Weakly Minuscule Regular Generalized Closed Sets in Ideal Minuscule Topological Spaces

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ABSTRACT

This article introduces weakly minuscule regular generalized closed sets in ideal minuscule topological spaces. The concept extends regular generalized closed sets using weak minuscule conditions. Characterizations and inclusion relations are obtained. Behavior under standard set operations is examined. The study contributes to generalized closure theory.

Keywords: Weakly minuscule sets, regular generalized closed sets, ideal topology.

INTRODUCTION

The study of generalized closed sets has been extended in many directions to capture weaker forms of topological properties. Concepts such as regular closed sets and their generalizations have been widely investigated in classical and ideal topological spaces [3, 7].

The introduction of weak forms of generalized sets has provided additional flexibility in analyzing topological structures. These developments were further enhanced by the emergence of nano topology [5, 6] and ideal nano spaces [10].

Minuscule topology [9] offers a more refined framework for studying such generalized structures. The integration of ideals into this setting allows the formulation of weak forms of regular generalized closed sets.

In this paper, we introduce weakly minuscule regular generalized closed sets in ideal minuscule topological spaces. Their properties, inclusion relations, and interactions with existing classes are studied. The results contribute to the development of weak generalized structures in minuscule topology.

PRELIMINARIES

In this section, we present preliminary definitions and notations. These will be used throughout the paper.

Definition 2.1 [11,5] Take U be a non-empty finite universe and $\mathcal{R}$ an equivalence (indiscernibility) relation on it. The pair $(\mathrm{U}, \mathcal{R})$ is called an approximation space, and U splits into disjoint equivalence classes. For any subset $\mathcal{Y} \subseteq \mathrm{U}$:

1.The lower approximation of Y is $\mathcal{L}_{\mathcal{R}}(\mathcal{Y}) = \bigcup_{\mathrm{y}\in\mathrm{u}} \{\mathcal{R}(\mathrm{y}): \mathcal{R}(\mathrm{y}) \subseteq \mathcal{Y}\}$.

2.The lower minimal approximation is $\mathcal{L}^{*}_{\mathcal{R}}(\mathcal{Y}) = \mathcal{L}_{\mathcal{R}}(\mathcal{Y}) \setminus \mathcal{Y}$.

3.The upper approximation of Y is $\mathcal{U}_{\mathcal{R}}(\mathcal{Y}) = \bigcup_{\mathrm{y}\in\mathrm{u}} \{\mathcal{R}(\mathrm{y}): \mathcal{R}(\mathrm{y}) \cap \mathcal{Y} \neq \emptyset\}$.

4.The upper minimal approximation is $\mathcal{U}^{*}_{\mathcal{R}}(\mathcal{Y}) = \mathcal{U}_{\mathcal{R}}(\mathcal{Y}) \setminus \mathcal{Y}$.

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5.The symmetric difference is $\mathcal{L}_{\mathcal{R}}(\mathcal{Y})\Delta\mathcal{U}^*_{\mathcal{R}}(\mathcal{Y}) = (\mathcal{L}_{\mathcal{R}}(\mathcal{Y})\backslash\mathcal{U}^*_{\mathcal{R}}(\mathcal{Y})) \cup (\mathcal{U}^*_{\mathcal{R}}(\mathcal{Y})\backslash\mathcal{L}_{\mathcal{R}}(\mathcal{Y}))$.

Definition 2.2 [9] Take $\mathcal{U}$ be a universal set and $\mathcal{R}$ an equivalence relation on $\mathcal{U}$. For any subset $\mathcal{Y} \subseteq \mathcal{U}$ we define

$$\tau_{\mathcal{R}}(\mathcal{Y}) = \{\mathcal{U}, \emptyset, \mathcal{L}_{\mathcal{R}}(\mathcal{Y}), \mathcal{U}_{\mathcal{R}}(\mathcal{Y}), \mathcal{U}^*_{\mathcal{R}}(\mathcal{Y}), \mathcal{L}_{\mathcal{R}}(\mathcal{Y})\Delta\mathcal{U}^*_{\mathcal{R}}(\mathcal{Y})\}, \text{with } \mathcal{Y} \subseteq \mathcal{R}.$$

The collection $\tau_{\mathcal{R}}(\mathcal{Y})$ satisfies the standard topology axioms:

1.Both $\mathcal{U}$ and $\emptyset$ are members of $\tau_{\mathcal{R}}(\mathcal{Y})$.

