

$b - I_S -$ Open Sets and Decomposition of Continuity Via Idealization

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ABSTRACT

In this paper, we introduce the notion of $b - I_S -$ open sets and strong $B_{IS} -$ sets to obtain decomposition of continuity via idealization. Additionally, we investigate properties of $b - I_S -$ open sets and strong $B_{IS} -$ sets

Key words and Phrases: semi- $I_S -$ open sets, pre- $I_S -$ open sets, $\alpha - I_S -$ open sets, $b - I_S -$ open sets and strong $B_{IS} -$ sets

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INTRODUCTION AND PRELIMINARIES

Ideal in topological spaces have been considered since 1966 by Kuratowski [9] and Vaidyanathaswamy [16]. After several decades, in 1990, Jankovic and Hammett [7] investigated the topological ideals which is the generalization of general topology. Whereas in 2010, Khan and Noiri [8] introduced and studied the concept of semilocal functions. In 2014, Shanthi and Rameshkumar [14] introduced semi- $I_S -$ open sets, pre- $I_S -$ open sets and $\alpha - I_S -$ open sets. In this paper we introduce the notions of $b - I_S -$ open sets and strong $B_{IS} -$ sets to obtain decomposition of continuity. Let (X, τ) be a topological space and I is an ideal of subset of X . An ideal I on a topological space (X, τ) is a collection of nonempty subsets of X which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X, τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(\cdot)^* : \wp(X) \rightarrow \wp(X)$, called the local function of A with respect to τ and I , is defined as follows: for $A \subset X$, $A^*(I, \tau) = \{x \in X / U \cap A \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau / x \in U\}$ (Kuratowski 1966). A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(I, \tau)$, called the $*$ -topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I, \tau)$ (Vaidyanathaswamy, 1945). When there is no chance for confusion, we will simply write A^* for $A^*(I, \tau)$ and τ^* or $\tau^*(I)$ for $\tau^*(I, \tau)$. If I is an ideal on X , then (X, τ, I) is called an ideal space. $\beta = \{G - A / G \in \tau, A \in I\}$ is a basis for τ^* (Jankovic and Hammett, 1992). If $A \subset X$, $cl(A)$ and $int(A)$ will respectively denote the closure and the interior of A in (X, τ) and $int^*(A)$ will denote the interior of A in (X, τ^*) .

Definition 1.1. Let (X, τ) be a topological space. A subset A of X is said to be semiopen [10] if there exists an open set U in X such that $U \subset A \subset cl(U)$. The complement of a semi open set is said to be semi-closed. The collection of semi open (resp. semiclosed) sets in X is denoted by $SO(X)$ (resp. $SC(X)$). The semi closure of A in (X, τ) is denoted by the intersection of all semiclosed sets containing A and is denoted by $scl(A)$.

Definition 1. 2. For $A \subset X$, $A_*(I, \tau) = \{x \in X / U \cap A \notin I \text{ for every } U \in SO(X)\}$ is called the semi-local function [8] of A with respect to I and τ , where $SO(X, x) = \{U \in SO(X) : x \in U\}$. We simply write A_* instead of $A_*(I, \tau)$. It is given in [1] that $\tau^{*s}(I)$ is a topology on X , generated by the sub basis

$\{U - E : U \in SO(X) \text{ and } E \in I\}$ or equivalently $\tau^{*s}(I) = \{U \subset X : cl^{*s}(X - U) = X - U\}$. The closure operator $cl^{*s}(A) = A \cup A_*$ and $int^{*s}(A)$ denote the interior of the set A (X, τ^{*s}, I) . It is known that $\tau \subset \tau^*(I) \subset \tau^{*s}(I)$.

Lemma 1.3[8]. Let (X, τ, I) be an ideal topological space and $A, B \subset X$.

Then for the semi-local function the following properties hold:

- (i) If $A \subset B$, then $A_* \subset B_*$.
- (ii) If $U \in \tau$, then $U \cap A_* \subset (U \cap A)_*$.

Definition 1.4.

A subset A of a topological space X is said to be

- (i) α -open [12] if $A \subset int(cl(int(A)))$.
- (ii) pre-open [11] if $A \subset int(cl(A))$.
- (iii) semi-open [10] if $A \subset cl(int(A))$.
- (iv) t-set [13] if $int(A) = int(cl(A))$.
- (v) b-open set [3] if $A \subset int(cl(A)) \cup cl(int(A))$.
- (vi) strong B-set [4] if $A = U \cap V$, where U is open, V is t-set and $int(cl(A)) = cl(int(A))$.

Definition 1.5.

A subset A of an ideal topological space (X, τ, I) is said to be

- (i) $\alpha - I$ -open [6] if $A \subset int(cl^*(int(A)))$.
- (ii) pre-I-open [5] if $A \subset int(cl^*(A))$.
- (iii) semi-I-open [6] if $A \subset cl^*(int(A))$.
- (iv) b-I-open [2] if $A \subset int(cl^*(A)) \cup cl^*(int(A))$.
- (v) t-I-set [6] if $int(cl^*(A)) = int(A)$.
- (vi) B_I -set [6] if $A = U \cap V$, $U \in \tau$ and V is a t-I-set.
- (vii) Strong B_I -set [2] if $A = U \cap V$, $U \in \tau$ and V is a t-I-set and $int(cl^*(V)) = cl^*(int(V))$.

