Type II Topp-Leone Generalized Exponentiated Weibull Distribution: Properties and Application to Cancer Stem Cell Data

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ABSTRACT

This work presents a flexible lifetime distribution with increasing, increasing and decreasing and non-monotonic hazard rate called Type II Top-Leone Exponentiated Weibull ($TIITLEW$) distribution. The density function of $TIITLEW$ model has right-skewed and symmetrical shapes. Descriptive properties such as quantile function, moments, incomplete moments, probability weighted moments, moment generating functions, Renyl and Shannon entropies are theoretically established. Parameters of $TIITLEW$ distribution are estimated using maximum likelihood method. The potentiality/tractability of $TIITLEW$ distribution is demonstrated by its to cancer stem cell data.

Keywords: Quantile, Renyl and Shannon entropies, Probability Weighted Moments, Incomplete Moments.

INTRODUCTION

The Weibull (W) and exponentiated Weibull (EW) distributions give a close form solution to several problems in reliability studies. However, they do not possess a good/ reasonable parametric fit for real life applications; for example, when modeling phenomenon with non-monotonic failure rates, the Weibull distribution should not be considered because it does not provide a reasonable parametric fit. The unimodal and bathtub failure rate which are commonly observed in biological and reliability studies which cannot be modeled using the Weibull distribution. In recent decade, several attempts have been made to develop new families of distribution that extent the well-known families of distribution and also inducing flexibility which improves its modeling potentials of the baseline distribution in modeling real life data. Such work includes: the exponentiated Weibull distribution by Mudholkar and Srivastava (1993), Weibull-geometric (WG) distribution by Barreto-Souza et al. (2011), Exponentiated Weibull-geometric (EWG) distribution by Mahmoudi and Shiran (2012). Further, complementary versions of the Exponential Geometric and Weibull Geometric distributions, so-called Complimentary Exponential Geometric and Complimentary Weibull Geometric distribution, respectively, have been introduced by Louzada et a (2011) and Tojeiro et al. (2014). Marshall-Olkin Exponentiated Weibull distribution by Bidram et al. (2015), Transmuted Exponentiated Weibull distribution by Khan et al. (2019).

Motivation of study

The main purpose of the modification and extension forms of the Weibull distribution is to describe and fit the data sets with non-monotonic hazard rate, such as the bathtub, unimodal and modified unimodal hazard rate. Many modifications of the Weibull distribution have achieved the above purpose. On the other hand, unfortunately, the number of parameters has increased, the forms of the survival and hazard functions have been complicated and estimation problems have risen.

EW distribution; A brief review

The EW distribution is an extension of the Weibull family and was developed by Mudholkar and Srivastava (1993). The EW distribution exhibits a non-monotone failure rate which makes it a reliable model in modeling lifetime data. Mudholkar et al. (1993), Mudholkar and Huston (1996), Gupta and Kundu (2001), Nassar and Eissa (2003) and Choudhury (2005) applied the EW model to modeling reliability and survival data.

The random variable X follows an EW distribution if its cumulative density function (cdf) is given by

$$F(x;\alpha,\beta,\theta)=\left(1-e^{-(\beta x)^{\theta}}\right)^{\alpha} \quad ,x>0 \tag{1}$$

Where α and θ are positive shape parameters and β is a positive scale parameter. The associated probability density function (pdf) corresponding to (1) is given as

$$f(x;\alpha,\beta,\theta)=\alpha\beta^{\theta}x^{\theta-1}\theta e^{-(\beta x)^{\theta}}\left(1-e^{-(\beta x)^{\theta}}\right)^{\alpha} \quad ,x>0 \tag{2}$$

The reliability, $R(x;\alpha,\beta,\theta)$ and hazard rate $h(x;\alpha,\beta,\theta)$ functions of the EW distribution are respectively given as

$$R(x;\alpha,\beta,\theta)=1-\left(1-e^{-(\beta x)^{\theta}}\right)^{\alpha} \ . \tag{3}$$