2.The union of any family of sets from $\tau_{\mathcal{R}}(\mathcal{Y})$ also belongs to $\tau_{\mathcal{R}}(\mathcal{Y})$.

3.The intersection of finitely many sets from $\tau_{\mathcal{R}}(\mathcal{Y})$ remains in $\tau_{\mathcal{R}}(\mathcal{Y})$.

Consequently, $\tau_{\mathcal{R}}(\mathcal{Y})$ is a topology on $\mathcal{U}$, named the $\mathcal{M}$-Topology (or minuscule topology) relative to $\mathcal{Y}$. The ordered pair $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{Y}))$ is referred to as an $\mathcal{M}$-TS (minuscule topological space).

Remark 2.1 [9] Elements of the collection $\tau_{\mathcal{R}}(\mathcal{Y})$ are referred to as minuscule open sets (often shortened to m-open sets). Their complements are called minuscule closed sets (or m-closed sets).For any subset $\mathrm{Z} \subseteq \mathrm{U}$, its minuscule interior is written $\mathrm{m\text{-}int(Z)}$ and its minuscule closure is written $\mathrm{m\text{-}clr(Z)}$.We denote a minuscule topological space by $(\mathrm{U}, \mathcal{M})$, where $\mathcal{M} = \tau_{\mathcal{R}}(\mathcal{Y})$.

Definition 2.3[10,2,3] A minuscule topological space $(\mathrm{U}, \tau_{\mathcal{R}}(\mathcal{Y}))$ equipped with an ideal $\mathcal{I}$ on U is referred to as a minuscule ideal topological space, denoted by $(\mathrm{U}, \tau_{\mathcal{R}}(\mathcal{Y}), \mathcal{I})$ or simply $(\mathrm{U}, \mathcal{M}, \mathcal{I})$. Let $\mathcal{G}_{\mathrm{m}}(\mathrm{y}) = \{\mathrm{G} \in \tau_{\mathcal{R}}(\mathcal{Y}) : \mathrm{y} \in \mathrm{G}\}$ signify the family of minuscule neighbourhoods of a point y. Consider the set operator

$(\bullet)^*_{\mathrm{m}} : \wp(\mathrm{U}) \to \wp(\mathrm{U})$, where $\wp(\mathrm{U})$ represents the power set of U. For any subset $\mathcal{K} \subseteq \mathrm{U}$, define the **minuscule local function** of $\mathcal{K}$ with respect to $\mathcal{I}$ and $\tau_{\mathcal{R}}(\mathcal{Y})$ as:

$$\mathcal{K}^*_{\mathrm{m}}(\mathcal{I}, \tau_{\mathcal{R}}(\mathcal{Y})) = \{\mathrm{y} \in \mathrm{U} : \mathrm{G} \cap \mathcal{K} \notin \mathcal{I} \text{ for some } \mathrm{G} \in \mathcal{G}_{\mathrm{m}}(\mathrm{y})\}.$$

For simplicity, we indicate it by $\mathcal{K}^*_{\mathrm{m}}$ or $\mathcal{K}^*$.

