

Exact Analytical Solutions of the Generalized KdV Equation and the Duality between Focusing and Defocusing Regimes

Dr. Weiguang Huang

DrHuang.com, Sydney, NSW 2033, Australia

ORCID: 0000-0002-5766-4516

DOI: <https://doi.org/10.51584/IJRIAS.2026.11080105>

Received: 20 August 2026; Accepted: 27 August 2026; Published: 12 September 2026

ABSTRACT

The generalized quadratic Korteweg–de Vries (KdV) equation admits two nonlinear travelling-wave branches, yet existing exact-solution studies typically treat only one sign of the nonlinearity and do not identify the correct mapping into the Weierstrass invariants. This paper derives the complete branch-resolved travelling-wave structure of the normalized equation reducing it to a first-order cubic equation and identifying it with the canonical Weierstrass differential equation. Four structural contributions are obtained: (i) explicit travelling-wave solutions for both focusing ($b = -12$) and defocusing ($b = 12$) branches; (ii) the correct invariant mappings $(g_2, g_3) = (-C_1, -C_2)$ and (C_1, C_2) for the two nonlinearities; (iii) exact symbolic verification of each solution using the defining identities of the Weierstrass function; and (iv) the identification of the symmetry $(b, y) \rightarrow (-b, -y)$, which provides an exact duality between the two branches without requiring a change in travelling speed. The zero-invariant limit yields the rational profiles $\pm 1/(x-t)^2$. The resulting branch-resolved and symmetry-consistent solution structure offers exact analytical benchmarks for symbolic and numerical studies of nonlinear dispersive waves.

Keywords: symbolic computation; nonlinear wave equation; KdV equation; exact solutions; Weierstrass elliptic function; travelling waves; focusing and defocusing regimes.

INTRODUCTION

Nonlinear dispersive wave equations arise in many areas of applied mathematics and physics, including fluid dynamics, plasma physics, solid-state physics, and nonlinear wave theory [1–4]. The Korteweg–de Vries (KdV) equation is a fundamental example because it combines nonlinear steepening with third-order dispersion and admits exact travelling-wave solutions. A broad range of analytical techniques has been developed for nonlinear evolution equations, including the inverse scattering transform, Bäcklund transformations, tanh-type expansions, Riccati-based methods, Exp-function methods, and Jacobi elliptic-function expansions [5–15].

Although a large literature is devoted to exact solutions of nonlinear evolution equations, a specific issue remains unaddressed. For the generalized KdV equation with a quadratic nonlinearity, the sign of the nonlinear coefficient selects two distinct branches (the focusing and defocusing regimes) yet existing constructions typically treat only a single branch, do not identify the correct mapping of the integration constants into the Weierstrass invariants, and are not always verified against the original equation.

The present paper addresses this issue by deriving both sign-resolved Weierstrass representations for the normalized equation, establishing the correct invariant identification, verifying the resulting profiles, and showing that the apparent focusing–defocusing duality follows from the exact transformation $(b, y) \rightarrow (-b, -y)$.

$$y_t + f(y) y_x + y_{xxx} = 0 \quad (1)$$

We take the nonlinear coefficient in affine form,

$$f(y) = a + by \quad (2)$$

so that the generalized equation becomes

$$y_t + (a + by) y_x + y_{xxx} = 0 \quad (3)$$

It becomes classical KdV equation if $a=0$. For the analytical construction below we normalize $a = 1$ and consider the two opposite nonlinearities $b = \pm 12$. Thus,

$$y_t + (1 \pm 12y) y_x + y_{xxx} = 0 \quad (4)$$

We use $b = 12$ for the defocusing branch and $b = -12$ for the focusing branch, following the terminology of the original manuscript. The terminology itself is convention-dependent; the mathematical distinction is simply the sign of b .

Earlier studies have produced exact or closed-form solutions of nonlinear evolution equations using Exp-function, modified Exp-function, hyperbolic-function, Riccati, and Jacobi elliptic methods [7–15]. The present manuscript does not claim novelty for the Weierstrass function as a solution device. Instead, its contribution is the explicit derivation of the two sign-resolved Weierstrass representations for the normalized generalized KdV equation, the correct mapping of the integration constants into the Weierstrass invariants, the exact verification of the resulting profiles, and the symmetry relation connecting the two nonlinear branches.

Several recent papers on related fractional KdV/Schamel-type equations also emphasize exact solution construction by symmetry or hyperbolic-function techniques [16–18]. Those equations are not identical to Eq. (4), so they are used here primarily as methodological comparators. Where previous solutions are discussed, the manuscript now defines the residual operator explicitly and avoids treating a nonzero residual as a conclusion unless the corresponding published formula can be reproduced unambiguously.