And

$$h(x;\alpha,\beta,\theta)=\frac{\alpha\beta^{\theta}x^{\theta-1}\theta e^{-(\beta x)^{\theta}}\left(1-e^{-(\beta x)^{\theta}}\right)^{\alpha}}{1-\left(1-e^{-(\beta x)^{\theta}}\right)}. \tag{4}$$

The p^{th} moment about the origin of the EW distribution is given by

$$E(X^{p})=\alpha\beta^{-p}\Gamma\left(\frac{p}{\theta}+1\right)\mathcal{N}_{p}(\theta),$$

where

$$\mathcal{N}_{p}(\theta)=1+\sum_{l=1}^{\alpha-1}(-1)^{l}\binom{\alpha-1}{l}(l+1)^{-\left[{p}/{\theta}+1\right]}$$

Where $\Gamma\left(\frac{p}{\theta}+1\right)$ represents the incomplete gamma function. For more detail, see Nassar and Eissa (2003).

The Type II Topp-Leone Exponentiated Weibull distribution

Using the generalization by Elgarhy et al. (2018), the cdf of $TIITLEW$ distribution is given by

$$\boldsymbol{F}(x;\alpha,\beta,\theta,v)=1-\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v}, \tag{5}$$

The corresponding pdf to (5), is given by

$$\boldsymbol{f}(x;\alpha,\beta,\theta,v)=\alpha\beta^{\theta}x^{\theta-1}\theta e^{-(\beta x)^{\theta}}\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v-1}. \tag{6}$$

Where α, θ, and v are positive shape parameters and β is a positive scale parameter. The graph of the cdf and the pdf for various values of the parameters of the distribution are given in figures 1.

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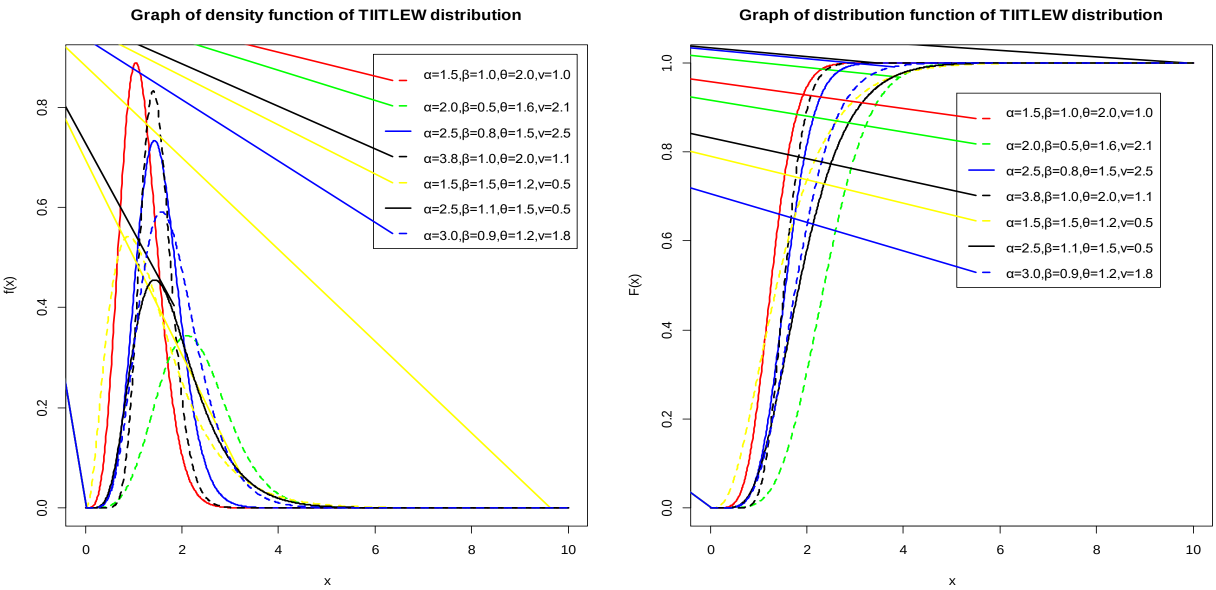

Figure 1. Graph of the density and distribution functions of $TIITLEW$ distribution