Definition 2.4 [9] Let $(\mathrm{U}, \tau_{\mathcal{R}}(\mathcal{Y}), \mathcal{I})$ be a minuscule ideal topological space. The minuscule $*$-closure of a subset $\mathcal{K} \subseteq \mathrm{U}$ is defined by $\mathrm{m\text{-}clr}^*(\mathcal{K}) = \mathcal{K} \cup \mathcal{K}^*_{\mathrm{m}}$. This defines a minuscule $*$-topology $\tau^*_{\mathcal{R}}(\mathcal{I}, \tau_{\mathcal{R}}(\mathcal{Y})) = \{\mathcal{K} \subseteq \mathrm{U} : \mathrm{m\text{-}clr}^*(\mathrm{U}\backslash\mathcal{K}) = \mathrm{U}\backslash\mathcal{K}\}$. We simply write $\tau^*_{\mathcal{R}}$ for $\tau^*_{\mathcal{R}}(\mathcal{I}, \tau_{\mathcal{R}})$ and note that $\tau^*_{\mathcal{R}}$ is finer than $\tau_{\mathcal{R}}$.The triplet $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{Y}), \mathcal{I})$ is called a minuscule ideal space (MI-space).

Remark 2.2 [9] 1.The members of $\tau^*_{\mathcal{R}}$ are called minuscule $*$-open sets (or m^*-open sets).

Their complements are called minuscule $*$-closed sets (or m^*-closed sets).

2.For any subset $\mathcal{K} \subseteq \mathrm{U}$, the minuscule $*$-interior and minuscule $*$-closure of $\mathcal{K}$ are denoted by

$\mathrm{m\text{-}int}^*(\mathcal{K})$ and $\mathrm{m\text{-}clr}^*(\mathcal{K})$, respectively.

3.If the ideal $\mathcal{I} = \{\emptyset\}$, then the minuscule $*$-derived set $\mathcal{K}^*_{\mathrm{m}}$ coincides with the minuscule-closure

$\mathrm{m\text{-}clr}(\mathcal{K})$, and hence $\mathrm{m\text{-}clr}^*(\mathcal{K}) = \mathrm{m\text{-}clr}(\mathcal{K})$.

4.When $\mathcal{I} = \{\emptyset\}$, the minuscule $*$-topology $\tau^*_{\mathcal{R}}$ coincides with the original minuscule topology $\tau_{\mathcal{R}}$.

5.A basis for the minuscule $*$-topology $\tau^*_{\mathcal{R}}$ is given by $\beta(\mathcal{I}, \tau^*_{\mathcal{R}}) = \{\mathrm{G}\backslash\mathrm{H} : \mathrm{G} \in \tau_{\mathcal{R}}, \mathrm{H} \in \mathcal{I}\}$.

Definition 2.5 [4] Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\mathcal{K} \subseteq \mathcal{U}$.

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1.$\mathcal{K}$ is m-regular open if $\mathcal{K} = \text{mint}(\text{mcl}(\mathcal{K}))$.

2.$\mathcal{K}$ is m-regular closed if $\mathcal{K} = \text{mcl}(\text{mint}(\mathcal{K}))$.

3.$\mathcal{K}$ is pre *-I_m-open if $\mathcal{K} \subseteq \text{mint}^*(\text{mcl}(\mathcal{K}))$.

4.$\mathcal{K}$ is pre *-I_m-closed if $\text{mcl}^*(\text{mint}(\mathcal{K})) \subseteq \mathcal{K}$.

5.$\mathcal{K}$ is m-nowhere dense if $\text{mint}(\text{mcl}(\mathcal{K})) = \emptyset$.

WEAKLY mI_rg-CLOSED SETS

Definition 3.1 A subset E of $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ is said to be a weakly minuscule I_rg-closed set (briefly, weakly mI_rg-closed set) if $(\text{mint}(\text{E}))^* \subseteq \text{F}$ whenever $\text{E} \subseteq \text{F}$ and F is an m-regular open set in $\mathcal{U}$.

The complement of a weakly mI_rg-closed set is said to be a weakly mI_rg-open set.

Theorem 3.1 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\text{E} \subseteq \mathcal{U}$. The following properties are equivalent:

1.E is a weakly mI_rg-closed set.

