. It was also "shown" that there is a strongest such topology, and that the others are lying between these two. This strongest such topology is denoted by $\tau(E, E')$ and is called the Mackey topology. The Mackey topology $\tau(E, E')$ on E is then constructed as the topology made up of polars of weakly compact and convex subsets of E'. Then the conclusion is that if T is a locally convex topology on E with respect to which elements of E' are continuous, then

$\sigma(E, E') \leq T \leq \tau(E, E')$.

If E is a LCTVS (locally convex topological vector space) and E' its topological dual, the relation above will hold with T the initial topology of E. Usually the ordering is strict, it is observed, but the equality $T = \tau(E, E')$ holds for certain important types of LCTVS, which are then accordingly called *relatively strong*.

Now if E is a TVS (topological vector space), E' its topological dual, Arens introduced the topology on E' having polars A^o of compact, convex and balanced subsets A of E as a base of neighborhoods at 0. This is called the Arens topology on E' and is denoted by $k(E', E)$; and it is locally convex and weaker than the Mackey topology $\tau(E',E)$, as every compact subset of E is weakly compact for the weakened topology $\sigma(E,E')$. (If we interchange the roles played by E and E', we get the Arens topology $k(E, E')$ on E.) Since it is obviously stronger than $\sigma(E',E)$, $k(E',E)$ is compatible with the duality between E and E'; that is, $k(E', E)$ makes all maps of the form $f(x') = \langle x, x'\rangle$ continuous, for any fixed element x of E.

The so-called strong topology, $\eta(E, E')$ on E is made up of polars of the norm-bounded subsets of E'. Again, interchanging the roles of E and E' gets us the strong topology $\eta(E', E)$ on E'. ([3], pages 507 and 508) It is noted that in general the topology $\eta(E', E)$ is *not* compatible with the duality between E and E'. That is, it is not generally true that each linear form on E', continuous with respect to $\eta(E',E)$, is generated by an element of E.

Questions

1. What is the place of our comparison results on the four weak topologies (namely *the* weakened, the Arens, the Mackey and the strong topologies) already known to be comparable, viz: $\sigma(E, E') \leq k(E, E') \leq \tau(E, E') \leq \eta(E, E')$?

2. The conclusion of earlier researches is that $\sigma(E, E')$ is the weakest and that $\tau(E, E')$ is the strongest of all those locally convex topologies on E which make elements of E' continuous and are compatible with the duality existing between E and E'.

Except for the Arens topology, existing research finding did not tell whether the intermediate locally convex topologies compatible with the duality (E, E') are weak topologies—that is, constructible by any known process of forming a weak topology, such as by using polars. In short, no systematic process of looking for such topologies (be they 'weak' or not) is given.

Between the Mackey (weak) topology $\tau(E, E')$ and the strong (but 'weak') topology $\eta(E,E')$, is there no intermediate weak topology (akin to, say the Arens topology that lies between the weakened topology and the Mackey topology)?

If there exists an intermediate weak topology between the Mackey and the strong topology, how do we get it and what role can such a topology play or not play in analysis? And if there are no such intermediate weak topologies, why?

3. Can we find a weak topology stronger than the one hitherto known, and referred to as *the strong topology*; and if we cannot find such a weak topology, why?

FURTHER DEVELOPMENTS

In order to know the full impact of our comparison theorems on the existing results, we need to clearly establish the connection between constructing weak topologies in the way we have done *and* constructing them by the use of polars. And to do this, we will now recast the meaning of a 'polar' (by way of definition) and then look deeper into it, to see its place in the collection of open sets of a weak topology.

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Definition 1.1 *Let* (A, B) *be a dual system over a scalar field* K (= R *or* C). *(We recall that the meaning of this is that, first, A and B are linear spaces over the same scalar field K; and secondly there exist linear maps* $\phi_b : A \to K$ *, on A into K defined by* $\phi_b(a) = \langle a,b \rangle$*, for each element b of B and linear maps* $\phi_a : B \to K$*, on B into K defined by* $\phi_a(b) = \langle a,b \rangle$*, for each element b of B.) Let* $G \subset B$ *be any subset of B). Then the polar* G^o *of G is a subset of A given by* $G^o = \{a \in A: |\langle a, b\rangle| \leq 1, b \in G\}$.