An expression for the Reliability, hazard, reversed, and cumulative hazard functions is respectively given by

$$\boldsymbol{R}(x;\alpha,\beta,\theta,v)=\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v}, \tag{7}$$

$$\boldsymbol{R}(x;\alpha,\beta,\theta,v)=\alpha\beta^{\theta}x^{\theta-1}\theta e^{-(\beta x)^{\theta}}\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{-1}, \tag{8}$$

$$\Phi(x;\alpha,\beta,\theta,v)=\frac{\alpha\beta^{\theta}x^{\theta-1}\theta e^{-(\beta x)^{\theta}}\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha-1}\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v-1}}{1-\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v}}, \tag{9}$$

And

$$\boldsymbol{H}(x;\alpha,\beta,\theta,v)=log\left(1-\left[1-\left(1-e^{-(\beta x)^{\theta}}\right)^{2\alpha}\right]^{v}\right). \tag{10}$$

The graph of the hazard function is given in figures

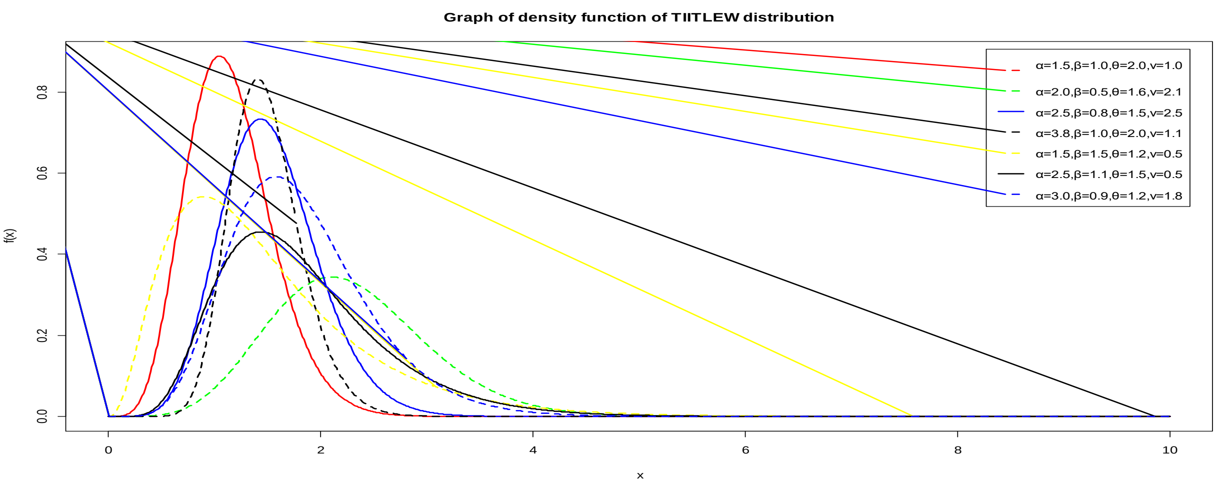

Figure 2. Graph of the hazard functions of $TIITLEW$ distribution

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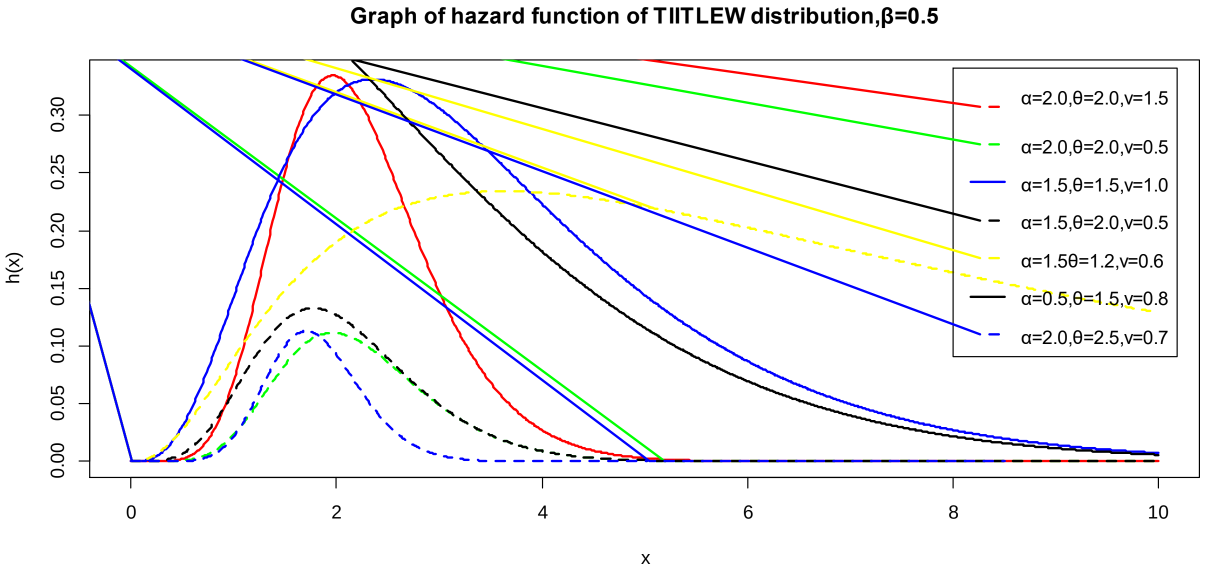

Figure 3. Graph of the hazard functions of $TIITLEW$ distribution

Important representation

In this subsection, an important tool for the expansion of the pdf and cdf for $TIITLEW$ is provided. From the generalized binomial series given by

$$(1-k)^c = \sum_{i=0}^{\infty} (-1)^i \binom{b}{i} k^i \tag{11}$$

For $|k| < 1$ and c is a positive real non-integer. Then, by applying the binomial theorem (11) in (6), the density function of $TIITLEW$ distribution can be written as