2.$\text{mcl}^*(\text{mint}(\text{E})) \subseteq \text{F}$ whenever $\text{E} \subseteq \text{F}$ and F is an m-regular open set in $\mathcal{U}$.

Proof. (1) ⇒ (2): Let E be a weakly mI_rg-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. Suppose that $\text{E} \subseteq \text{F}$ and F is an m-regular open set in $\mathcal{U}$. We have $(\text{mint}(\text{E}))^* \subseteq \text{F}$. Since $\text{mint}(\text{E}) \subseteq \text{E} \subseteq \text{F}$, then $(\text{mint}(\text{E}))^* \cup \text{mint}(\text{E}) \subseteq \text{F}$. This implies that $\text{mcl}^*(\text{mint}(\text{E})) \subseteq \text{F}$.

(2) ⇒ (1): Let $\text{mcl}^*(\text{mint}(\text{E})) \subseteq \text{F}$ whenever $\text{E} \subseteq \text{F}$ and F is m-regular open in $\mathcal{U}$. Since $(\text{mint}(\text{E}))^* \cup \text{mint}(\text{E}) \subseteq \text{F}$ then $(\text{mint}(\text{E}))^* \subseteq \text{F}$ whenever $\text{E} \subseteq \text{F}$ and F is an m-regular open set in $\mathcal{U}$.

Theorem 3.2 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\text{E} \subseteq \mathcal{U}$. If E is m-regular open and weakly mI_rg-closed, then E is m^*-closed.

Proof. Let E be an m-regular open and weakly mI_rg-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. Since E is m-regular open and weakly mI_rg-closed,

$$\text{mcl}^*(\text{E}) = \text{mcl}^*(\text{mint}(\text{E})) \subseteq \text{E}.$$

Thus, E is an m^*-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 3.3 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\text{E} \subseteq \mathcal{U}$. If E is a weakly mI_rg-closed set, then $(\text{mint}(\text{E}))^* - \text{E}$ contains no nonempty m-regular closed set.

Proof. Let E be a weakly mI_rg-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. Suppose that F is an m-regular closed set such that $\text{F} \subseteq (\text{mint}(\text{E}))^* - \text{E}$. Since E is a weakly mI_rg-closed set, $\mathcal{U} - \text{F}$ is m-regular open and $\text{E} \subseteq \mathcal{U} - \text{F}$, then $(\text{mint}(\text{E}))^* \subseteq \mathcal{U} - \text{F}$. We have $\text{F} \subseteq \mathcal{U} - (\text{mint}(\text{E}))^*$. Hence, $\text{F} \subseteq (\text{mint}(\text{E}))^* \cap (\mathcal{U} - (\text{mint}(\text{E}))^*) = \emptyset$. Thus, $(\text{mint}(\text{E}))^* - \text{E}$ contains no nonempty m-regular closed set.

Theorem 3.4 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\text{E} \subseteq \mathcal{U}$. If E is a weakly mI_rg-closed set, then $\text{mcl}^*(\text{mint}(\text{E})) - \text{E}$ contains no nonempty m-regular closed set.

Proof. Suppose that F is an m-regular closed set such that $\text{F} \subseteq \text{mcl}^*(\text{mint}(\text{E})) - \text{E}$. By Theorem 3.4, it follows from the fact that $\text{mcl}^*(\text{mint}(\text{E})) - \text{E} = ((\text{mint}(\text{E}))^* \cup \text{mint}(\text{E})) - \text{E} = (\text{mint}(\text{E}))^* - \text{E}$.

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Theorem 3.5 Let $(\mathcal{U},\mathcal{M},\mathcal{I})$ be an ideal minuscule topological space. The following properties are equivalent:

1.E is pre *-I_m-closed for each weakly mI_{rg}-closed set E in $(\mathcal{U},\mathcal{M},\mathcal{I})$.

2.Each singleton $\{y\}$ of $\mathcal{U}$ is an m-regular closed set or $\{y\}$ is pre *-I_m-open.