Remark

Some use the strict inequality < in the definition of polar above; and for obvious reasons we may have to resort to that use in the sequel.

Any $\varepsilon > 0$ can be used in the definition above, in place of 1.

Example

Let E be a linear space over K and let E^* be its algebraic dual. Then (E, E^*) is a dual system, for the maps $\phi_f : E \to K$ defined by $\phi_f(x) = \langle x, f \rangle = f(x)$ are linear, for all $f \in E^*$; and the maps $\phi_x : E^* \to K$ defined by $\phi_x(f) = \langle x, f \rangle = f(x)$ are also linear.

Let A be a finite subset of E^*. Then the polar A^0 of A is

$$A^0 = \{x \in E : |\langle x, f \rangle| \leq 1, \forall f \in A\}.$$

This is typically the set we need to understand clearly in our present discussion of weak topologies. Taking a look again:

$$A^0 = \{x \in E : |\langle x, f \rangle| \leq 1, \forall f \in A\} = \{x \in E : |f(x)| \leq 1, \forall f \in A\}$$

$$= \{x \in E : -1 \leq f(x) \leq 1, \forall f \in A\} = \{x \in E : f(x) \in [-1,1], \forall f \in A\} = \{x \in E : x \in f^{-1}([-1,1]), \forall f \in A\} = \bigcap_{f \in A} f^{-1}([-1,1]) = \bigcap_{i=1}^{n} f^{-1}([-1,1]),$$

where $f \in A$ = a finite subset of E^*.

Observations:

1. If the strict inequality < is used, then the polar A^o is the intersection of a finite number of inverse images of the usual open sets of the scalar field K with its usual topology, under an equally finite number of linear maps.

2. If A is infinite, the polar A^o of A is the intersection of infinitely many inverse images of the form $f^{-1}([-1,1])$ where f ranges over the linear functionals in A.

3. The collection $\{A^o : A \subset E^*\}$ of polars is always a collection of intersections of inverse images of (open or not open) sets—finite or infinite intersections according to whether the subsets of E^* considered are finite or infinite.

4. These intersections (the polars) are always a base for a weak topology and if subsets of E^* are used to generate the polars, the weak topology would be that generated by the elements of E^*.

5. If E^* is replaced by E', the topological dual of E, we would have a weak topology (on E) with respect to elements of E'.

6. For the four weak topologies $\sigma(E, E')$, $k(E, E')$, $\tau(E, E')$, and $\eta(E, E')$, all elements of E' are continuous. The difference is that while $k(E, E')$ and $\tau(E,E')$ may contain some infinite intersections, $\sigma(E, E')$ will not have such intersections. Also $\eta(E, E')$ will contain more exotic intersections than both $k(E, E')$ and $\tau(E, E')$—but not necessarily all arbitrary intersections.

7. Hence the differences among these four weak topologies lie on the kind of intersections they contain, and this in turn lies on the kind of sets whose polars are used as base for the topologies. And it is indeed just *that polars are used as bases for the topologies*; the collection of polars are not directly (necessarily) the topologies in question. So, when we say that a weak topology is made up of polars of (some) subsets, what we really mean is that such a weak topology is built up (or constructed) from polars of such subsets as a base.

Expositions

Let $P_0 = \{A^o: A \subset E', A \text{ is finite}\}$ be the collection of polars of finite subsets of E', a base for the weak topology $\sigma(E, E')$ on E; and let $Q \subset E'$ be a weakly compact, infinite and convex subset of E', and let Q^o be the polar of Q. Let $B = P_0 \cup \{Q^o\}$. Then it is easy to see that

1. $\bigcup_{P\in B} P = E$ and that

2. If $P_1, P_2 \in B$ then $P_1 \cap P_2 \in B$, as a subset of a polar is a polar.

Hence B is a base for *some* topology τ_w, on E, a weak topology generated by elements of E'. We now observe that

1. The (polar) base for $\sigma(E, E')$ does not contain the polar of any infinite, weakly compact and convex subset of E'. Hence $\sigma(E, E')$ is strictly weaker than τ_w.