$$g(x;\eta,\varphi,\rho) = 2v\alpha\theta\beta^\theta \sum_{i,j,k=0}^{\infty} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} x^{\theta-1} e^{-(j+1)(\beta x)^\theta} \tag{12}$$

This indicates that the $TIITLEW$ model can be written as an infinite mixture of the Weibull model

The quantiles, median and the upper quartile

A mathematical expression for the quantile and the median of $TIITLEW$ model is obtained in this subsection.

The quantile x_u of the $TIITLEW$ is given as follows

$$x_u = \frac{1}{\beta}\left(-log\left[1-\left(1-(1-u)^{1/v}\right)^{1/2\alpha}\right]\right)^{1/\theta} \tag{13}$$

The median and the upper quartile of $TIITLEW$ are found by putting $q = 0.5$ and 0.75 in (14), respectively, as follows:

$$x_{0.5} = \frac{1}{\beta}\left(-log\left[1-\left(1-(0.5)^{1/v}\right)^{1/2\alpha}\right]\right)^{1/\theta} \tag{15}$$

and

$$x_{0.75} = \frac{1}{\beta}\left(-log\left[1-\left(1-(0.25)^{1/v}\right)^{1/2\alpha}\right]\right)^{1/\theta} \quad (16)$$

The r^{th} Ordinary moment

If $X \sim TIITLEW(\gamma)$, then the $\boldsymbol{r^{th}}$ moment of X can be derived using

$$\mu_r^{'} = E(X^r) = \int_0^\infty x^r g(x)dx. \quad (17)$$

By substituting from (12) in (17), we obtain the $\boldsymbol{r^{th}}$ moment as follows

$$\mu_r^{'} = 2v\alpha \sum_{i,j,k=0}^{\infty} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-\left(1+r/\theta\right)} \beta^{-r} \Gamma\left(1+r/_{\theta}\right) \quad (18)$$

Where $\Gamma\left(1+r/_{\theta}\right)$ is a gamma function. By setting r = 1 in (18), we obtain the mean of X as

$$\mu_1^{'} = 2v\alpha \sum_{i,j,k=0}^{\infty} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-\left(1+1/\theta\right)} \beta^{-r} \Gamma\left(1+1/_{\theta}\right)$$