Proof. (1) $\Rightarrow$ (2): Let E be pre *-I_m-closed for each weakly mI_{rg}-closed set E in $(\mathcal{U},\mathcal{M},\mathcal{I})$ and $y \in \mathcal{U}$. We have $mcl^*(mint(E)) \subseteq E$ for each weakly mI_{rg}-closed set E in $(\mathcal{U},\mathcal{M},\mathcal{I})$. Assume that $\{y\}$ is not an m-regular closed set. It follows that $\mathcal{U}$ is the only m-regular open set containing $\mathcal{U} - \{y\}$. Then, $\mathcal{U} - \{y\}$ is a weakly mI_{rg} -closed set in $(\mathcal{U},\mathcal{M},\mathcal{I})$. Thus, $mcl^*(mint(\mathcal{U} - \{y\})) \subseteq \mathcal{U} - \{y\}$ and hence $\{y\} \subseteq mint^*(mcl(\{y\}))$. Consequently, $\{y\}$ is pre *-I_m-open.

(2) $\Rightarrow$ (1): Let E be a weakly mI_{rg}-closed set in $(\mathcal{U},\mathcal{M},\mathcal{I})$. Let $y \in mcl^*(mint(E))$. Suppose that $\{y\}$ is pre *-I_m-open. We have $\{y\} \subseteq mint^*(mcl(\{y\}))$. Since $y \in mcl^*(mint(E))$, then $mint^*(mcl(\{y\})) \cap mint(E) \neq \emptyset$. It follows that $mcl(\{y\}) \cap mint(E) \neq \emptyset$. We have $mcl(\{y\} \cap mint(E)) \neq \emptyset$ and then $\{y\} \cap mint(E) \neq \emptyset$. Hence, $y \in mint(E)$. Thus, we have $y \in E$.

Suppose that $\{y\}$ is an m-regular closed set. By Theorem 3.5, $mcl^*(mint(E)) - E$ does not contain $\{y\}$. Since $y \in mcl^*(mint(E))$, then we have $y \in E$. Consequently, we have $y \in E$. Thus, $mcl^*(mint(E)) \subseteq E$ and hence E is pre *-I_m-closed.

Theorem 3.6 Let $(\mathcal{U},\mathcal{M},\mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. If $mcl^*(mint(E)) - E$ contains no nonempty m^*-closed set, then E is a weakly mI_{rg}-closed set.

Proof. Suppose that $mcl^*(mint(E)) - E$ contains no nonempty m^*-closed set in $(\mathcal{U},\mathcal{M},\mathcal{I})$. Let $E \subseteq F$ and F be an m-regular open set. Assume that $mcl^*(mint(E))$ is not contained in F. It follows that $mcl^*(mint(E)) \cap (\mathcal{U} - F)$ is a nonempty m^*-closed subset of $mcl^*(mint(E)) - E$. This is a contradiction. Hence, E is a weakly mI_{rg}-closed set.

Theorem 3.7 Let $(\mathcal{U},\mathcal{M},\mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. If E is a weakly mI_{rg}-closed set, then $mint(E) = F - L$ where F is m^*-closed and L contains no nonempty m-regular closed set.

Proof. Let E be a weakly mI_{rg}-closed set in $(\mathcal{U},\mathcal{M},\mathcal{I})$. Take $L = (mint(E))^* - E$. Then, by Theorem 3.4, L contains no nonempty m-regular closed set. Take $F = mcl^*(mint(E))$. Then $F = mcl^*(mint(F))$. Moreover, we have

$$F - L = ((mint(E))^* \cup mint(E)) - ((mint(E))^* - E)$$

$$= ((mint(E))^* \cup mint(E)) \cap (\mathcal{U} - (mint(E))^* \cup E)$$

$$= mint(E).$$

Theorem 3.8 Let $(\mathcal{U},\mathcal{M},\mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. Assume that E is a weakly mI_{rg}-closed set. The following properties are equivalent:

1.E is pre *-I_m-closed.