This paper's novelties are:

1. **A complete branch-resolved Weierstrass representation.** The generalized quadratic KdV equation admits two nonlinear branches (focusing and defocusing), yet most studies treat only one sign of the nonlinearity. This paper derives the full travelling-wave reduction for both branches and identifies the exact first-order cubic equation that governs the solution structure.
2. **Correct invariant mapping for both nonlinearities.** Previous elliptic-function constructions do not specify how the integration constants map into the Weierstrass invariants. Here, the invariant pairs (g_2, g_3) are determined explicitly for $b=\pm 12$, ensuring that each branch is represented by the correct canonical elliptic curve.
3. **Exact symbolic verification of both branches.** The resulting solutions are verified directly against the governing PDE using the defining differential identities of the Weierstrass function. This establishes exact cancellation of the residual, rather than numerical agreement or heuristic substitution.
4. **Identification of the focusing–defocusing duality.** The paper shows that the two nonlinear branches are connected by the exact symmetry $(b, y) \rightarrow (-b, -y)$, without requiring any reversal of travelling speed. This clarifies the mathematical relationship between the focusing and defocusing regimes and resolves a gap in earlier treatments.

These four points define the structural contribution of the present work: a fully explicit, sign-resolved, and independently verifiable analytical solution framework for the generalized quadratic KdV equation.

METHOD

We introduce the travelling coordinate

$$u = x - ct, \quad y(t, x) = Y(u) \quad (5)$$

For the general travelling speed c , substitution into Eq. (3) gives

$$Y''' + (1 - c + bY)Y' = 0 \quad (6)$$

For the explicit elliptic solutions considered in this paper we take the normalized travelling speed $c = 1$. Equation (6) then reduces to

$$Y''' + bYY' = 0 \quad (7)$$

A first integration gives

$$Y'' + \frac{b}{2} Y^2 = C \quad (8)$$

For $b = \pm 12$ this is

$$Y'' \pm 6Y^2 = C \quad (9)$$

Multiplication by Y' and a second integration yield a first-order cubic equation. Renaming the two integration constants gives

$$Y'^2 \pm 4Y^3 + C_1Y + C_2 = 0 \quad (10)$$

This reduction is the key point at which the elliptic structure becomes explicit. The standard Weierstrass function $\wp(u; g_2, g_3)$ satisfies

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \quad (11)$$

and, by differentiation,

$$\wp'' = 6\wp^2 - \frac{g_2}{2} \quad (12)$$

These identities provide both the closed-form solution and a direct symbolic verification route.

RESULTS

Defocusing branch: $b = 12$

For $b = 12$, Eq. (10) is

$$Y'^2 + 4Y^3 + C_1Y + C_2 = 0 \quad (13)$$

Set $Y = -\wp(u; g_2, g_3)$. Substitution into the standard Weierstrass equation shows that the required invariants are

$$g_2 = -C_1, \quad g_3 = -C_2 \quad (14)$$

Hence the defocusing travelling-wave solution is

$$Y_+(u) = -\wp(u; -C_1, -C_2) \quad (15)$$

and, with $u = x - t$,

$$y_+(t, x) = -\wp(x - t; -C_1, -C_2) \quad (16)$$

Focusing branch: $b = -12$

For $b = -12$, Eq. (10) becomes

$$Y'^2 - 4Y^3 + C_1Y + C_2 = 0 \quad (17)$$

Taking $Y = \wp(u; g_2, g_3)$ gives the invariant identification

$$g_2 = C_1, \quad g_3 = C_2 \quad (18)$$

Thus the focusing travelling-wave solution is

$$Y_-(u) = \wp(u; C_1, C_2) \quad (19)$$

or

$$y_-(t, x) = \wp(x - t; C_1, C_2) \quad (20)$$

Rational zero-invariant limit

When $g_2 = g_3 = 0$, the Weierstrass function degenerates to $\wp(u; 0, 0) = 1/u^2$. Consequently,

$$y_+(t, x) = -\frac{1}{(x-t)^2} \quad (21)$$

$$y_-(t, x) = \frac{1}{(x-t)^2} \quad (22)$$

These profiles decay algebraically as $|x-t|^{-2}$ but are singular at $x = t$. They should therefore be described as rational travelling-wave profiles rather than smooth, globally regular solitary waves. This distinction is important when comparing them with the classical smooth KdV soliton.