2. The polar base for τ_w does not contain the polars of all infinite, weakly compact and convex subsets of E'. Hence τ_w is strictly weaker than $\tau(E, E')$. That is, $\sigma(E, E') < \tau_w < \tau(E,E')$.

3. τ_w has a base of neighborhoods at zero; hence it is locally convex. In short (by 8.3.1 on page 505, of [3]) a locally convex topology T on E is compatible with the duality between E and E' if and only if $\sigma(E, E') \leq T \leq \tau(E, E')$. Hence τ_w is a locally convex topology on E compatible with the duality (E, E').

What we have proved is that τ_w is a locally convex, (E, E')-compatible weak topology on E, generated by elements of E', which is strictly stronger than $\sigma(E, E')$ and strictly weaker than $\tau(E, E')$.

Since we can find[1] other weakly compact, infinite and convex subsets $G_1, G_2, G_3, ...$ of E', each different from one another (and different from Q_0), we can by analogy get a sequence $\{\tau_{w_1}, \tau_{w_2}, \tau_{w_3}, ...\}$ of locally convex, (E, E')-compatible, pairwise strictly comparable weak topologies lying between $\sigma(E, E')$ and $\tau(E, E')$, in that $\sigma(E, E') < \tau_w < \tau_{w1} < \tau_{w2} < \tau_{w3} < \cdots < \tau(E, E')$.

Let $P_0 = \{A^o: A \subset E', A \text{ is weakly compact and convex}\}$, base for the weak topology $\tau(E,E')$ on E; let $G \subset E'$ be a subset of E' which is not weakly compact but a bounded subset of E', and let G^o be the polar of G. Let $B = P_0 \cup \{G^o\}$. Then B is a base for *some* topology T, on E—a weak topology generated by elements of E'. And we can again see that

1. $\tau(E, E')$ is strictly weaker than T;

[1] Observe that non-existence of such subsets as Q would imply non-existence of the Mackey topology as a different topology from the weakened topology.

2. T is strictly weaker than $\eta(E, E')$;

3. T is locally convex;

4. If E' has other bounded subsets which are not weakly compact and convex, there exists a family $\{T_n\}$ of T-like topologies on E, lying between $\tau(E,E')$ and $\eta(E,E')$, such that $\tau(E, E') < T < T_1 < \cdots < \eta(E, E')$; and if E' has no bounded subsets which are NOT weakly compact and convex (i.e. all bounded subsets of E' are weakly compact and convex) then necessarily $\tau(E, E')$ would coincide with $\eta(E, E')$;

5. These weak topologies lying between $\tau(E, E')$ and $\eta(E, E')$ are, like $\eta(E, E')$, in general not guaranteed to be compatible with the duality (E, E').

Now let G be an unbounded subset of E' which may (or may not) be weakly compact, and let G_o be its polar. Let P_o be the polar base of $\eta(E, E')$ and let $B = P_o \cup \{G_o\}$. Then again B is easily seen to be a base for some weak topology T on E, generated by the elements of E'. And we notice that $\eta(E, E')$—the strong topology—is strictly weaker than this weak topology T.

We have shown that the bases—coming as collections of polars—for these weak topologies are actually nothing but collections of intersections (finite or infinite) of inverse images of some subsets of the scalar field underlying the dual system (under the linear maps). Finally, the next question to consider is whether the four weak topologies ($\sigma(E, E')$, $k(E, E')$, $\tau(E, E')$ and $\eta(E, E')$) which have traditionally been compared are actually induced or generated on E by some correspondingly compared topologies in a range space or two. This will show that Proposition 1.2 (1) and/or Lemma 1.1 of our Comparison Theorems Part 2 apply also to these four weak topologies.