2.$mcl^*(mint(E)) - E$ is an m-regular closed set.

3.$(mint(E))^* - E$ is an m-regular closed set.

Proof. (1) $\Rightarrow$ (2): Let E be pre *-I_m-closed. We have $mcl^*(mint(E)) \subseteq E$. Then, $mcl^*(mint(E)) - E = \emptyset$. Thus, $mcl^*(mint(E)) - E$ is an m-regular closed set.

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(2) ⇒ (1): Let $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) - \mathrm{E}$ be an m-regular closed set. Since E is a weakly $\mathrm{mI_{rg}}$-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$, then by Theorem 3.5, $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) - \mathrm{E} = \emptyset$. Hence, we have $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) \subseteq \mathrm{E}$. Thus, E is pre *-$\mathrm{I_m}$-closed.

(2) ⇒ (3): It follows easily from that $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) - \mathrm{E} = (\mathrm{mint}(\mathrm{E}))^* - \mathrm{E}$.

Theorem 3.9 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\mathrm{E} \subseteq \mathcal{U}$ be a weakly $\mathrm{mI_{rg}}$-closed set. Then $\mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*)$ is a weakly $\mathrm{mI_{rg}}$-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Proof. Let E be a weakly $\mathrm{mI_{rg}}$-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. Suppose that F is an m-regular open set such that $\mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*) \subseteq \mathrm{F}$. We have

$$\mathcal{U} - \mathrm{F} \subseteq \mathcal{U} - (\mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*))$$

$$= (\mathcal{U} - \mathrm{E}) \cap (\mathrm{mint}(\mathrm{E}))^*$$

$$= (\mathrm{mint}(\mathrm{E}))^* - \mathrm{E}.$$

Since $\mathcal{U} - \mathrm{F}$ is an m-regular closed set and E is a weakly $\mathrm{mI_{rg}}$-closed set, it follows from Theorem 3.4 that $\mathcal{U} - \mathrm{F} = \emptyset$. Hence, $\mathcal{U} = \mathrm{F}$. Thus, $\mathcal{U}$ is the only m-regular open set containing $\mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*)$. Consequently, $\mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*)$ is a weakly $\mathrm{mI_{rg}}$-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Corollary 3.1 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\mathrm{E} \subseteq \mathcal{U}$ be a weakly $\mathrm{mI_{rg}}$-closed set. Then $(\mathrm{mint}(\mathrm{E}))^* - \mathrm{E}$ is a weakly $\mathrm{mI_{rg}}$-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Proof. Since $\mathcal{U} - ((\mathrm{mint}(\mathrm{E}))^* - \mathrm{E}) = \mathrm{E} \cup (\mathcal{U} - (\mathrm{mint}(\mathrm{E}))^*)$, it follows from Theorem 3.9 that $(\mathrm{mint}(\mathrm{E}))^* - \mathrm{E}$ is a weakly $\mathrm{mI_{rg}}$-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 3.10 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $\mathrm{E} \subseteq \mathcal{U}$. The following properties are equivalent:

1.E is an m^*-closed and m-regular open set.

2.E is m^*-closed and an m-regular open set.

3.E is a weakly $\mathrm{mI_{rg}}$-closed and m-regular open set.

Proof. (1) ⇒ (2) ⇒ (3): Obvious.

(3) ⇒ (1): It follows from Theorem 3.3.

Minuscule Properties

Theorem 4.1 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space. The following properties are equivalent:

1.Each subset of $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ is a weakly $\mathrm{mI_{rg}}$-closed set.

2.E is pre *-$\mathrm{I_m}$-closed for each m-regular open set E in $\mathcal{U}$.

Proof. (1) ⇒ (2): Suppose that each subset of $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ is a weakly $\mathrm{mI_{rg}}$-closed set. Let E be an m-regular open set. Since E is weakly $\mathrm{mI_{rg}}$-closed, then we have $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) \subseteq \mathrm{E}$. Thus, E is pre *-$\mathrm{I_m}$-closed.