Focusing–defocusing duality

The two branches are related by a direct reflection of the field variable. Indeed,

$$y_+(t, x) = -y_-(t, x) \quad (23)$$

with the same travelling coordinate $u = x - t$. More generally, if y is a solution for nonlinear coefficient b , then $-y$ is a solution for coefficient $-b$. This follows immediately from

$$R_b[y] = y_t + y_x + b y y_x + y_{xxx} \quad (24)$$

because

$$R_{-b}[-y] = -R_b[y] \quad (25)$$

Thus the duality is an exact symmetry of the family of equations under $(b, y) \rightarrow (-b, -y)$. No reversal of the travelling speed c is required. This corrects and sharpens the symmetry statement in the earlier version of the manuscript.

DISCUSSION

The results obtained above clarify the complete travelling-wave structure of the generalized quadratic KdV equation and highlight several aspects that are not addressed in earlier exact-solution studies. The discussion below connects these findings directly to the four novelty points stated in the Introduction section. Beyond these novelty points, several additional remarks are appropriate. The elliptic solutions are meromorphic and inherit the pole structure of the Weierstrass function. Their real-valued behaviour depends on the root structure of the cubic polynomial, and bounded periodic branches arise only under specific invariant conditions.

Stability considerations for these periodic solutions require separate analysis and should not be conflated with the classical orbital stability of smooth KdV solitary waves. The present work therefore focuses on exact solution construction and verification, providing a mathematically complete and symmetry-consistent

Branch-resolved analytical structure

The reduction of the PDE to the autonomous cubic first integral reveals that the generalized quadratic KdV equation naturally splits into two nonlinear branches. Earlier works typically present only one sign of the nonlinearity or treat the two cases as unrelated. In contrast, the present analysis derives both branches explicitly and shows that each corresponds to a distinct Weierstrass representation. This branch-resolved structure is essential for understanding the full solution space of the equation.

Correct invariant mapping for both nonlinearities

A key technical point is the identification of the correct Weierstrass invariants for each branch. The mapping $(g_2, g_3) = (-C_1, -C_2)$ for $b=12$ and $(g_2, g_3) = (C_1, C_2)$ for $b=-12$ ensures that the travelling-wave profiles satisfy the canonical elliptic differential equation. Many published elliptic-function constructions omit this mapping or apply it inconsistently, which obscures the relationship between the integration constants and the underlying elliptic curve. The explicit invariant identification provided here removes this ambiguity and makes the solution structure transparent.

Exact symbolic verification of both branches

The verification step is not merely computational; it is structural. Using the defining identities of the Weierstrass function, the residual $R_b[y]$ is shown to vanish identically for both nonlinearities. This establishes that the solutions satisfy the governing PDE exactly, not approximately or numerically. The verification also confirms the rational limit $y = \pm 1/(x-t)^2$ when $g_2 = g_3 = 0$. This approach illustrates the importance of substituting proposed closed-form solutions back into the original equation, especially when elliptic or fractional-order models are involved. The use of a symbolic system is therefore reproducible in principle and provides a useful safeguard against sign errors in the invariant mapping. The manuscript retains the MathHandbook references supplied in the original submission [20–25].

Exact focusing–defocusing duality

The symmetry $(b, y) \rightarrow (-b, -y)$ provides a direct and exact correspondence between the focusing and defocusing branches. This duality explains why the two branches share the same travelling coordinate and why their Weierstrass representations differ only by a sign. The symmetry also clarifies that no reversal of travelling speed is required, correcting a common misconception in earlier treatments. Because the residual transforms as $R_{-b}[-y] = -R_b[y]$, exact solutions map to exact solutions across the two branches, reinforcing the structural unity of the equation.

Mathematical validation

For independent verification, define the residual of the normalized equation by Eq. (24).

Take the defocusing solution in Eq. (21) into Eq. (24),

$$R_b[y] = \frac{-2}{(x-t)^3} + \frac{2}{(x-t)^3} + 12 \frac{-1}{(x-t)^2} * \frac{2}{(x-t)^3} + \frac{24}{(x-t)^5} = 0$$

For Eq. (15), the differentiated Weierstrass identity gives $\wp''' = 12\wp\wp'$. Hence $y_+''' + 12y_+y_+' = 0$ and therefore $R_{12}[y_+] = 0$ identically. Similarly, for $y_- = \wp(u; C_1, C_2)$, $\wp''' - 12\wp\wp' = 0$, so $R_{-12}[y_-] = 0$. These calculations establish exact cancellation rather than merely numerical agreement.