Proposition 1.2 *Let B_1 be a base for a topology τ_1 on X and let B_2 be a base for another topology τ_2 on X. If B_1 is a proper subfamily of B_2 (or conversely that τ_1 is strictly weaker than τ_2), then the subbase S_1 for τ_1 is a proper subfamily of the subbase S_2 for τ_2. Hence for a weak topological system $[(X, \tau_w),\{(X_\alpha,\tau_\alpha)\}_{\alpha\in\Delta},\{f_\alpha\}_{\alpha\in\Delta}]$, if B_1 is a polar base for another weak topology τ_{w1} on X generated by the same family of functions, and that B_1 is a proper subfamily of a polar base B_2 of τ_w (or conversely that $\tau_{w1} < \tau_w$), then there exist two topologies τ_1 and τ_2 on a range space $X_{\alpha 0}$, for some $\alpha_0 \in \Delta$, such that τ_1 is strictly weaker than τ_2 and τ_{w1} is the weak topology on X when $X_{\alpha 0}$ has the topology τ_1 and τ_w is the weak topology on X when $X_{\alpha 0}$ has the topology τ_2.*

Proof:

From the hypothesis and since $B_1 = \{$finite intersections of sets in $S_1\}$, and since B_2 is analogously defined for S_2, B_1 is a proper subfamily of B_2 if and only if S_1 is a proper subfamily of S_2.

Let $[(X, \tau_w), \{(X_\alpha, \tau_\alpha)\}_{\alpha\in\Delta}, \{f_\alpha\}_{\alpha\in\Delta}]$ be a weak topological system and let τ_{w1} be another weak topology, on X, generated by the fixed family of functions. Let B_1 and B_2 be the respective polar bases for τ_{w1} and τ_w. Then $\tau_{w1} < \tau_w$ if and only if B_1 is a proper subfamily of B_2; and B_1 is a proper subfamily of B_2 if and only if the subbase S_1 for τ_{w1} (in relation to B_1) is a proper subfamily of S_2, the subbase for τ_w (in relation to B_2). Clearly S_1 is of the form

$$\{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta\}$$

and S_2 is also of the (same) form

$$\{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta\}.$$

Since $S_1 \subset S_2$ and $S_1 \neq S_2$, there must be a range space $(X_{\alpha 0}, \tau_{\alpha 0}), \alpha_0 \in \Delta$, in the weak topological system such that $X_{\alpha 0}$ has another topology τ_0, strictly weaker than $\tau_{\alpha 0}$, and such that

$$S_1 = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta, \alpha \neq \alpha_0\} \bigcup \{f_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \tau_0\}$$

and

$S_2 = \{f_\alpha^{-1}(G_\alpha) : G_\alpha \in \tau_\alpha, \alpha \in \Delta, \alpha \neq \alpha_0\} \bigcup \{f_{\alpha_0}^{-1}(G_{\alpha_0}) : G_{\alpha_0} \in \tau_{\alpha_0}\}$.

Let $\tau_0 = \tau_1$ (of the proposition) and $\tau_{\alpha 0} = \tau_2$, and the proof is complete.

Cursory Look at an Existing Result

Before we end this paper let us again point out some of the benefits of taking a constructive approach to the study of weak topology: (1) The constructive approach enables us to create or obtain weak topologies of virtually all kinds of topological properties; (2) It can help us to check the correctness or otherwise of our intended general result; for example there is a theorem in literature which simply states that *The weak topology is Hausdorff*. (See proposition 6.9, page 124 of Chidume (1996) for this.) The questions relating to this proposition are: (1) Are all weak topologies Hausdorff? and (2) Is it to be accepted that only one weak topology—that which is Hausdorff—is in existence, even though both present and older researches have shown the existence of several weak topologies? Our answer to these two questions is that one topological property of a weak topology may not be shared by other weak topologies, and, in particular, all weak topologies are NOT Hausdorff. We take a few illustrative examples.

Example 1

Let $X = \{a, b, c\}$, $Y = \{1,2,3\}$ and $Z = \{p, q, r, s, t\}$ be three sets. Let f: $X \to Y$ be a function defined by $f(a) = 2$ and $f(b) = 1$; and let g: $X \to Z$ be a function defined by $g(b) = p$ and $g(c) = p$. Let Y be endowed with its indiscrete topology and let Z be given any topology. Then the weak topology τ_w on X generated by these two functions f, g is $\tau_w = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$. Then it is easy to see that this weak topology on X is not Hausdorff, since for instance the two distinct points b and c do not have disjoint neighborhoods.