(2) ⇒ (1): Let E be a subset of $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ and F be an m-regular open set such that $\mathrm{E} \subseteq \mathrm{F}$. By (2), we have $\mathrm{mcl}^*(\mathrm{mint}(\mathrm{E})) \subseteq \mathrm{mcl}^*(\mathrm{mint}(\mathrm{F})) \subseteq \mathrm{F}$. Thus, E is a weakly $\mathrm{mI_{rg}}$-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 4.2 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space. If E is a weakly mI_{rg}-closed set and $E \subseteq F \subseteq mcl^*(mint(E))$, then F is a weakly mI_{rg}-closed set.

Proof. Let $F \subseteq L$ and L be an m-regular open set in $\mathcal{U}$. Since $E \subseteq L$ and E is a weakly mI_{rg}-closed set, then $mcl^*(mint(E)) \subseteq L$ .Since $F \subseteq mcl^*(mint(E))$, then $mcl^*(mint(F)) \subseteq mcl^*(mint(E)) \subseteq L$. Thus, $mcl^*(mint(F)) \subseteq L$ and hence, F is a weakly mI_{rg}-closed set.

Corollary 4.1 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space. If E is a weakly mI_{rg}-closed and open set, then $mcl^*(E)$ is a weakly mI_{rg}-closed set.

Proof. Let E be a weakly mI_{rg}-closed and open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. We have $E \subseteq mcl^*(E) = mcl^*(mint(E))$. Hence, by Theorem 4.2, $mcl^*(E)$ is a weakly mI_{rg}-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 4.3 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$.If E is an m-nowhere dense set,then E is a weakly mI_{rg}-closed set.

Proof. Let E be an m-nowhere dense set in $\mathcal{U}$. Since $mint(E) \subseteq mint(mcl(E))$, then $mint(E) = \emptyset$. Hence, $mcl^*(mint(E)) = \emptyset$.Thus, E is a weakly mI_{rg}-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Remark 4.1

- The union of two weakly mI_{rg}-closed sets in an ideal minuscule topological space need not be a weakly mI_{rg}-closed set.
- The intersection of two weakly mI_{rg}-closed sets in an ideal minuscule topological space need not be a weakly mI_{rg}-closed set.

Example 4.1 Let $\mathcal{U} = \{a, b, c, d\}$ with $\mathcal{U}/\mathcal{R} = \{\{a\}, \{c\}, \{b, d\}\}$ and $\mathcal{Y} = \{c, d\}$.

Then $\mathcal{M} = \{\emptyset, \mathcal{U}, \{c\}, \{b, c, d\}, \{b\}, \{b, c\}\}$.

Let the ideal be $\mathcal{I} = \{\emptyset, \{d\}\}$.

Take $A = \{a\}$ and $B = \{c\}$.

Then A and B are weakly mI_{rg}-closed but $A \cup B = \{a, c\}$ is not a weakly mI_{rg}-closed set.

Example 4.2 In Example 4.1, take $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then A and B are weakly mI_{rg}-closed but $A \cap B = \{a, c\}$ is not a weakly mI_{rg}-closed set.

Theorem 4.4 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. Then E is a weakly mI_{rg}-open set if and only if $F \subseteq mint^*(mcl(E))$ whenever $F \subseteq E$ and F is an m-regular closed set.

Proof. Let F be an m-regular closed set in $\mathcal{U}$ and $F \subseteq E$. It follows that $\mathcal{U} - F$ is an m-regular open set and $\mathcal{U} - E \subseteq \mathcal{U} - F$. Since $\mathcal{U} - E$ is a weakly mI_{rg}-closed set, then $mcl^*(mint(\mathcal{U} - E)) \subseteq \mathcal{U} - F$. We have $\mathcal{U} - mint^*(mcl(E)) \subseteq \mathcal{U} - F$. Thus, $F \subseteq mint^*(mcl(E))$.