The r^{th} incomplete moments

If $X \sim TIITLEW(\gamma)$, then the $\boldsymbol{r^{th}}$ incomplete moments of X can be derived using

$$\varsigma_r(t) = \int_0^t x^r g(x)dx. \quad (19)$$

By substituting from (12) in (19), we obtain the $\boldsymbol{r^{th}}$ moment as follows

$$\varsigma_r(t) = 2v\alpha \sum_{i,j,k=0}^{\infty} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-\left(1+\frac{r}{\theta}\right)} \beta^{-r} \Gamma\left(1+\frac{r}{\theta}, (j+1)(\beta t)^{\theta}\right) \quad (20)$$

Where $\Gamma\left(1+\frac{r}{\theta}, (j+1)(\beta t)^{\theta}\right)$ is an incomplete gamma function. Bet setting $r = 1$ in (20), we obtain the first incomplete moment of $TIITLEW$ model as

$$\varsigma_1(t) = 2v\alpha \sum_{i,j,k=0}^{\infty} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-\left(1+\frac{1}{\theta}\right)} \beta^{-1} \Gamma\left(1+\frac{1}{\theta}, (j+1)(\beta t)^{\theta}\right) \quad (21)$$

Moment generating function (MGF)

The MGF of $TIITLEW(\zeta)$, say $M_X(t)$ is found using

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tX} f(x)dx = \sum_{v=0}^{\infty} \frac{t^r}{r!} E(X^r) \quad (22)$$

Substituting (18) into (22), we obtain

$$M_X(t) = 2v\alpha \sum_{i,j,r=0}^{\infty} \frac{t^r}{r!} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-\left(1+\frac{1}{\theta}\right)} \beta^{-1} \Gamma\left(1+\frac{1}{\theta}\right) \quad (23)$$

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Characteristics function

The characteristic function can be derived by replacing t with it in (23). Thus, the characteristic moments for $TIITLEW$ distribution is given as

$$\varphi_X t = E\left(e^{itx}\right) = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} E(X^r)$$

Then we obtain

$$\varphi_X t = 2v\alpha\beta^{-r} \sum_{i,j,r,1}^{\infty} \frac{(it)^r}{r!} (-1)^{i+j} \binom{v-1}{i} \binom{2\alpha(i+1)-1}{j} (j+1)^{-(1+\frac{r}{\theta})}\ \beta^{-r}\Gamma\left(1+\frac{1}{\theta}\right) \quad (24)$$

The probability weighted moment (PWM)

Taking the expectation of a function of X, which can be used to obtain the parameters of a certain distribution for which the inverse form can be obtained, this is defined as the probability-weighted moment (PWM). The PWM of X cdf, $G(x)$, say $\zeta_{r,s}$, is obtained by

$$\zeta_{r,s} = E(x^r F^s(x)) = \int_0^{\infty} x^r G^s(x) g(x) dx \quad (25)$$

If X $\sim TIITLEW(\zeta)$, then $\zeta_{r,s}$ is given by

$$\zeta_{r,s} = 2\alpha v\beta^{-r} \sum_{i,j}^{\infty} (-1)^{i+j} \binom{v(i+j)-1}{i} \binom{2\alpha(i+1)-1}{j} (1+j)^{1-r-\theta}\Gamma(r+1) \quad (26)$$

Rényi Entropy Function and $\boldsymbol{\rho}$ −Entropy

The entropy function can be used to evaluate the level randomness or uncertainty related to X whose pdf $g(x)$. It plays a fundamental role in reliability, engineering, and others. The Rényi entropy of X, say $I_\rho(X)$, is determined by

$$I_\rho = \frac{1}{1-\rho} log \int_{-\infty}^{\infty} g^\rho(x) dx, \quad (27)$$

If X $\sim TIITLEW$ (ζ), then $I_\delta(X)$ is obtained by

$$I_\rho = \frac{1}{1-\rho} log\left(2^\rho v^\rho \alpha^\rho \theta^{\rho-1} \beta^{\rho-1} W^* \Gamma\left(\frac{(\theta-1)(\rho-1)}{\theta}+1\right)\right). \quad (28)$$

where

$$W^* = \sum_{\square=\square=0}^{\infty} (-1)^{\square+\square} \binom{\square(\square-1)}{\square} \binom{2\square(\square+\square)-\square}{\square} (\square+\square)^{-\left(\frac{\square(\square-1)}{\square}+1\right)}$$

Consequently, the $\square$-entropy of $\square$, say $\square_{\square}(\square)$ is given by

$$\square_{\square}(\square) = \frac{1}{1-\square} \square\square\square[1-(1-\square)\square_{\square}(\square)]. \quad (29)$$

Where an expression for $\square_{\square}(\square)$ can be found in (29).