Definition 1.6.

A subset A of an ideal space (X, τ, I) is said to be

- (i) $\alpha - I_s$ -open [14] if $A \subset int(cl^{*s}(int(A)))$.

- (ii) $\text{pre} - I_S - \text{open}$ [14] if $A \subset \text{int}(cl^{*s}(A))$.
- (iii) $\text{semi} - I_S - \text{open}$ [14] if $A \subset cl^{*s}(\text{int}(A))$.
- (iv) $t - I_S - \text{set}$ [14] $\text{int}(cl^{*s}(A)) = \text{int}(A)$.
- (v) $B_{I_S} - \text{set}$ [14] if $A = U \cap V$, where $U \in \tau$ and V is an $t - I_S - \text{set}$.

The family of all $\alpha - I_S - \text{open}$ (resp. $\text{Semi} - I_S - \text{open}$, $\text{Pre} - I_S - \text{open}$) sets in an ideal topological space (X, τ, I) is denoted by $\alpha ISO(X)$ (resp. $SISO(X)$, $PISO(X)$).

Lemma 1.8[15]. Let (X, τ, I) be an ideal topological space and $A \subset X$.

If U is open in (X, τ, I) , then $U \cap cl^{*s}(A) \subset cl^{*s}(U \cap A)$.

$b - I_S - \text{OPEN SET}$

Definition 2.1 A subset A of an ideal space (X, τ, I) is said to be a $b - I_S - \text{open set}$

if $A \subset \text{int}(cl^{*s}(A)) \cup cl^{*s}(\text{int}(A))$.

Proposition 2.1. Let A be a $b - I_S - \text{open set}$ such that $\text{int}(A) = \emptyset$, then A is $\text{pre} - I_S - \text{open}$.

Proof : Let A be a $b - I_S - \text{open set}$. Then we have $A \subset \text{int}(cl^{*s}(A)) \cup cl^{*s}(\text{int}(A))$.

If $\text{int}(A) = \emptyset$, then $cl^{*s}(\text{int}(A)) = \emptyset$. Therefore, $A \subset \text{int}(cl^{*s}(A)) \cup cl^{*s}(\text{int}(A))$ becomes

$$A \subset \text{int}(cl^{*s}(A))$$

Proposition 2.2. For a subset of an ideal space (X, τ, I) the following hold.

- (i) Every open set is $b - I_S - \text{open}$.
- (ii) Every $\text{semi} - I_S - \text{open set}$ is $b - I_S - \text{open}$.
- (iii) Every $\text{pre} - I_S - \text{open set}$ is $b - I_S - \text{open}$.
- (iv) Every $b - I_S - \text{open set}$ is $b - \text{open}$.

Proof: (i),(ii),(iii) Obvious.

(iv) Let A be $b - I_S - \text{open}$. Then we have

$$\begin{aligned} A \subset \text{int}(cl^{*s}(A)) \cup cl^{*s}(\text{int}(A)) &\subset \text{int}(A_* \cup A) \cup (\text{int}(A))_* \cup \text{int}(A) \subset \text{int}(scl(A) \cup A) \cup scl(\text{int}(A)) \cup \text{int}(A) \\ &\subset \text{int}(cl(A)) \cup cl(\text{int}(A)) \cup \text{int}(A) \subset \text{int}(cl(A)) \cup cl(\text{int}(A)). \end{aligned}$$

This shows that A is $b - \text{open}$.

Remark 2.1. Converse of the Proposition 2.2 need not be true as seen from the following examples.

Example 2.1. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then

(i) $A = \{a, b, c\}$ is $b - I_S$ - open but it is not semi- I_S - open.

(ii) $A = \{a, b, c\}$ is $b - I_S$ - open but it is not open.

Example 2.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $I = \{\emptyset, \{c\}\}$.

(i) $A = \{b, c\}$ is $b - I_S$ - open but it is not pre- I_S - open.

Example 2.3. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{b\}, \{a, d\}, \{a, b, d\}, X\}$ and $I = \{\emptyset, \{b\}\}$.

Then $A = \{a, c, d\}$ is not b open but it is $b - I_S$ - open.

STRONG B_{IS} - SET

Definition 3.1. A subset A of an ideal space (X, τ, I) is called strong B_{IS} - set if $A = U \cap V$, where $U \in \tau$ and V is a $t - I_S$ - set and $\text{int}(cl^{*s}(V)) = cl^{*s}(\text{int}(V))$.

Proposition 3.1. Let (X, τ, I) be an ideal space and $A \subset X$. If A is a strong B_{IS} - set, then A is a B_{IS} - set.

Proof: Obvious

Remark 3.1. Converse of the Proposition 3.1 need not be true as seen from the following example.