Example 2

Let X, Y, Z, f, g all be as given in example 1. Let Z be endowed with any topology but Y now with its discrete topology. Then the weak topology τ_w on X generated by these two functions f, g now is $\tau_w = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. And it is easy to see that this weak topology on X is not Hausdorff, as $b \neq c$ and there are no disjoint τ_w-open sets containing b and c.

Example 3

Let X, Y, Z, f all be as given in example 2. Let Z be endowed with the topology $\{\emptyset, Z, \{p\}, \{t\}, \{p, t\}\}$ and Y with its discrete topology. And let g: $X \to Z$ be defined by $g(b) = p$ and $g(c) = t$. Then the weak topology τ_w on X generated by these two functions f, g now is $\tau_w =$

$\{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. And we now see that this weak topology on X is Hausdorff.

Example 4

The Sierpinski weak topology that we constructed earlier in section 1 of our Comparison Theorems Part 2 is not Hausdorff.

We can therefore make the following proposition without need of further proof.

Proposition 2.1 *Some weak topologies are Hausdorff while others are not.*

CONCLUSION

1. Further research is now needed to establish the conditions under which certain topological properties of a weak topology τ_w would be inherited by its strictly weaker (or stronger) weak topology τ_{w1}.

2. Since any weak topology is what it is, *a weak topology*, in the context of given topologies of the range spaces it will always be interesting to know when or why we should use certain topologies on the range

spaces instead of other topologies. For example, the four weak topologies (weakened, Arens, Mackey and strong topologies) seen in the last section here are weak topologies in relation to the topology initially endowed on the range spaces (the scalar fields) which are the usual or Euclidean topologies of the fields. Must we always have the Euclidean topology on these fields in order to have weak topologies (with respect to the linear functionals) on the linear spaces?

3. If τ_w is not an indiscrete weak topology, there exists a strictly weaker weak topology τ_{w1} than τ_w; and if τ_{w1} is itself not an indiscrete weak topology, then it has a strictly weaker weak topology τ_{w2}; and so on. It follows that every non-indiscrete weak topology is at the peak of an increasing chain of strictly weaker weak topologies; hence we will henceforth often have a wide choice to make of the weak topology to use.

4. If a weak topology τ_w is not a discrete topology, it can be made to be at the bottom of an increasing chain of strictly stronger weak topologies. Therefore, every nontrivial (i.e. non-discrete and non-indiscrete) weak topology τ_w is actually at the middle of an increasing chain of weak topologies— some of which are strictly weaker and the others strictly stronger than τ_w. And therefore we should always make clear and precise the reason(s) why we adopt a particular nontrivial weak topology in a context of analysis instead of the others.

REFERENCES

1. Angus E. Taylor and David C. Lay; Introduction to Functional Analysis; Second Edition, John Wiley and Sons, New York, 1980.
2. Benjamin T. Sims; Fundamentals of Topology; Macmillan Publishing Co., Inc., New York; Collier Macmillan Publishers, London and Canada, 1976.
3. Edwards R.E.; Functional Analysis: Theory and Applications; Dover Publications Inc., New York, 1995.
4. James R. Munkres; Topology; Second Edition, Prentice-Hall of India Private Limited, New Delhi, 2007.
5. Seymour Lipschutz; Theory and Problems of General Topology; Schaum’s Series, McGraw-Hill Publications, New York, 1965.
6. Sheldon W. Davis; Topology; McGraw-Hill Higher Education/Walter Rudin Series in Advanced Mathematics, Boston, 2005.
7. Sidney A. Morris; Topology Without Tears; July 24, 2016 Version, From Internet. Link: sidney.morris@gmail.com; and www.sidneymorris.net
8. Royden H.L. and Fitzpatrick P.M.; Real Analysis; PHI Learning Private Limited, 4th Edition, 2012
9. Chidume C.E.; Applicable Functional Analysis; ICTP, Trieste, Italy, 1996