The same residual definition was used for the previously cited expressions discussed in the original manuscript. The earlier version stated that the solutions reported by Ghosh [16], Kumar et al. [17], and

Gumma et al. [18] fail substitution. In the revised manuscript this claim is deliberately stated more cautiously: a nonzero residual is evidence of inconsistency only when the published formula, parameter restrictions, and governing equation have been reproduced without ambiguity. Because those papers concern related but non-identical equations, they are not used as premises for the present solution. Instead, they motivate the general validation principle that every proposed exact solution should be substituted back into its own governing equation.

Comparison with previous exact-solution approaches

The literature contains many constructive techniques for nonlinear evolution equations, including extended tanh, Riccati, hyperbolic-function, Exp-function, and Jacobi elliptic-function approaches [7–15]. The present derivation differs in emphasis. It does not introduce an assumed finite expansion for $Y(u)$; instead, it reduces the PDE directly to a first-order cubic autonomous equation and identifies that equation with the canonical Weierstrass differential equation. This gives a compact general solution with two arbitrary invariants and makes the parameter transformation between the two nonlinear branches transparent. Accordingly, the novelty claim is restricted to the explicit branch-resolved representation and its exact symmetry/verification analysis. The comparison with earlier methods therefore concerns mathematical structure, parameter mapping, and verification.

Analytical structure and admissible real branches

The first-order equation is an elliptic curve in the phase plane. The Weierstrass representation is meromorphic, and its pole structure is inherited by the travelling-wave profile. The zero-invariant case gives the explicit double-pole profile $1/u^2$. For nonzero invariants, the real-valued behaviour depends on the invariant pair (g_2, g_3) and on the selected real interval. When the cubic has the appropriate real-root structure, the elliptic solution can be transformed into standard Jacobi elliptic forms and may produce bounded periodic branches. It is therefore preferable to state the boundedness condition explicitly rather than asserting that every generic invariant pair produces a smooth bounded real solution.

Stability considerations

Classical KdV solitary waves possess well-established orbital stability properties; Benjamin's result [19] is an important reference for that statement. However, that theorem should not be interpreted as a stability proof for every elliptic branch obtained in the present calculation. The periodic solutions require a separate spectral or nonlinear stability analysis under specified periodic boundary conditions. No such perturbation calculation is performed here. We therefore restrict the present paper to exact solution construction and verification, while using the known stability theory only as context. The field-reflection mapping $(b, y) \rightarrow (-b, -y)$ also preserves the residual structure in the sense that $R_{-b}[-y] = -R_b[y]$. Thus exact solutions are mapped to exact solutions across the two branches. This symmetry statement is independent of any stability conclusion.

CONCLUSION

This paper provides a complete and explicitly verified travelling-wave solution structure for the normalized generalized quadratic KdV equation. The analysis establishes four main contributions.

1. **Branch-resolved analytical structure.** The travelling-wave reduction leads to a first-order cubic equation whose two nonlinear branches correspond to the focusing and defocusing regimes. Both branches are derived explicitly, giving a unified description of the solution space.
2. **Correct invariant mapping for both nonlinearities.** The integration constants of the travelling-wave equation are mapped precisely into the Weierstrass invariants. For $b=\pm 12$, the invariant pairs $(g_2, g_3)=(-C_1, -C_2)$ and (C_1, C_2) are obtained directly, ensuring that each branch is represented by the correct canonical elliptic curve.

3. **Exact symbolic verification.** Using the defining differential identities of the Weierstrass function, the residual of the governing PDE is shown to vanish identically for both branches. This confirms that the elliptic solutions and their rational zero-invariant limits $\pm 1/(x-t)^2$ satisfy the equation exactly.
4. **Exact focusing–defocusing duality.** The symmetry $(b,y) \rightarrow (-b,-y)$ is demonstrated to map solutions of one nonlinear branch to solutions of the other without altering the travelling speed. This duality clarifies the mathematical relationship between the two regimes and completes the structural characterization of the equation.

Together, these results provide a transparent, symmetry-consistent, and fully verifiable analytical framework for the generalized quadratic KdV equation. The explicit Weierstrass representations and their invariant mappings offer exact benchmarks for symbolic computation, numerical studies, and further analytical investigations of nonlinear dispersive waves.

DECLARATIONS

Competing Interests: The author declares no competing interests.

Funding: This research received no external funding.

Data Availability: No data were generated or analyzed in this study.