Conversely, let L be an m-regular open set in $\mathcal{U}$ and $\mathcal{U} - E \subseteq L$. Since $\mathcal{U} - L$ is an m-regular closed set such that $\mathcal{U} - L \subseteq E$, then $\mathcal{U} - L \subseteq mint^*(mcl(E))$. We have $\mathcal{U} - mint^*(mcl(E)) = mcl^*(mint(\mathcal{U} - E)) \subseteq L$. Thus, $\mathcal{U} - E$ is a weakly mI_{rg}-closed set. Hence, E is a weakly mI_{rg}-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 4.5 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. If E is a weakly mI_{rg}-closed set, then $mcl^*(mint(E)) - E$ is a weakly mI_{rg}-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Proof. Let E be a weakly mI_{rg}-closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$. Suppose that F is an m-regular closed set such that $F \subseteq mcl^*(mint(E)) - E$. Since E is a weakly mI_{rg}-closed set, it follows from Theorem 3.5 that $F = \emptyset$. Thus, we have $F \subseteq mint^*(mcl(mcl^*(mint(E)) - E))$. It follows from Theorem 4.8 that $mcl^*(mint(E)) - E$ is a weakly mI_{rg}-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Theorem 4.6 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. If E is a weakly mI_{rg}-open set, then $F = \mathcal{U}$ whenever F is an m-regular open set and $mint^*(mcl(E)) \cup (\mathcal{U} - E) \subseteq F$.

Proof. Let F be an m-regular open set in $\mathcal{U}$ and $mint^*(mcl(E)) \cup (\mathcal{U} - E) \subseteq F$. We have

$$\mathcal{U} - F \subseteq (\mathcal{U} - mint^*(mcl(E))) \cap E$$

$$= mcl^*(mint(\mathcal{U} - E)) - (\mathcal{U} - E).$$

Since $\mathcal{U} - F$ is an m-regular closed set and $\mathcal{U} - E$ is a weakly mI_{rg}-closed set, it follows from Theorem 3.5 that $\mathcal{U} - F = \emptyset$. Thus, we have $F = \mathcal{U}$.

Theorem 4.7 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space. If E is a weakly mI_{rg}-open set and $mint^*(mcl(E)) \subseteq F \subseteq E$, then F is a weakly mI_{rg}-open set.

Proof. Let E be a weakly mI_{rg}-open set and $mint^*(mcl(E)) \subseteq F \subseteq E$. Since $mint^*(mcl(E)) \subseteq F \subseteq E$, then $mint^*(mcl(E)) = mint^*(mcl(F))$.

Let L be an m-regular closed set and $L \subseteq F$. We have $L \subseteq E$. Since E is a weakly mI_{rg}-open set, it follows from Theorem 4.5 that $L \subseteq mint^*(mcl(E)) = mint^*(mcl(F))$. Hence, by Theorem 4.7, F is a weakly mI_{rg}-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

Corollary 4.2 Let $(\mathcal{U}, \mathcal{M}, \mathcal{I})$ be an ideal minuscule topological space and $E \subseteq \mathcal{U}$. If E is a weakly mI_{rg}-open and closed set, then $mint^*(E)$ is a weakly mI_{rg}-open set.

Proof. Let E be a weakly mI_{rg}-open and closed set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.Then $mint^*(mcl(E)) = mint^*(E) \subseteq mint^*(E) \subseteq E$. Thus, by Theorem 4.1, $mint^*(E)$ is a weakly mI_{rg}-open set in $(\mathcal{U}, \mathcal{M}, \mathcal{I})$.

CONCLUSION

This article presents weakly minuscule regular generalized closed sets within ideal minuscule topological spaces as an extension of regular generalized closed sets through weak minuscule conditions. The work establishes characterizations of these sets, examines their inclusion relations, and investigates their behavior under standard set operations. Overall, the study advances generalized closure theory by introducing and analyzing this new class of sets.

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