Stress strength Reliability

Here, we derived an expression for the stress-strength parameter of □□□□□□□ distribution. Suppose $\square_1$ stand for the strength of a structure with stress $\square_2$, and if $\square_1$ follows □□□□□□□ $(\square_1, \square, \square_1, \square)$ and $\square_2$ follows $TIITLEW\ (\alpha_2, \beta, v_2, \theta)$, given that $X_1$and X_2 are independent random variables, then the Stress-strength Reliability ($\mathfrak{K}$) of □□□□□□□ is obtained as follows:

$$\square = \square(\square_2 < \square_1) = \int_0^\infty \square_1(\square; \square_1, \square, \square_1, \square)\square_2(\square; \square_2, \square, \square_2, \square)\square\square \qquad (30)$$

If $\square \sim$ □□□□□□□(□), then □ is given by

$$\square = \square(\square_2 < \square_1) = \square_1(\square; \square_1, \square, \square_1, \square) - \square^{\square,\square,\square}$$

Where

$$\square^{\square,\square,\square} = 2\square_1\square_1\square^{1-\square} \sum_{\square=0}^{\square_1-1} \sum_{\square=0}^{\square_2} \sum_{\square=0}^{\infty} \binom{\square_1 - 1}{\square}\binom{\square_2}{\square}\binom{2[\square_1(\square+1)+\square_2\square]-1}{\square}(-1)^{\square+\square+\square}$$

Maximum Likelihood Estimator of □□□□□□□ Distribution

Let $\square_1, \square_2, \ldots, \square_{\square}$ be a random sample drawn from □□□□□□□ $(\square, \square, \square, \square)$ then the log-likelihood function is given by

$$\square = \log(2\square\square\square\square^{\square}) + (\square - 1)\sum_{\square=1}^{\square}\log(\square_{\square}) - \sum_{\square=1}^{\square}(-\square\square_{\square})^{\square} + (2\square - 1)\sum_{\square=1}^{\square}\log(1 - \square^{-(\square\square_{\square})^{\square}}$$

$$+(\square - 1)\sum_{\square=1}^{\square}\log\left(1 - \left(1 - \square^{-(\square\square_{\square})^{\square}}\right)^{2\square}\right) \qquad (31)$$

We differentiate (31) with respect $(\square, \square, \square, \square)$ to obtain the element of the score vector $\left(\square_{\square} = \frac{\square\square}{\square\square}, \square_{\square} = \frac{\square\square}{\square\square}, \square_{\square} = \frac{\square\square}{\square\square}, \square_{\square} = \frac{\square\square}{\square\square}\right)^{\square}$. The elements of the score vector is given by

$$\frac{\square\square}{\square\square} = \frac{\square}{\square} + 2\sum_{\square=1}^{\square}\log\left(1 - \square^{-(\square\square_{\square})^{\square}}\right) + 2(\square - 1)\sum_{\square=1}^{\square}\frac{\left(1 - \square^{-(\square\square_{\square})^{\square}}\right)^{2\square}\log(1 - \square^{-(\square\square_{\square})^{\square}}}{\left(1 - \left(1 - \square^{-(\square\square_{\square})^{\square}}\right)^{2\square}\right)} \qquad (32)$$

$$\frac{\square\square}{\square\square} = \frac{\square}{\square} + \square^{\square}\square\square\square\square + (2\square - 1)\sum_{\square=1}^{\square}\frac{(\square\square_{\square})^{\square}\log(\square\square_{\square})}{\left(1 - \square^{-(\square\square)^{\square}}\right)}$$

$$+2(\square - 1)\sum_{\square=1}^{\square}\frac{\square(\square\square)^{\square}\left(1 - \square^{-(\square\square)^{\square}}\right)^{2\square}\log(\square\square_{\square})}{\left(1 - \left(1 - \square^{-(\square\square)^{\square}}\right)^{2\square}\right)\left(1 - \square^{-(\square\square)^{\square}}\right)} \qquad (33)$$