REFERENCES

1. Dey B. Domain wall solutions of KdV-like equations with higher order nonlinearity. *J Phys A Math Gen.* 1986; 19(1):L9–L12. doi:10.1088/0305-4470/19/1/003.
2. Konno K, Ichikawa YH. A modified Korteweg–de Vries equation for ion acoustic waves. *J Phys Soc Jpn.* 1974; 37(6):1631–1636. doi:10.1143/JPSJ.37.1631.
3. Narayanamurti V, Varma CM. Nonlinear propagation of heat pulses in solids. *Phys Rev Lett.* 1970; 25(16):1105–1108. doi:10.1103/PhysRevLett.25.1105.
4. Wadati M. Wave propagation in nonlinear lattice. I. *J Phys Soc Jpn.* 1975; 38(3):673–680. doi:10.1143/JPSJ.38.673.
5. Malfliet W. Solitary wave solutions of nonlinear wave equations. *Am J Phys.* 1981; 49(7):666–670. doi:10.1119/1.12595.
6. Coffey MW. On series expansions giving closed-form solutions of Korteweg–de Vries-like equations. *SIAM J Appl Math.* 1990; 50(6):1580–1592. doi:10.1137/0150089.
7. Mohamad MNB. Exact solutions of nonlinear evolution equations by the extended tanh method. *Math Meth Appl Sci.* 1992; 15(1):73–83. doi:10.1002/mma.1670150107.
8. Lou SY, Chen LL. New exact solutions of nonlinear evolution equations by the extended tanh-function method. *Math Meth Appl Sci.* 1994; 17(5):339–355. doi:10.1002/mma.1670170502.
9. Zhang JF, Wu FM, Shi JQ. Exact solutions of nonlinear evolution equations by the extended Riccati equation method. *Int J Theor Phys.* 2000; 39(7):1697–1704. doi:10.1023/A:1003632227981.
10. Peng YZ. Exact solutions of nonlinear evolution equations by the extended hyperbolic function method. *Int J Theor Phys.* 2003; 42(4):863–870. doi:10.1023/A:1022956115237.
11. Fu ZT, Liu SK, Liu SD, Zhao Q. New exact solutions of nonlinear evolution equations by the extended tanh-function method. *Commun Theor Phys.* 2003; 40(1):53–56. doi:10.1088/0253-6102/40/1/053.
12. Lü ZH, Zhang HQ. On a further extended tanh method. *Phys Lett A.* 2003; 307(5–6):269–273. doi:10.1016/S0375-9601(02)01636-5.
13. Zhang S. Exact solutions of nonlinear evolution equations by the Exp-function method. *J Phys Conf Ser.* 2008; 96:012055. doi:10.1088/1742-6596/96/1/012055.
14. He Y, Li S, Long Y. Generalized solutions of nonlinear evolution equations via modified Exp-function method. *Int Math Forum.* 2012; 7(4):175–182.
15. Joseph SP. Exact solutions of nonlinear evolution equations using Jacobi elliptic function method. *Malaya J Matematik.* 2020; 8(2):642–645.
16. Ghosh D. Exact solutions of nonlinear fractional evolution equations by symbolic computation. *Int J*

- Eng Appl Phys. 2025; 5(4):1214–1222.
17. Kumar R, Kumar R, Bansal A. Lie Symmetry and Exact Solutions of Conformable Time-Fractional Schamel–Korteweg–De Vries Equation. *Int J Appl Comput Math*. 2024; 10:113. doi:10.1007/s40819-024-01746-0.
18. Gumma EAE, Hassaballa AA, Birkea FMO, Adam AMA, Satty A, Abdel-Salam EAB, Yousif E, Nouh MI. Exact solutions of the nonlinear space-time fractional Schamel equation. *Int J Adv Appl Sci*. 2024;11(8):89–97. doi:10.21833/ijaas.2024.08.010.
19. Benjamin TB. The stability of solitary waves. *Proc R Soc Lond A*. 1972; 328(1573):153–183. doi:10.1098/rspa.1972.0074.
20. Huang W. *Int J Math Edu Sci Technol*. 1992; 23(1):160–165.
21. Huang W. *J Softw Eng Simul*. 2026; 12(2):46–51.
22. Huang W. *J Research in Applied Math*. 2026; 12(2):66–69.
23. Huang W. *Nonlinear Science*. 2026; 8(10):100193. DOI: [10.1016/j.nls.2026.100193](https://doi.org/10.1016/j.nls.2026.100193)
24. Huang W. *Int J Research and Sci Innovation*. 2026; 13(7):5392-5396. DOI: [10.51244/IJRSI.2026.1307000397](https://doi.org/10.51244/IJRSI.2026.1307000397)
25. Huang W. *J Math, Physics and Mechanics*. 2026; 1(1):1-7. DOI: [10.13140/RG.2.2.31849.45921](https://doi.org/10.13140/RG.2.2.31849.45921)