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $I = \{\emptyset\}$.

If $A = \{b\}$, then $\text{int}(cl^{*s}(A)) = \{\emptyset\}$ and $\text{int}(A) = \{\emptyset\}$. Hence A is a $t - I_S$ - set. Clearly A is a B_{IS} - set. But $\text{int}(cl^{*s}(A)) = \{\emptyset\}$ and $cl^{*s}(\text{int}(A)) = \{b\}$. Hence $\text{int}(cl^{*s}(V)) \neq cl^{*s}(\text{int}(V))$. So A is not a strong B_{IS} - set.

Theorem 3.1. Let (X, τ, I) be an ideal space and $A \subset X$. Then the following conditions are equivalent:

- (i) A is open;
- (ii) A is $b - I_S$ - open and a strong B_{IS} - set.

Proof: (i) \Rightarrow (ii) By Proposition 2.2, every open set is $b - I_S$ - open. On the other hand every open set is strong B_{IS} - set, because X is $t - I_S$ - set and $\text{int}(cl^{*s}(X)) = cl^{*s}(\text{int}(X))$.

(ii) \Rightarrow (i) Let A be $b - I_S$ - open and a strong B_{IS} - set. Then $A \subset \text{int}(cl^{*s}(A)) \cup cl^{*s}(\text{int}(A)) = \text{int}(cl^{*s}(U \cap V)) \cup cl^{*s}(\text{int}(U \cap V))$ where U is open and V is a $t - I_S$ - set and $\text{int}(cl^{*s}(V)) = cl^{*s}(\text{int}(V))$. Hence $A \subset (\text{int}(cl^{*s}(U)) \cap \text{int}(cl^{*s}(V))) \cup (cl^{*s}(\text{int}(U)) \cap cl^{*s}(\text{int}(V)))$

$$\Rightarrow A \subset U \cap (\text{int}(cl^{*s}(V)) \cup cl^{*s}(\text{int}(V)))$$

$$\Rightarrow A \subset U \cap \text{int}(cl^{*s}(V))$$

$$\Rightarrow A \subset U \cap \text{int}(V) = \text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V) = \text{int}(A).$$

So A is open.

Remark 3.2. The notion of A is $b - I_S$ - openness is different from that of strong B_{IS} - sets.

- (i) In Example 2.1 $A = \{b, c\}$ is not $b - I_S$ - open. But $\text{int}(cl^{*s}(A)) = cl^{*s}(\text{int}(A)) = \text{int}(A) = \{b\}$. So A is a strong B_{IS} - set.
- (ii) In Example 2.1 $A = \{a, b, c\}$ is $b - I_S$ - open. But $\text{int}(cl^{*s}(A)) = X, cl^{*s}(\text{int}(A)) = \{a, b, d\}, \text{int}(A) = \{a, b\}$. So A is not a strong B_{IS} - set.

DECOMPOSITION OF CONTINUITY

Definition 4.1. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be b continuous [3] if for every $V \in \sigma, f^{-1}(V)$ is b open set of (X, τ) .

Definition 4.2. A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be B_{IS} - continuous [14] (resp. semi - I - continuous [6], pre - I - continuous [5]) if for every $V \in \sigma, f^{-1}(V)$ is a B_{IS} - set (resp. semi - I - open set, pre - I - open set) of (X, τ, I) .

Definition 4.3. A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be $\alpha - I_S$ - continuous [14] (resp. semi - I_S - continuous [14], pre - I_S - continuous [14]) if for every $V \in \sigma, f^{-1}(V)$ is an $\alpha - I_S$ - open set (resp. semi - I_S - open set, pre - I_S - open set) of (X, τ, I) .

Definition 4.4. A function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is said to be $b - I_S$ - continuous, (resp. strong B_{IS} - continuous) if for every $V \in \sigma, f^{-1}(V)$ is a $b - I_S$ - set (resp. a strong B_{IS} - set) of (X, τ, I) .

Proposition 4.1. Let (X, τ, I) be an ideal space. If a function $f : (X, \tau, I) \rightarrow (Y, \sigma)_{IS}$ semi - I_S - continuous (res. pre - I_S - continuous), then f is $b - I_S$ - continuous.

Proof: This is an immediate consequence of Proposition 2.2 (ii) and (iii).

Proposition 4.2 Let (X, τ, I) be an ideal space. If a function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ $b - I_S$ - continuous, then f is b continuous.

Proof: This is an immediate consequence of Proposition 2.2 (iv).

Proposition 4.3 Let (X, τ, I) be an ideal space. If a function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ strong B_{IS} - continuous, then f is B_{IS} - continuous.

Proof: This is an immediate consequence of Proposition 3.1.

Theorem 4.1. Let (X, τ, I) be an ideal space. For a function $f : (X, \tau, I) \rightarrow (Y, \sigma)$ the following conditions are equivalent:

- (i) f is continuous;

(ii) f is $b - I_s -$ continuous and strong $B_{I_s} -$ continuous.

Proof: This is an immediate consequence of Theorem 3.1.

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