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$$\frac{\square\square}{\square\square} = \frac{\square}{\square} + \square\square^{\square}\square\square\square\square - \square\square^{\square}\sum_{\square=1}^{\square}\log(\square)\,\square^{\square} + 2(\square - 1)\square\square^{\square}\sum_{\square=1}^{\square}\frac{\square^{\square}\square^{-(\square\square)^{\square}}\log(\square)}{\left(1 - \square^{-(\square\square)^{\square}}\right)}$$

$$+(\square - 1)\sum_{\square=1}^{\square}\frac{\square\square^{\square}\log(\square)\,\square^{\square}\square^{-(\square\square)^{\square}}\left(1 - \square^{-(\square x)^{\square}}\right)^{2\square}}{\left(1-\left(1 - \square^{-(\square\square)^{\square}}\right)^{2\square}\right)\left(1 - \square^{-(\square\square)^{\square}}\right)} \qquad (34)$$

$$\frac{\square\square}{\square\square} = \frac{\square}{\square} - \sum_{\square=1}^{\square}\square\square\square\left[\left(1 - \left(1 - \square^{-(\square\square)^{\square}}\right)^{2\square}\right)\right] \qquad (35)$$

Application of □□□□□□□ model

In this section, the □□□□□□□ model is compared with Type II Topp-Leone Exponentiated Exponential (□□□□□□□), Type II Top-Leone Weibull (□I□□□□), Weibull (W) and Exponential € distributions. Different goodness of fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov (KS) statistics with Probability values (P-v), Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC). The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. For previous study see Lee and Wang (2003). That data are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. Table 1.0 gives the exploratory data analysis of the cancer data which shows that the data is over-dispersed and leptokurtic. Figure 4.0 represents the boxplot for the cancer data which shows that the data is positively skewed. Total time on test plot is given in Figure 5.0 which shows that the cancer data exhibits bathtub failure rate. The better fit corresponds to smaller W, A, KS, AIC, CAIC, BIC, HQIC and the larger the □ − □. The Maximum Likelihood Estimates (MLEs) of the unknown parameters and values of goodness of fit measures are computed for □□□□□□□ distribution and its sub-models.

Table 4.4: Exploratory data Analysis of Bladder cancer patients

□□□.	$\square_{\square}$	□□□□□□	□□□□	$\square_{\square}$	□□□.	□□□□□□□□	□□□□□□□□	□□□□□□□□
0.080	*3.35*	*6.40*	*9.37*	*11.868*	*79.050*	*18.48*	*3.29*	*110.43*

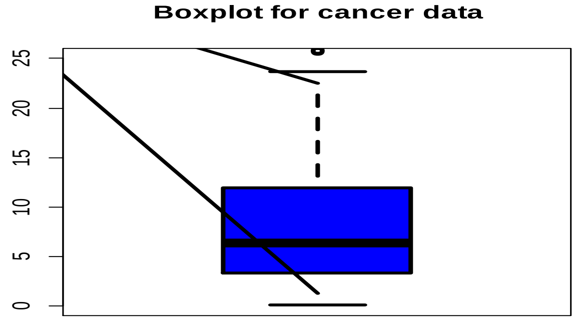

Figure 4.0 Boxplot for cancer data

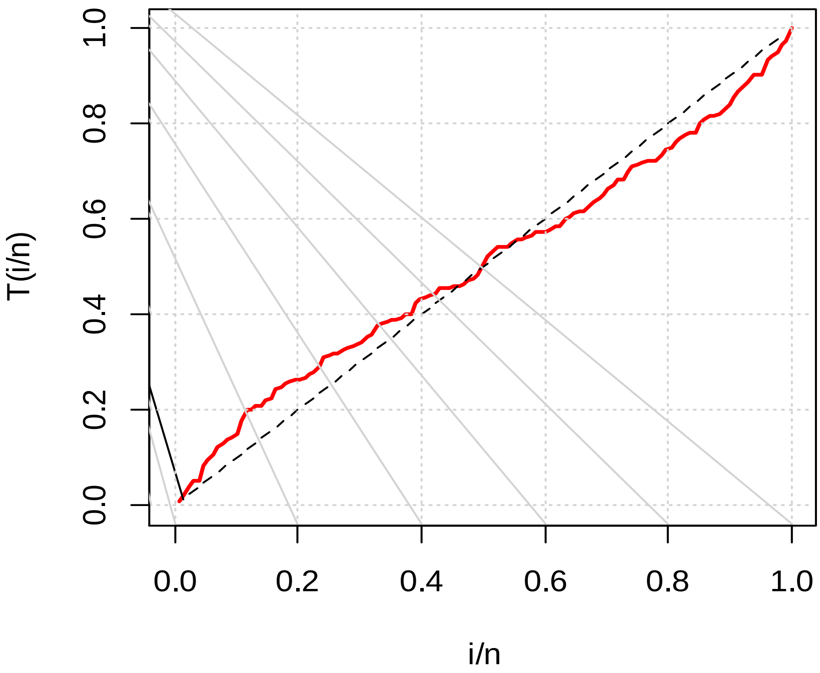

Figure 5. Total Time on Test (TTT) plot

Table 2: Result of the MLEs and standard error for cancer

□□□□□	□	□	□	□
□□□□□	*1.7214* (*0.6209*)	*0.2595* (*0.1290*)	*0.5404* (*0.1210*)	*1.6701* (*0.8545*)
□□□□□	*0.7247* (*0.1360*)	*0.2605* (*0.1251*)		*0.4552* (*0.0747*)
□□□□□	− (−)	*3.0190* (*0.1925*)	*0.5891* (*0.1094*)	*0.4065* (*1.2761*)

□□	*7.7960* (*4.3039*)	*0.0128* (*0.0025*)	*1.7873* (*0.2754*)	− (−)
□	− (−)	*0.1045* (*0.0093*)	*1.0539* (*0.0678*)	− (−)
□	− (−)	*0.1068* (*0.0094*)	− (−)	− (−)

Table 3: Goodness-of-fit statistics for bladder cancer data set

□□□*e*□	−□	□□□	□□□	□□□□	□□□□	□□	□□	□□	□□
□□□□□	*410.64*	*829.29*	835.69	829.62	833.22	*0.2752*	*0.0413*	*0.0455*	*0.9535*
□□□□□	412.17	830.34	838.89	830.53	833.82	*0.5684*	*0.0930*	*0.0581*	*0.7801*
□□□□□	*410.992*	*829.99*	*836.54*	829.18	*834.46*	*0.3512*	*0.0579*	*0.0497*	*0.9101*
□	*486.05*	*832.18*	*846.23*	*832.27*	*978.21*	*0.796*	*0.1331*	*0.0707*	*0.5469*
□	414.34	830.683	*836.53*	*830.72*	831.*84*	*0.7159*	*0.1193*	*0.0846*	*0.3182*

From Tables, we observe that □□□□□□□ model has a better fit than its existing sub-model models which includes □□□□□□□, □□□□□□, □ and □ model because it possesses the smallest □□□, □□□□, □□□, □, □□□ □, and also possesses the highest P-value.

CONCLUSION

A new four-parameter distribution called the □□□□□□□ distribution is developed. This distribution is a generalization of the EW distribution and contains several lifetime sub-models such as: □□□□□, □□, □□, □ and □. A characteristic of the □□□□□□□ distribution is that its failure rate function can be decreasing, increasing, bathtub-shaped and unimodal depending on its parameter values. Several statistical properties of the new distribution such as its probability density function, its cumulative density function, quantiles, moments, incomplete moments, moments generating functions, probability weighted moments, stress-strength reliability function, Renyi and ρ-entropies are obtained. Fitting the □□□□□□□ model to areal data sets indicates the flexibility and usefulness of the proposed distribution in modeling cancer remission times data because it provides a good fit when compared with other competing models considered in this study.

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